# Macdonald processes: Integrability in the Kardar-Parisi-Zhang universality class

**Ivan Corwin**

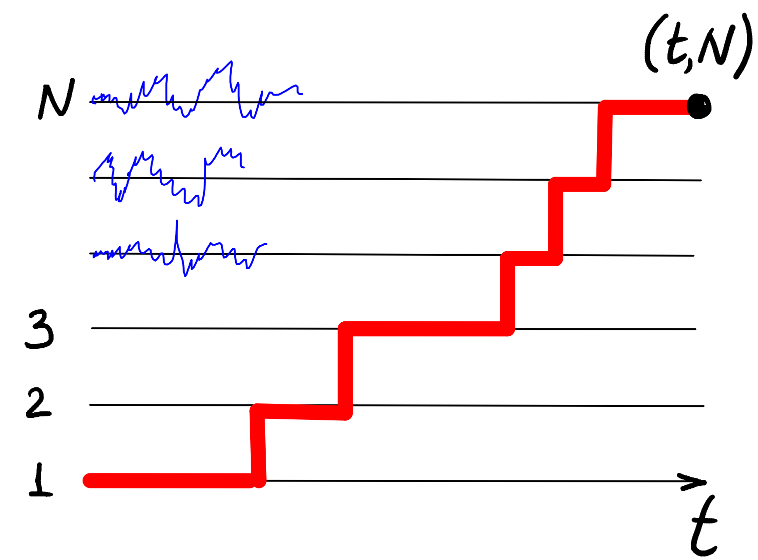
(Clay Mathematics Institute, MIT and Microsoft Research)

# Partition function for a semi-discrete directed random polymer

$$Z_t^N = \int_{0<s_1<\dots<s_{N-1}<t} e^{B_1(0,s_1)+B_2(s_1,s_2)+\dots+B_N(s_{N-1},t)}\, ds_1 \dots ds_{N-1}$$

$B_1, \dots, B_N$ are independent Brownian motions

$$B_k(\alpha,\beta) := B_k(\beta) - B_k(\alpha) = \int_\alpha^\beta \dot{B}_k(x)\,dx$$




[O'Connell-Yor 2001]

$$u(t,N) := e^{-3t/2} Z_t^N = e^{-3t/2} \int_{0<s_1<\ldots<s_{N-1}<t} e^{B_1(0,s_1)+\ldots+B_N(s_{N-1},t)}\, ds$$

satisfies

$$\frac{\partial u(t,N)}{\partial t} = \big(u(t,N-1) - u(t,N)\big) + \dot{B}_N(t)\cdot u(N,t)$$

with $u(0,N) = \delta_{1N}$.

This is a discrete analog of the <u>stochastic heat equation</u>

$$u_t = \tfrac{1}{2}\Delta u + \dot{W}\cdot u$$

where $\dot{W}$ is the space-time white noise.

