

Nonequilibrium dynamics of a quasi-disordered integrable quantum system after a quench

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in transition to

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Supported by:



Quenches in disordered quantum systems

1 Introduction

• Motivation

- Unitary evolution and thermalization
- Results for nonintegrable and integrable systems

2 Non-equilibrium dynamics in the presence of disorder

- Exact solution
- Time evolution
- Results after relaxation vs statistical mechanics

3 Summary

Foundations of quantum statistical mechanics

Quantum ergodicity: John von Neumann '29
(Proof of the ergodic theorem and the
H-theorem in quantum mechanics)



Related to ideas on typicality and eigenstate thermalization:

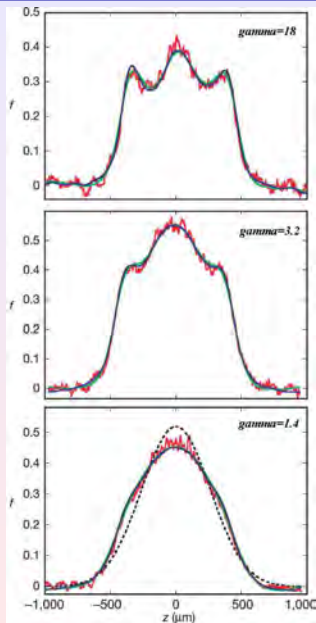
Goldstein, Lebowitz, Tumulka, and Zanghi '06
(Canonical Typicality)

Popescu, Short, and A. Winter '06
(Entanglement and the foundation of statistical mechanics)

Goldstein, Lebowitz, Mastrodonato, Tumulka, and Zanghi '10
(Normal typicality and von Neumann's quantum ergodic theorem)

MR and Srednicki '12
(Alternatives to Eigenstate Thermalization)

Experiments with ultracold gases in 1D geometries



T. Kinoshita, T. Wenger, and D. S. Weiss,
Nature **440**, 900 (2006).

Effective 1D δ potential

M. Olshanii, PRL **81**, 938 (1998).

$$U_{1D}(x) = g_{1D}\delta(x)$$

where

$$g_{1D} = \frac{2\hbar a_s \omega_{\perp}}{1 - C a_s \sqrt{\frac{m\omega_{\perp}}{2\hbar}}}$$

Lieb-Liniger Parameter

$$\gamma = \frac{mg_{1D}}{\hbar^2 \rho}$$

Observables (density and momentum
distribution functions) equilibrated to
nonthermal distributions

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Results from quantum mechanics (sudden quench)

If the initial state is not an eigenstate of $\hat{H}$

$$|\psi_0\rangle \neq |\alpha\rangle \quad \text{where} \quad \hat{H}|\alpha\rangle = E_\alpha|\alpha\rangle \quad \text{and} \quad E_0 = \langle\psi_0|\hat{H}|\psi_0\rangle,$$

then a generic (few-body) observable O will evolve in time following

$$O(\tau) \equiv \langle\psi(\tau)|\hat{O}|\psi(\tau)\rangle \quad \text{where} \quad |\psi(\tau)\rangle = e^{-i\hat{H}\tau}|\psi_0\rangle.$$

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What is it that we call thermalization?

$$\overline{O(\tau)} = O(E_0) = O(T) = O(T, \mu).$$

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One can rewrite

$$O(\tau) = \sum_{\alpha', \alpha} C_{\alpha'}^* C_{\alpha} e^{i(E_{\alpha'} - E_{\alpha})\tau} O_{\alpha' \alpha} \quad \text{where} \quad |\psi_0\rangle = \sum_{\alpha} C_{\alpha} |\alpha\rangle,$$

and taking the infinite time average (diagonal ensemble)

$$\overline{O(\tau)} = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^{\tau} d\tau' \langle\Psi(\tau')|\hat{O}|\Psi(\tau')\rangle = \sum_{\alpha} |C_{\alpha}|^2 O_{\alpha\alpha} \equiv \langle\hat{O}\rangle_{\text{diag}},$$

which depends on the initial conditions through $C_{\alpha} = \langle\alpha|\psi_0\rangle$.

