

# Exact and simple results for the XYZ spin chain

Paul Fendley

Work with Christian Hagendorf

One of most amazing results in statistical mechanics was announced in a conference discussion section:

## DISCUSSIONE E OSSERVAZIONI

Prof. J. G. KIRKWOOD, *Pasadena, Cal.*:

Asks RUSHBROOKE if he has evaluated the critical temperature for a plane cubic lattice, for which we know the exact answer.

Prof. G. S. RUSHBROOKE, *Oxford*:

DOMB has derived analogous high- and low-temperature expansions for the plane square lattice, but used the known critical temperature in discussing their behaviour. It would be useful to forget all about the known value and see how accurately it could be predicted from the expansions.

Prof. L. ONSAGER, *New Haven, Conn.*:

Mathematically, the composition-temperature curve in a solid solution presents the same problem as the degree of order in a ferromagnetic with scalar spin. B. KAUFMAN and I have recently solved the latter problem (unpublished) for a two-dimensional rectangular net with interaction energies  $J, J'$ . If we write  $\sinh (2J/kT) \sinh (2J'/kT) = 1/k$ , then the degree of order, for  $k < 1$ , is simply  $(1 - k^2)^{1/8}$ .

Prof. J. E. MAYER, *Chicago, Ill.*:

As I pointed out in my paper, there is a formal exact mathematical correspondence

# The quantum Ising chain

The Hilbert space is a two-state quantum system on each site of a chain.

The Hamiltonian has **flip terms** and **nearest-neighbor interactions**:


$$H = - \sum_{j=1}^L \left( k \sigma_j^x + \sigma_j^z \sigma_{j+1}^z \right)$$

For  $k < 1$ , the interactions win: the spins favor **alignment**. The spontaneous magnetization in the ground state  $M(k) = \langle 0 | \sigma_j^z | 0 \rangle$  **does not vanish**.

We now know a great deal about critical behavior from Landau theory, the renormalization group, conformal invariance, large N, integrability, etc.

But none of these explain why Onsager, Kaufman and Yang's result is both **ridiculously simple** and **exact**!

$$M(k) = (1 - k^2)^{1/8}$$

# What's so special about the Ising chain?

Yes, I know it can be mapped onto free fermions.

But the map from spins to fermions is **non-local**!

The computation of the magnetization formula is still complicated.

# Another example: dimers on the triangular lattice

By a horrible mess using the determinant of a matrix that is Toeplitz+Hankel, the limit of monomer-monomer correlator at large distances is

$$0.14942924536134225401731517482693\dots = \sin(\pi/12)/\sqrt{3}$$

Fendley, Moessner and Sondhi

After an incredibly complicated analysis of the asymptotics of these matrices:

$$\frac{1}{2} \sqrt{\frac{t}{2t(2+t^2) + (1+2t^2)\sqrt{2+t^2}}}$$

where  $t=0$  is the square lattice and  $t=1$  is the triangular.

Basor and Ehrhardt

Are there non free-fermi systems  
with simple and exact formulas?

Yes!



The chiral Potts model is an  $N$ -state generalization of the Ising model with some amazing properties.

Howes, Kadanoff and den Nijs; von Gehlen and Rittenberg

It is definitely **not** free-fermionic.

Yet the order parameters are given by

$$\langle e^{2\pi i r \sigma_j / N} \rangle = (1 - k^2)^{r(N-r)/(2N^2)}$$

conjectured in 1989 by Albertini, McCoy, Perk and Tang ;  
proved in 2005/2010 by Baxter

# Supersymmetric field theories have similar nice properties.

- For example, in  $N=(2,2)$  theories in 1+1 dimensions, the potential is **not renormalized** beyond naïve scaling.
- Thus if the potential has multiple minima, the **gap/mass of the (BPS) kinks** can be computed by a pedestrian computation.
- We should look at quantum chains that turn into supersymmetric field theories in the scaling limit.

# The XYZ spin chain

$$H = - \sum_{j=1}^L \left[ J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z \right]$$

yields a **supersymmetric field theory** when

$$J_x J_y + J_x J_z + J_y J_z = 0$$

This is easy to see along the **critical line**  $J_x = J_y$ ,  $|J_z| \leq 1$ , where the XXZ chain becomes a free boson. The off-critical perturbation turns this into the **sine-Gordon model** at its supersymmetric coupling.