The path integral is the <u>Feynman-Kac solution</u>

If $u(t,x)$ satisfies $u_t = \frac{1}{2}\Delta u + \dot{W}\cdot u$ then

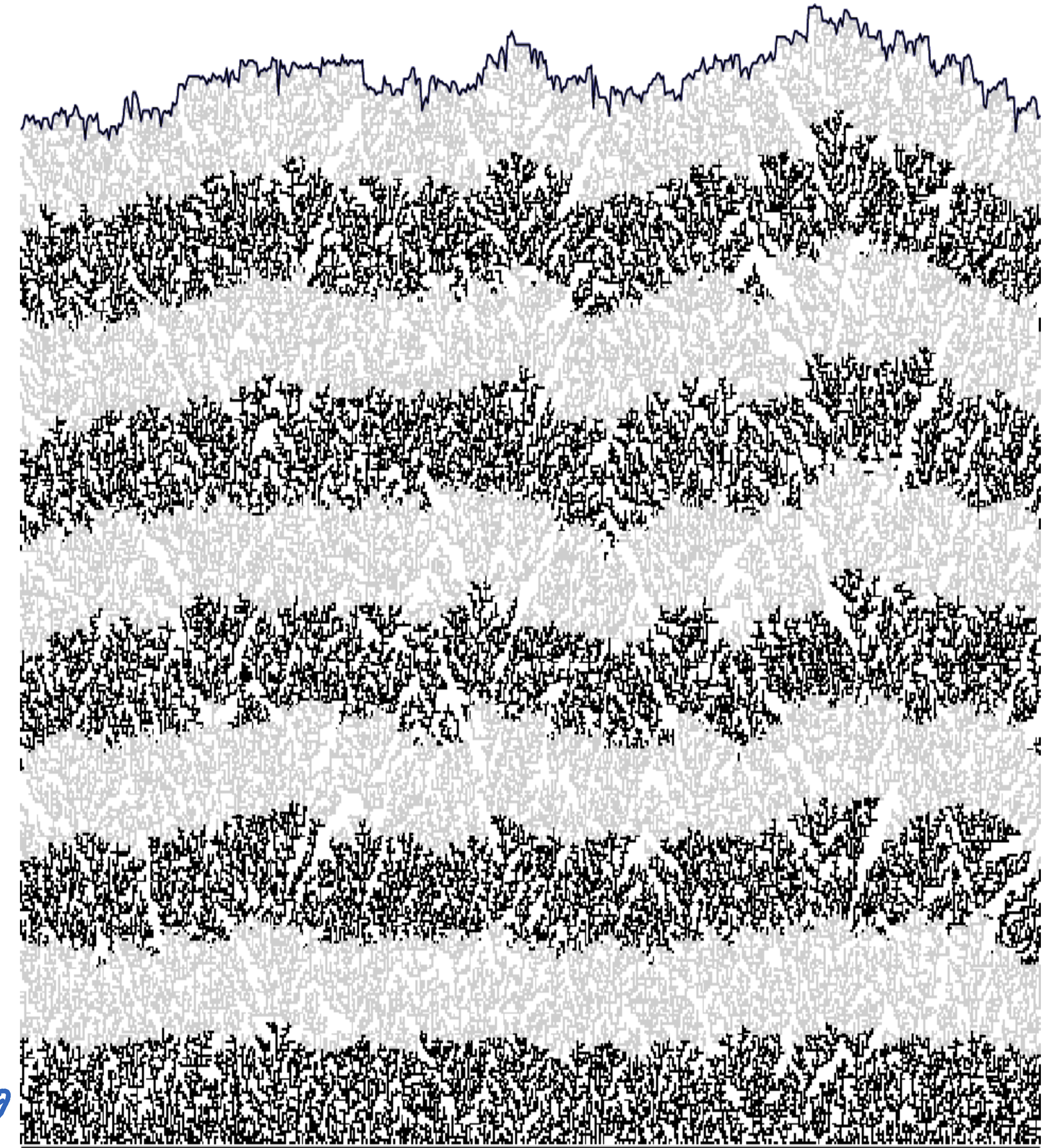
$h := \log(u)$ formally satisfies

$$h_t = \frac{1}{2}\Delta h + \frac{1}{2}(\nabla h)^2 + \dot{W}$$

The Kardar-Parisi-Zhang equation:

Invented in 1986 to describe random interface growth.

Thus, log(u) and random interfaces should be in the same universality class.



Ballistic deposition

# The semi-discrete Brownian directed polymer is exactly solvable.

<u>Theorem (Borodin-Corwin, 2011)</u> The Laplace transform of the polymer partition function $Z_t^N$ can be written as a Fredholm determinant

$$\left\langle e^{-\zeta Z_t^N} \right\rangle = \det\left(\mathbb{1} + K_\zeta\right)_{L^2(\bigcirc)}$$

where

$$K_\zeta(v,v') = \frac{i}{2} \int_{-i\infty+\frac{1}{2}}^{i\infty+\frac{1}{2}} \left(\frac{\Gamma(v-1)}{\Gamma(s+v-1)}\right)^N \frac{\zeta^s e^{vts+\frac{ts^2}{2}}}{s+v-v'} \frac{ds}{\sin \pi s}.$$

<u>Corollary (B-C, B-C-Ferrari, 2011-12)</u> Set $F_t^N = \log Z_t^N$. For any $\varkappa > 0$

$$\lim_{N\to\infty} \mathbb{P}\left\{ \frac{F_{\varkappa N}^N - N\bar{f}_\varkappa}{N^{1/3}} \le r \right\} = F_{GUE}\left( \left(\frac{\bar{g}_\varkappa}{2}\right)^{-1/3} r \right)$$