Width of the energy density (sudden quench)

Initial state $|\psi_0\rangle = \sum_{\alpha} C_{\alpha} |\alpha\rangle$ is an eigenstate of $\hat{H}_0$. At $\tau = 0$

$$\hat{H}_0 \rightarrow \hat{H} = \hat{H}_0 + \hat{W} \quad \text{with} \quad \hat{W} = \sum_{j \in \sigma} \hat{w}(j) \quad \text{and} \quad \hat{H}|\alpha\rangle = E_{\alpha}|\alpha\rangle.$$

The width of the energy density ΔE is then

$$\Delta E = \sqrt{\sum_{\alpha} E_{\alpha}^2 |C_{\alpha}|^2 - (\sum_{\alpha} E_{\alpha} |C_{\alpha}|^2)^2} = \sqrt{\langle \psi_0 | \hat{W}^2 | \psi_0 \rangle - \langle \psi_0 | \hat{W} | \psi_0 \rangle^2},$$

or

$$\Delta E = \sqrt{\sum_{j_1, j_2 \in \sigma} [\langle \psi_0 | \hat{w}(j_1) \hat{w}(j_2) | \psi_0 \rangle - \langle \psi_0 | \hat{w}(j_1) | \psi_0 \rangle \langle \psi_0 | \hat{w}(j_2) | \psi_0 \rangle]} \stackrel{L \rightarrow \infty}{\propto} L^{d_{\sigma}/2}$$

Since the width of the full spectrum diverges as L^{d_L}

$$\Delta \epsilon = \frac{\Delta E}{L^{d_L}} \stackrel{L \rightarrow \infty}{\propto} \frac{1}{L^{d_L - d_{\sigma}/2}},$$

$d_L(d_{\sigma})$ is the dimensionality of the lattice (of the region affected by the quench).

since $d_L \geq d_{\sigma}$ then $\Delta \epsilon$ vanishes in the thermodynamic limit.

MR, V. Dunjko, and M. Olshanii, Nature **452**, 854 (2008).

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Description after relaxation (lattice models)

Hard-core boson (spinless fermion) Hamiltonian

$$\hat{H} = \sum_{i=1}^L -t \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + V \hat{n}_i \hat{n}_{i+1} - t' \left(\hat{b}_i^\dagger \hat{b}_{i+2} + \text{H.c.} \right) + V' \hat{n}_i \hat{n}_{i+2}$$

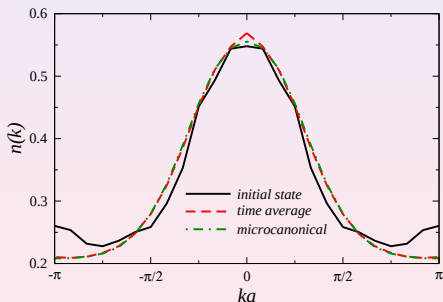
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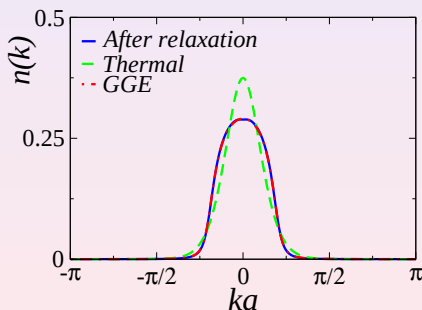
Dynamics vs statistical ensembles

Nonintegrable: $t' = V' \neq 0$



MR, PRL **103**, 100403 (2009);
PRA **80**, 053607 (2009), ...

Integrable: $t' = V' = 0$



MR, V. Dunjko, V. Yurovsky, and
M. Olshanii, PRL **98**, 050405 (2007), ...