# Along this line of couplings, the XYZ chain satisfies some remarkable properties.

- As  $L$  gets large, the ground state energy satisfies

$$E_0 = -L(J_x + J_y + J_z)$$

Baxter 1972

- This relation is conjectured (and partially proved) to be **exact for any  $L$  when  $L$  is odd**.

Stroganov; Fendley, Nienhuis and Schoutens

- Coefficients of the terms in the ground state are related to the numbers of **alternating sign matrices**.

Razumov and Stroganov

$$H = - \sum_{j=1}^L \left[ J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z \right]$$

We parameterize this **sXYZ** chain as

$$J_x = 2s(s - 3), \quad J_y = 2s(s + 3), \quad J_z = 9 - s^2$$

It has critical points at  $s = \pm 1, \infty$ .

The **symmetries** of permuting the couplings are

$$s \rightarrow (3 - s)/(s + 1), \quad s \rightarrow -s$$

# The glories of experimental mathematics:

Find the ground state explicitly for small systems using Maple.

Doing this for 5 sites yields the magnetization

$$M_5(s) = \frac{(1 - s^2)^2(1 + 5s^2)}{(1 + 6s^2 + 25s^4)(1 + s^2)}$$

This is already slightly special: the simple ground state energy means the coefficients of the ground state in the spin basis are **polynomials in  $s$  with integer coefficients**. But just wait...

# A new **amazing property** of the sXYZ chain

Expand  $M_L(s)$  in a **series** around the **completely aligned limit  $s=0$** :

$$M_5 = 1 - 4 \tilde{s}^2 - 12 \tilde{s}^4 + 188 \tilde{s}^6 - 844 \tilde{s}^8 + 380 \tilde{s}^{10} + \dots$$

$$M_7 = 1 - 4 \tilde{s}^2 - 12 \tilde{s}^4 - 52 \tilde{s}^6 + 2516 \tilde{s}^8 - 18004 \tilde{s}^{10} + \dots$$

$$M_9 = 1 - 4 \tilde{s}^2 - 12 \tilde{s}^4 - 52 \tilde{s}^6 - 284 \tilde{s}^8 + 33516 \tilde{s}^{10} + \dots$$

$$M_{11} = 1 - 4 \tilde{s}^2 - 12 \tilde{s}^4 - 52 \tilde{s}^6 - 284 \tilde{s}^8 - 1764 \tilde{s}^{10} + \dots$$

where  $\tilde{s} = s/3$ .

We call quantities **scale free** when terms up to order  $L$  are **independent of  $L$** .

Thus by solving the system at small  $L$ , we presumably obtain the **exact expansion in the scaling limit!**

$$M_{\infty}(s) = 1 - 4\tilde{s}^2 - 12\tilde{s}^4 - 52\tilde{s}^6 - 284\tilde{s}^8 - 1764\tilde{s}^{10} + \dots$$

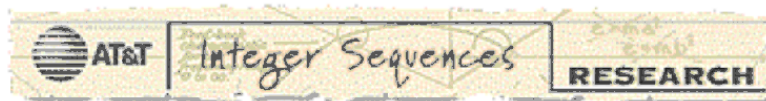
So then we do what any good combinatorialist would do...



# From the On-Line Encyclopedia of Integer Sequences:

1,3,13,71,441 - OEIS Search Results

<http://www.research.att.com/~njas/sequences/in>



**Greetings from The On-Line Encyclopedia of Integer Sequences!**

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Let  $y = y(x)$  satisfy  $g(x, y(x)) = 0$ . The sequence  $a(n)$  is the number of terms in the expansion of the divided difference  $[x_0, \dots, x_n]y$  in terms of bivariate divided differences of  $g$ .