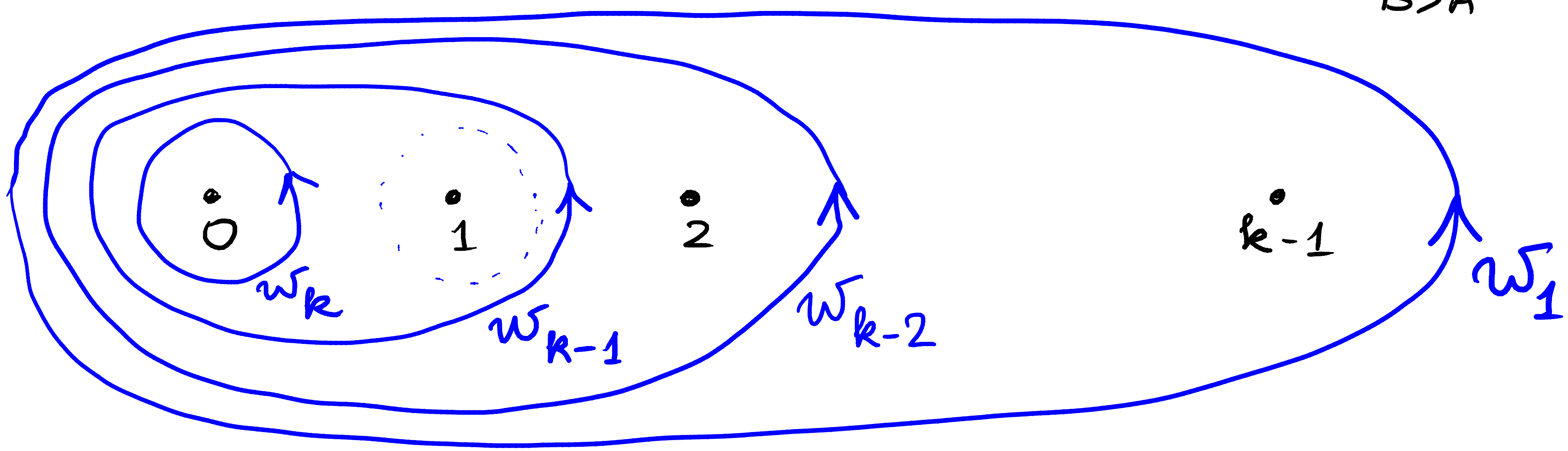
# The semi-discrete Brownian directed polymer is exactly solvable.

<u>Theorem (Borodin-Corwin, 2011)</u> The moments of the polymer partition function have the following integral representation

$$\left\langle Z_t^{N_1} \cdots Z_t^{N_k} \right\rangle = \frac{e^{\frac{kt}{2}}}{(2\pi i)^k} \oint \cdots \oint \prod_{1 \le A < B \le k} \frac{w_A - w_B}{w_A - w_B - 1} \prod_{j=1}^{k} \frac{e^{t w_j}}{w_j^{N_j}} \, dw_j$$

where $N_1 \geq N_2 \geq \ldots \geq N_k \geq 1$.

The contours are such that $w_A$- contour contains 0 and $\{w_B + 1\}_{B>A}$



The replica approach is based on showing that

$$\overline{\mathcal{U}}(t, x_1, \dots, x_k) = \left\langle \mathcal{U}(t, x_1) \cdots \mathcal{U}(t, x_k) \right\rangle$$

satisfies

$$\frac{\partial \overline{\mathcal{U}}}{\partial t} = H \overline{\mathcal{U}}, \qquad H = \frac{1}{2}\Delta + \frac{1}{2}\sum_{i \neq j} \delta(x_i - x_j),$$

finding eigenbasis of H via Bethe ansatz [Lieb-Liniger, 1963], [McGuire, 1964], and using the expansion

$$\left\langle e^{-\zeta\, \mathcal{U}(t,x)} \right\rangle \text{"="} \sum_{k=0}^{\infty} \frac{(-\zeta)^k}{k!} \left\langle \mathcal{U}^k(t,x) \right\rangle$$

Such series always diverge due to intermittency! Drawing conclusions is risky, originally an incorrect answer was obtained.