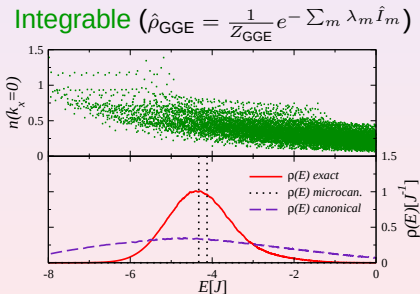
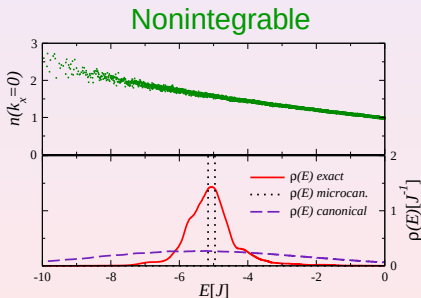
Eigenstate thermalization

Eigenstate thermalization hypothesis

[Deutsch, PRA **43** 2046 (1991); Srednicki, PRE **50**, 888 (1994).]

- The expectation value $\langle \alpha | \hat{O} | \alpha \rangle$ of a few-body observable $\hat{O}$ in an eigenstate of the Hamiltonian $|\alpha\rangle$, with energy E_α , of a many-body system equals the thermal average of $\hat{O}$ at the mean energy E_α :

$$\langle \alpha | \hat{O} | \alpha \rangle = \langle \hat{O} \rangle_{\text{microcan.}}(E_\alpha).$$



MR, V. Dunjko, and M. Olshanii, Nature **452**, 854 (2008).

What changes in the presence of disorder?

Many-body localization

- D. M. Basko, I. L. Aleiner, and B. L. Altshuler, Ann. Phys. **321**, 1126 (2006).
- V. Oganesyan and D. A. Huse, Phys. Rev. B **75**, 155111 (2007).
- A. Pal and D. A. Huse, Phys. Rev. B **82**, 174411 (2010).
- ...

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- ...

Some questions we would like to address

- How is the relaxation dynamics?
- Will observables fail to equilibrate?

$$O(\tau) \neq \overline{O(\tau)}$$

- If an observable equilibrates, will it fail to thermalize?

$$\overline{O(\tau)} \neq O(E_0) = O(T) = O(T, \mu)$$

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Bose-Fermi mapping

Hard-core boson Hamiltonian in an external potential

$$\hat{H} = -J \sum_i \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + \sum_i v_i \hat{n}_i$$

Constraints on the bosonic operators

$$\hat{b}_i^{\dagger 2} = \hat{b}_i^2 = 0$$

Bose-Fermi mapping

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Map to spins and then to fermions (Jordan-Wigner transformation)

$$\sigma_i^+ = \hat{f}_i^\dagger \prod_{\beta=1}^{i-1} e^{-i\pi \hat{f}_\beta^\dagger \hat{f}_\beta}, \quad \sigma_i^- = \prod_{\beta=1}^{i-1} e^{i\pi \hat{f}_\beta^\dagger \hat{f}_\beta} \hat{f}_i$$



Non-interacting fermion Hamiltonian

$$\hat{H}_F = -J \sum_i \left(\hat{f}_i^\dagger \hat{f}_{i+1} + \text{H.c.} \right) + \sum_i v_i \hat{n}_i^f$$

Bose-Fermi mapping

Hard-core boson Hamiltonian in an external potential

$$\hat{H} = -J \sum_i \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + \sum_i v_i \hat{n}_i$$

Constraints on the bosonic operators

$$\hat{b}_i^{\dagger 2} = \hat{b}_i^2 = 0$$



Integrals of motion

(underlying noninteracting fermions)

$$\begin{aligned} \hat{H}_F \hat{\gamma}_m^{f\dagger} |0\rangle &= E_m \hat{\gamma}_m^{f\dagger} |0\rangle \\ \left\{ \hat{I}_m^f \right\} &= \left\{ \hat{\gamma}_m^{f\dagger} \hat{\gamma}_m^f \right\} \end{aligned}$$