**1, 3, 13, 71, 441**, 2955, 20805, 151695, 1135345, 8671763, 67320573, 529626839, 4213228969, 33833367963, 273892683573, 2232832964895, 18314495896545, 151037687326755, 1251606057754605, 10416531069771111, 87029307323766681 ([list](#); [graph](#); [listen](#))

OFFSET        1, 2

FORMULA       Let  $E = \mathbb{N} \times \mathbb{N} \setminus \{(0, 0), (0, 1)\}$  be a set of pairs of natural numbers. The number of terms  $a(n)$  is the coefficient of  $x^n y^{n-1}$  of the generating function

$$1 - \log(1 - \sum_{(s,t) \in E} x^s y^{s+t-1}) = 1 + \sum_{q \geq 1} (\sum_{(s,t) \in E} x^s y^{s+t-1})^q / q$$

## Let's think like physicists:

The dimension of the perturbing operator is  $4/3$ .

Baxter

Coulomb gas/CFT arguments give dimension  $1/3$  for the magnetization operator. Indeed, small- $L$  numerics yield

$$M_L(1) \approx .9552745 L^{-1/3}(1 + O(L^{-2}))$$

Thus as  $s \rightarrow 1$ , the critical behavior is

$$M_L(s) \propto (1 - s)^\beta$$

where  $\beta = \frac{1/3}{2 - 4/3} = 1/2$ .

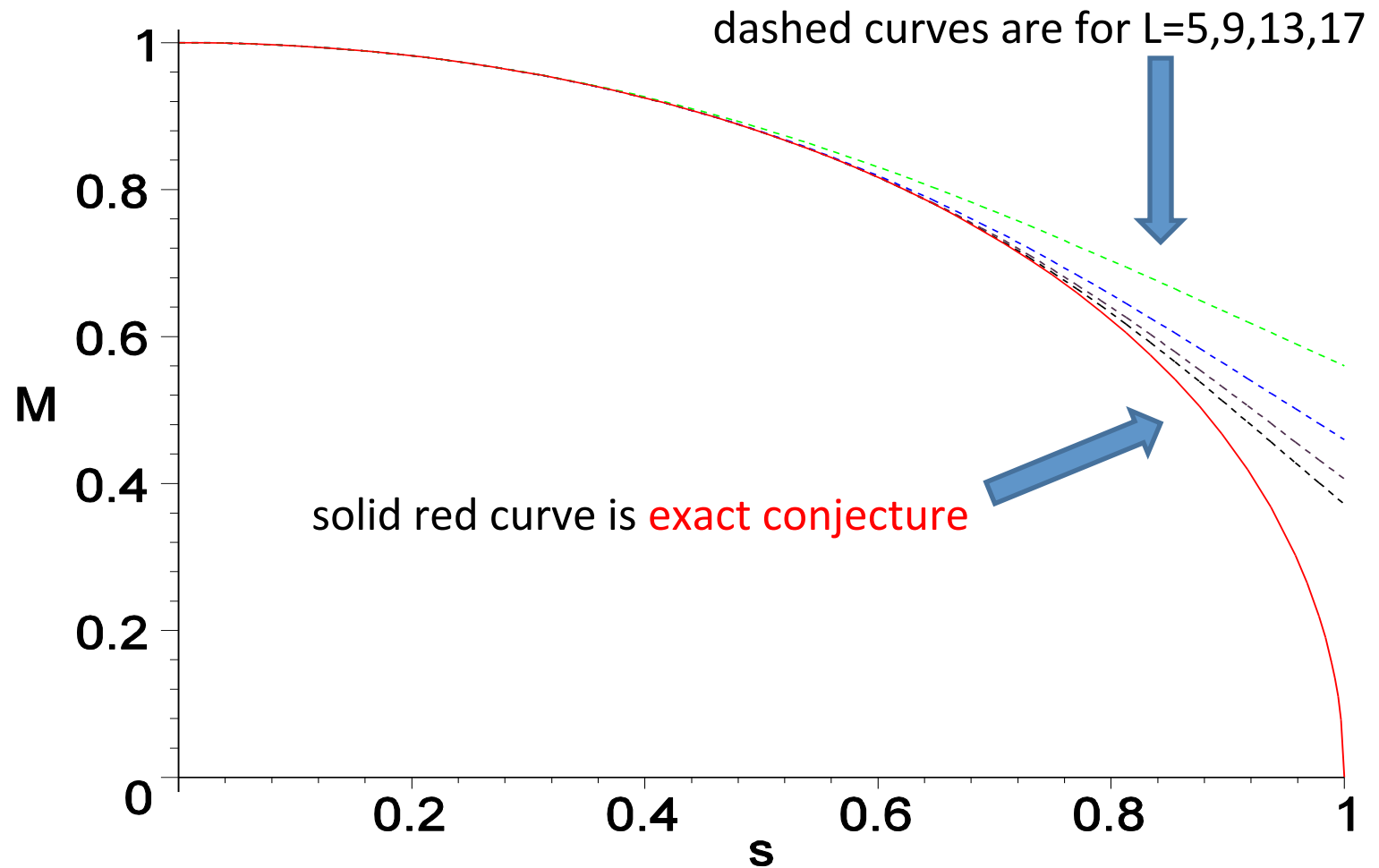
The square-root singularity suggests that we try **squaring the series**.