*We use a totally different approach. Comparison with Bose gas yields*

<u>Theorem (Borodin-Corwin, 2011)</u> For $x_1 < x_2 < \dots < x_k$, the integral

$$\mathcal{U}(t, x_1, \dots, x_k) := \int \cdots \int \prod_{A<B} \frac{z_A - z_B}{z_A - z_B - c} \prod_{j=1}^{k} e^{\frac{t}{2} z_j^2 + x_j z_j} \frac{dz_j}{2\pi i}$$

where the $z_j$ – integration is over $\alpha_j + i\mathbb{R}$ with $\alpha_1 > \alpha_2 + c > \alpha_3 + 2c > \dots$

yields the solution of the quantum many body system

$$\frac{\partial \mathcal{U}}{\partial t} = \frac{1}{2}\Big(\Delta + c \sum_{i \neq j} \delta(x_i - x_j)\Big)\mathcal{U}$$

with the delta initial condition $\mathcal{U}(0, x) = \delta(x)$.

*Note the symmetry between the attractive and repulsive cases (positive-negative c). Bethe eigenstates are very different!*

**Macdonald processes** $q, t \in [0, 1)$

Ruijsenaars–Macdonald system

Representations of Double Affine Hecke Algebras

**q-Whittaker processes** $t = 0$

q-TASEP, 2d dynamics

q-deformed quantum Toda lattice

Representations of $\widehat{\mathfrak{gl}}_N$, $U_q(\mathfrak{gl}_N)$

**General β RMT** $t = q^{\beta/2} \to 1$

Random matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$

Calogero–Sutherland, Jack polynomials

Spherical functions for Riem. Symm. Sp.

**Hall–Littlewood processes** $q = 0$

Random matrices over finite fields

Spherical functions for p-adic groups

**Whittaker processes** $t = 0$, $q \to 1$

Directed polymers and their hierarchies

Quantum Toda lattice, repr. of $GL(n, \mathbb{R})$

discussed so far

**Kingman partition structures** $q = 0$, $t = 1$

Cycles of random permutations

Poisson–Dirichlet distributions

**Schur processes** $q = t$

Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE

Characters of symmetric, unitary groups

(Ascending) Macdonald processes are probability measures on interlacing triangular arrays (Gelfand-Tsetlin patterns)

$$
\begin{array}{ccccccccc}
\lambda^{(N)}_N & & \lambda^{(N)}_{N-1} & & \cdots & & \lambda^{(N)}_2 & & \lambda^{(N)}_1 \\
& \lambda^{(N-1)}_{N-1} & & \lambda^{(N-1)}_{N-2} & \cdots & & & \lambda^{(N-1)}_1 & \\
& & \ddots & & \cdots & & \iddots & & \\
& & & \lambda^{(2)}_2 & & \lambda^{(2)}_1 & & & \\
& & & & \lambda^{(1)}_1 & & & &
\end{array}
$$

$$\lambda^{(m)}_j \in \mathbb{Z}_{\geq 0}$$

$$\mathbb{P}\left(\lambda^{(k)}\right) = \frac{P_{\lambda^{(k)}}(a_1, \dots, a_k)\, Q_{\lambda^{(k)}}(b_1, \dots, b_M)}{\Pi(\underbrace{a_1, \dots, a_k}; \underbrace{b_1, \dots, b_M})}$$

Macdonald polynomials

normalization constant

two groups of parameters

Macdonald polynomials $P_\lambda(x_1,\dots,x_N) \in \mathbb{Q}(q,t)[x_1,\dots,x_N]^{S(N)}$ with partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0)$ form a basis in symmetric polynomials in N variables over $\mathbb{Q}(q,t)$. They diagonalize

$$(\mathcal{D}_1 f)(x_1,\dots,x_N) = \sum_{i=1}^{n} \prod_{j\neq i} \frac{t x_i - x_j}{x_i - x_j} f(x_1,\dots,q x_i,\dots,x_N)$$

with (generically) pairwise different eigenvalues

$$\mathcal{D}_1 P_\lambda = (q^{\lambda_1} t^{N-1} + q^{\lambda_2} t^{N-2} + \dots + q^{\lambda_N}) P_\lambda .$$

They have many remarkable properties that include orthogonality (dual basis $Q_\lambda$ ), simple reproducing kernel (Cauchy type identity), Pieri and branching rules, index/variable duality, explicit generators of the algebra of (Macdonald) operators commuting with $\mathcal{D}_1$, etc.