Lagrange multipliers

(calculated analytically)

$$\lambda_m = \ln \left[\frac{1 - \langle \hat{I}_m \rangle_{\tau=0}}{\langle \hat{I}_m \rangle_{\tau=0}} \right]$$

One-particle density matrix

One-particle Green's function

$$G_{ij} = \langle \Psi_{HCB} | \sigma_i^- \sigma_j^+ | \Psi_{HCB} \rangle = \langle \Psi_F | \prod_{\beta=1}^{i-1} e^{i\pi \hat{f}_\beta^\dagger \hat{f}_\beta} \hat{f}_i \hat{f}_j^\dagger \prod_{\gamma=1}^{j-1} e^{-i\pi \hat{f}_\gamma^\dagger \hat{f}_\gamma} | \Psi_F \rangle$$



Time evolution

$$|\Psi_F(\tau)\rangle = e^{-i\hat{H}_F\tau/\hbar} |\Psi_F^I\rangle = \prod_{\delta=1}^N \sum_{\sigma=1}^L P_{\sigma\delta}(\tau) \hat{f}_\sigma^\dagger |0\rangle$$



Exact Green's function

$$G_{ij}(\tau) = \det \left[\left(\mathbf{P}^l(\tau) \right)^\dagger \mathbf{P}^r(\tau) \right]$$

Computation time $\sim L^2 N^3$

3000 lattice sites, 300 particles

MR and A. Muramatsu, PRL **93**, 230404 (2004); PRL **94**, 240403 (2005).

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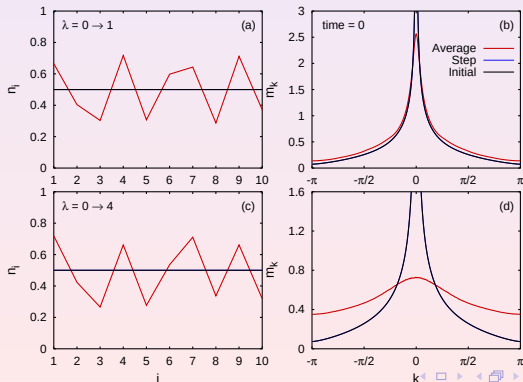
Model Hamiltonian and the localization transition

Hard-core boson Hamiltonian in 1D ($\lambda_c = 2J$)

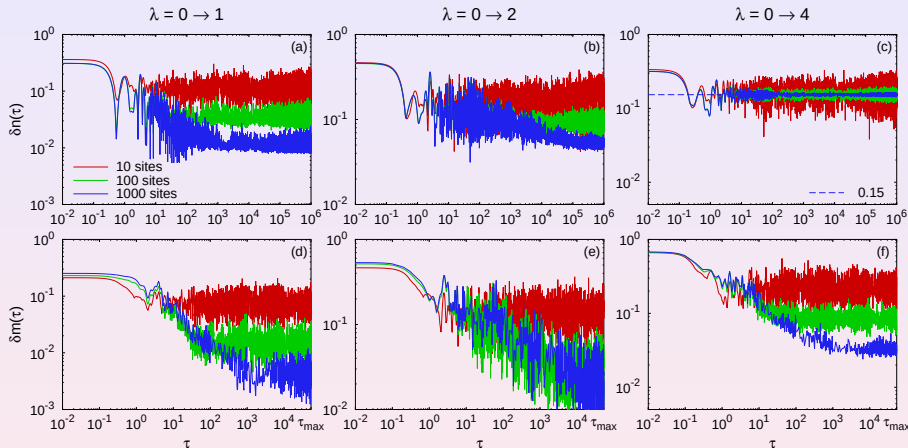
$$\hat{H} = -J \sum_{i=1}^{L-1} (\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.}) + \lambda \sum_i \cos(2\pi\sigma i + \delta) \hat{n}_i^b \quad \text{where} \quad \sigma = (\sqrt{5} - 1)/2$$