$$M_{\infty}(s) = 1 - 4\tilde{s}^2 - 12\tilde{s}^4 - 52\tilde{s}^6 - 284\tilde{s}^8 - 1764\tilde{s}^{10} + \dots$$

$$(M_{\infty}(s))^2 = 1 - 8\tilde{s}^2 - 8\tilde{s}^4 - 8\tilde{s}^6 - 8\tilde{s}^8 - 8\tilde{s}^{10} - \dots$$

So, we conjecture that

$$M_{\infty}(s) = 3 \left( \frac{1-s^2}{9-s^2} \right)^{1/2}$$



Using some looks-new theta-function identities, this is the same as Baxter's corner transfer matrix computation.

Hagendorf

We find **exact and simple** formulas for other quantities:

For  $s < 1$ ,

$$\begin{aligned}\langle 0 | \sigma_j^a \sigma_{j+1}^a | 0 \rangle &= 1 + 4\tilde{s}^2(-1 + \tilde{s}^2 + 3\tilde{s}^4 + 5\tilde{s}^6 + 7\tilde{s}^8 + \dots) \\ &= 1 + 12 \frac{s^2(s^2 - 3)}{(s^2 - 9)^2} + O(s^{L+1})\end{aligned}$$

while for  $s > 1$ , letting  $t = (3-s)/6$ ,

$$\begin{aligned}\langle 0 | \sigma_j^a \sigma_{j+1}^a | 0 \rangle &= \frac{1}{2}(2t + 3t^2 + 4t^3 + 5t^4 + \dots) \\ &= \frac{(s+9)(3-s)}{2(3+s)^2} + O(t^{L+1})\end{aligned}$$

These also can be found using Baxter's solution and theta-function identities.

# Why $L$ odd?

- The XYZ chain has a  $\mathbb{Z}_2$  spin-flip symmetry.
- In the  $J_z \rightarrow \infty$  limit, there are two ground states: all spins up, and all spins down.
- When  $L$  is odd, the two ground states are in different sectors. They do not mix!
- No external field or boundary condition is needed to spontaneously break the symmetry.

Even the gap has an exact and simple formula.