We are able to do two basic things:

- Construct relatively explicit Markov operators that map Macdonald processes to Macdonald processes;
- Evaluate averages of a broad class of observables.

The construction is based on commutativity of Markov operators

$$\mathbb{P}(\lambda \to \mu) = \frac{P_\mu(x_1,\dots,x_{n-1})}{P_\lambda(x_1,\dots,x_n)} P_{\lambda/\mu}(x_n), \qquad \mathbb{P}(\lambda \to \nu) = \frac{P_\nu(x_1,\dots,x_m)}{P_\lambda(x_1,\dots,x_m)} \frac{P_{\nu/\lambda}(u)}{\Pi(x;u)}$$

skew Macdonald polynomials

additional parameter

an idea from [Diaconis-Fill '90], and Schur process dynamics from [Borodin-Ferrari '08].

Evaluation of averages is based on the following observation.
Let $\mathcal{D}$ be an operator that is diagonalized by the Macdonald polynomials (for example, a product of Macdonald operators),

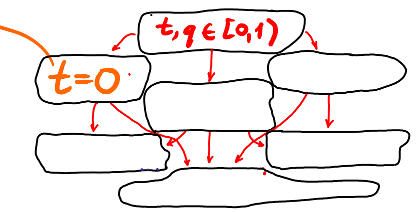
$$\mathcal{D} P_\lambda = d_\lambda P_\lambda .$$

Applying it to the Cauchy type identity $\sum_\lambda P_\lambda(a) Q_\lambda(b) = \Pi(a;b)$
we obtain

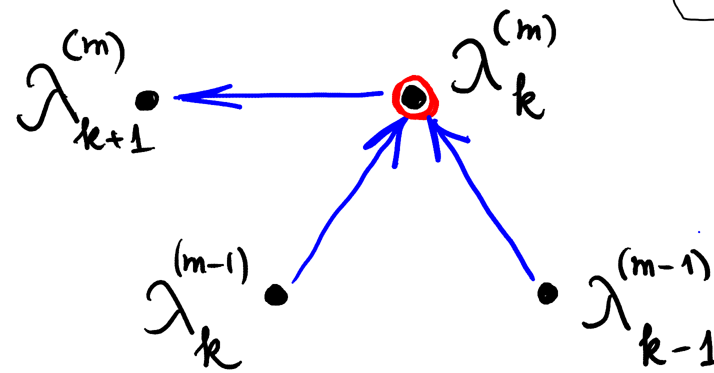
$$\langle d_\lambda \rangle = \frac{\mathcal{D}^{(a)} \Pi(a;b)}{\Pi(a;b)} .$$

If all ingredients are explicit (as for products of Macdonald operators), we obtain meaningful probabilistic information. Should be contrasted with the lack of explicit formulas for the Macdonald polynomials.

Here is an example of a Markov process preserving the class of the $q$-Whittaker processes (Macdonald processes with $t=0$).

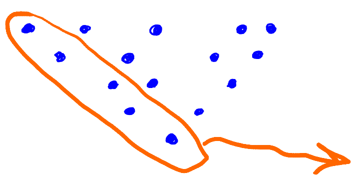


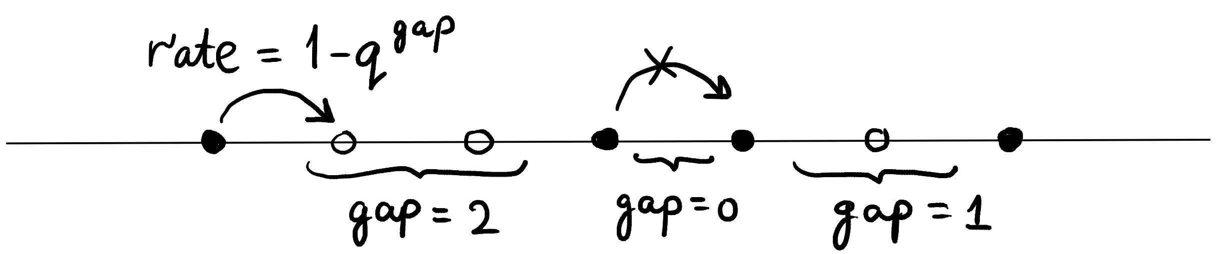

Each coordinate of the triangular array jumps by 1 to the right independently of the others with