C. Gramsch and MR, Phys. Rev. A **86**, 053615 (2012).

Dynamics after a quench from the ground state ($\lambda_I \neq 0 \rightarrow \lambda_F < \lambda_I$)



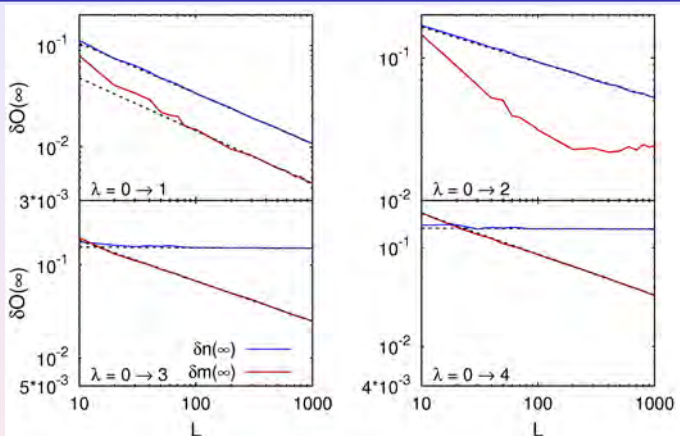
Dynamics after a quench from the ground state



$$\delta n(\tau) = \frac{\sum_i |n_i(\tau) - \overline{n_i(\tau)}|}{\sum_i \overline{n_i(\tau)}}$$

$$\delta m(\tau) = \frac{\sum_k |m_k(\tau) - \overline{m_k(\tau)}|}{\sum_k \overline{m_k(\tau)}}$$

Scaling of $\delta O(\tau)$ after relaxation



- Delocalized phase ($\lambda_F < 2$): $\delta n(\infty) \sim \delta m(\infty) \propto 1/\sqrt{L}$
- Critical point ($\lambda_F = 2$): $\delta n(\infty) \propto 1/L^{1/4}$
- Localized phase ($\lambda_F > 2$): $\delta n(\infty) = \text{const}$, $\delta m(\infty) \propto 1/\sqrt{L}$.
- Recent analytic results (proof for a specific class of observables):

L. Campos Venuti and P. Zanardi, arXiv:1208.1121.

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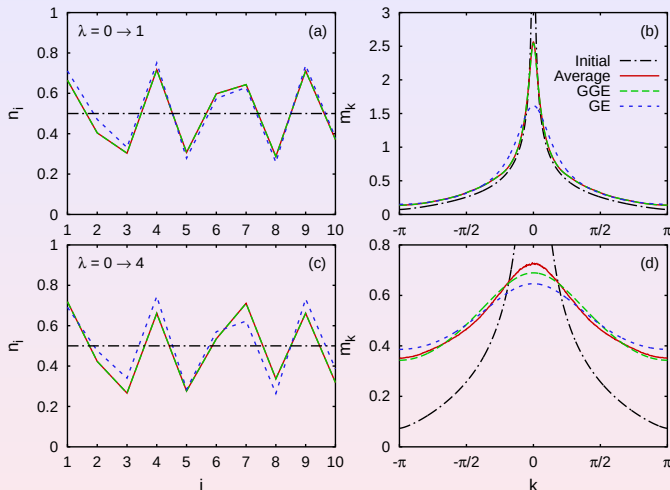
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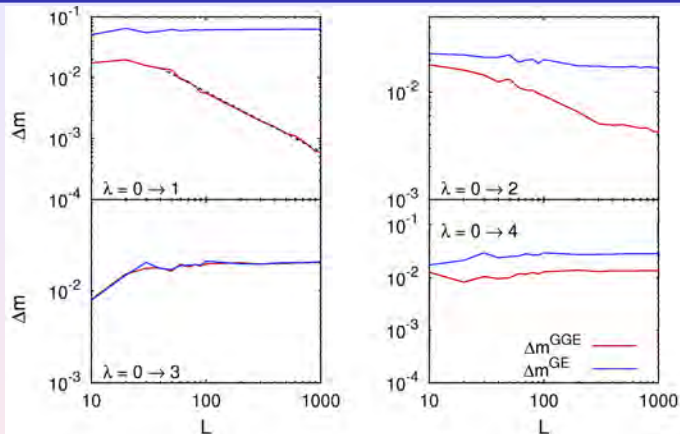
Summary