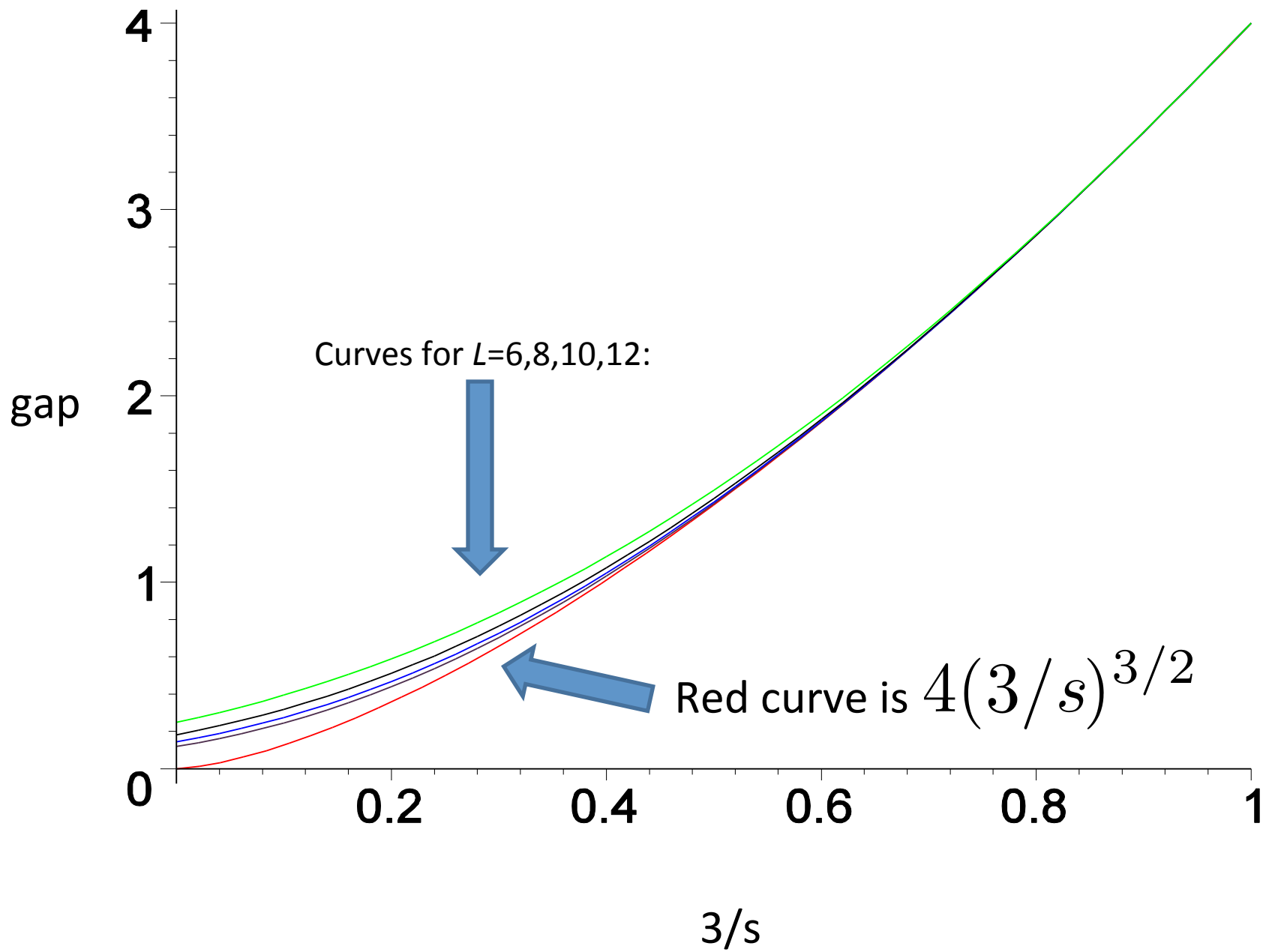
Go to an **even** number of sites and impose **twisted boundary conditions**, where the lowest-energy state is intuitively a **one-kink** state, separating the two ground states.

Expanding the gap around the completely aligned point  $s=3$ :

$$\begin{aligned}\Delta_L &= 4 - 6v + 3v^2/2 + v^3/4 + 3v^4/32 + \dots \\ &= 4 \left( \frac{3}{s} \right)^{3/2} + O(v^{L/2})\end{aligned}$$

where  $v=1-3/s$ .

The gap vanishes at the critical point  $1/s \rightarrow 0$  with exponent  $n=3/2=1/(2-4/3)$ , just like it should!





# So what's going on?

We found a hidden non-local supersymmetry in the XYZ. Its generators change the number of sites.

We expect that there's a symmetry algebra involving **integrability** and **supersymmetry**, akin to the Onsager algebra present in Ising and chiral Potts.

The situation is illuminated by ``fat-fermion'' model with an **explicit supersymmetry**. The same seeming miracles occur.

## But wait, there's more:

The ground states of the fat-fermion and sXYZ chains involve the **same** polynomial, when  $L$  in the latter is  $2f-1$  in the former.

These are the same polynomials that Bazhanov and Mangazeev related to the Painleve VI integrable non-linear differential equation!

# Many, many open questions

- Can we prove this directly in less than 15 years?
- Such models are the simplest generalization of free fermions in 1d. Can we couple them into 2d models?
- Are these really simpler than free fermions?
- What's with Painleve VI?
- Anyone have an announcement to make?