$$\mathrm{rate}\left(\lambda_k^{(m)}\right) = \frac{\left(1-q^{\lambda_{k-1}^{(m-1)}-\lambda_k^{(m)}}\right)\left(1-q^{\lambda_k^{(m)}-\lambda_{k+1}^{(m)}+1}\right)}{\left(1-q^{\lambda_k^{(m)}-\lambda_k^{(m-1)}}\right)}.$$

The set of coordinates $\left\{\lambda_m^{(m)}-m\right\}_{m\geq 1}$ forms $q$-TASEP





From dynamics of q-TASEP, easy to see that it satisfies:

$$dq^{\lambda_N^{(N)}(t)} = (1-q)\nabla q^{\lambda_N^{(N)}(t)}\,dt + q^{\lambda_N^{(N)}(t)}\,dM_N(t)$$

$(\nabla f)(N) = f(N-1) - f(N)$

Martingale

As $q = e^{-\varepsilon} \to 1$, at large times $\tau/\varepsilon^2$, this system scales to:

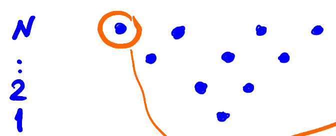
$$d\,u(\tau,N) = \nabla u(\tau,N)\,d\tau + u(\tau,N)\,dB_N(t)$$

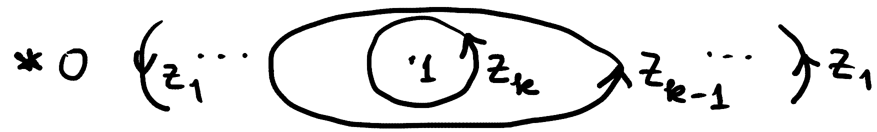
Brownian motion

And hence recovers the semi-discrete Brownian directed polymer

Taking the observables corresponding to powers of the first Macdonald operator yields

$$\mathbb{E}\left(q^{\lambda_N^{(N)}(\tau)}\right)^k = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint \cdots \oint \prod_{A<B} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{e^{(q-1)\tau z_j}}{(1-z_j)^N} \frac{dz_j}{z_j}$$




$$\mathbb{E} \underbrace{\sum_{k=0}^{\infty} \frac{\left(q^{\lambda_N^{(N)}}\right)^k \zeta^k}{(1-q)\cdots(1-q^k)}}_{q\text{-Laplace transform of } q^{\lambda_N^{(N)}}} = \mathbb{E} \frac{1}{\left(\zeta q^{\lambda_N^{(N)}}; q\right)_\infty} = \det(\mathbb{1} + K)_{L^2(\mathbb{N} \times \circlearrowleft 1)}$$

with

$$K(n_1, w_1; n_2, w_2) = \frac{f(w_1)\cdots f(q^{n_1 - 1} w_1)\, \zeta^{n_1}}{q^{n_1} w_1 - w_2}, \qquad f(w) = \frac{e^{(q-1)\tau w}}{(1-w)^N}$$

This is a perfectly legal q-version of the replica trick.

- Macdonald processes form a new class of exactly solvable probabilistically meaningful measures on sequence of partitions
- They generalize the Schur processes but they are not determinantal; integrability comes from structural properties of the Macdonald poly's
- Several directed polymer models are obtained as limits, new algebraic and analytic properties follow
- Massive amounts of joint moment formulas are available while many-point limiting distributions are still resisting
- A new integral ansatz for solving quantum many body problems applies in other settings (ASEP, parabolic Anderson model)
- Many things remain to be investigated as Macdonald processes have a variety of degenerations