Results after relaxation vs statistical mechanics



- **Delocalized phase ($\lambda_F < 2$):** GGE describes one-body observables, GE fails.
- **Localized phase ($\lambda_F > 2$):** GGE describes n_i but fails for m_k , GE fails.

Scaling of Δm with L



- **Delocalized phase ($\lambda_F < 2$):** GGE describes one-body observables ($\Delta m^{\text{GGE}} \propto 1/L$), GE fails.
- **Critical point ($\lambda_F = 2$):** GGE describes one-body observables, GE fails.
- **Localized phase ($\lambda_F > 2$):** GGE describes n_i but fails for m_k , GE fails.

Summary

- **Delocalized regime:** n_i and m_k equilibrate and they are described by GGE, despite the lack of translational invariance!
Power law relaxation?
- **Critical point:** Slower relaxation dynamics. GGE describes observables after relaxation.
- **Localized regime:** m_k equilibrates but GGE fails to describe it after relaxation. n_i does not equilibrate but GGE describes its time average.
- **One-body fermionic observables:** Their time average is always described by the GGE.

$$\hat{\rho}_F(\tau) = \sum_{\gamma', \gamma} c_{\gamma'}^* c_{\gamma} e^{i(e_{\gamma'} - e_{\gamma})\tau} |\gamma\rangle \langle \gamma'|$$

and

$$\overline{\hat{\rho}_F(\tau)} = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^{\tau} d\tau' \hat{\rho}_F(\tau') = \sum_{\gamma} |c_{\gamma}|^2 |\gamma\rangle \langle \gamma| \equiv \hat{\rho}_F^{\text{GGE}}$$

Dynamics after a quench

Quench protocol

- Start from an eigenstate of $\hat{H}$ ($|\psi_0\rangle$) in a certain disorder realization.
- Evolve under another disorder realization with the same α .
- $E = \langle \psi_0 | \hat{H}_{\text{fin}} | \psi_0 \rangle$ is the energy of a thermal state with temperature $T = 10$.

Eigenstate thermalization

Observables:

$$\hat{n}(k) = \frac{1}{L} \sum_{l,m} e^{ik(l-m)} \hat{f}_l^\dagger \hat{f}_m$$

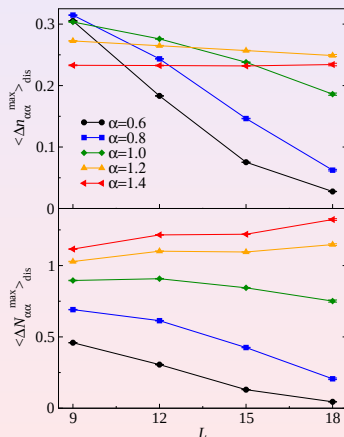
$$\hat{N}(k) = \frac{1}{L} \sum_{l,m} e^{ik(l-m)} \hat{n}_l \hat{n}_m$$

Maximal normalized difference:

$$\Delta O_{\alpha\alpha}^{\max} = \frac{\sum_k |O_{\alpha\alpha}^{\max}(k) - O_{\text{mic}}(k)|}{\sum_k O_{\text{mic}}(k)}$$

Disorder average:

$$\langle \Delta O_{\alpha\alpha}^{\max} \rangle_{\text{dis}}$$



E. Khatami et al., PRE **85**, 050102(R) (2012).