

Rare events in nonequilibrium quantum systems

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In collaboration with

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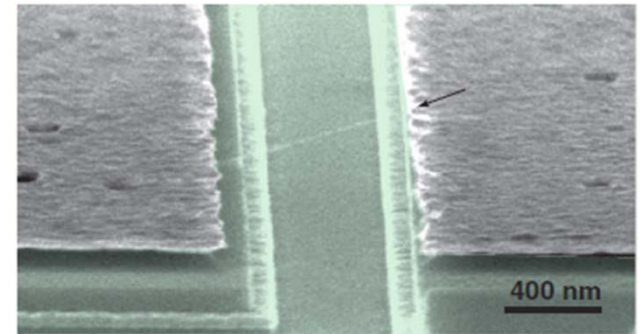
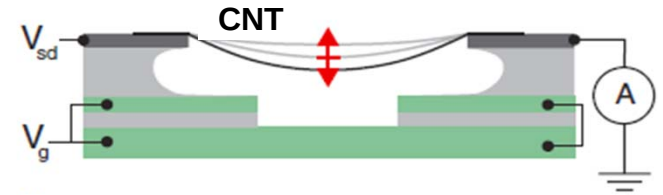
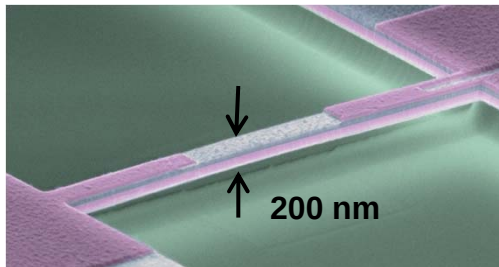
Vadim Smelyanskiy, *NASA Ames*



Mesoscopic oscillators

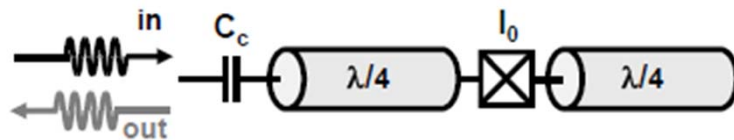
Nanomechanics, CQED, cQED: vibrational systems that are

- sufficiently large to be individually accessed
- small, so that classical and quantum fluctuations are substantial



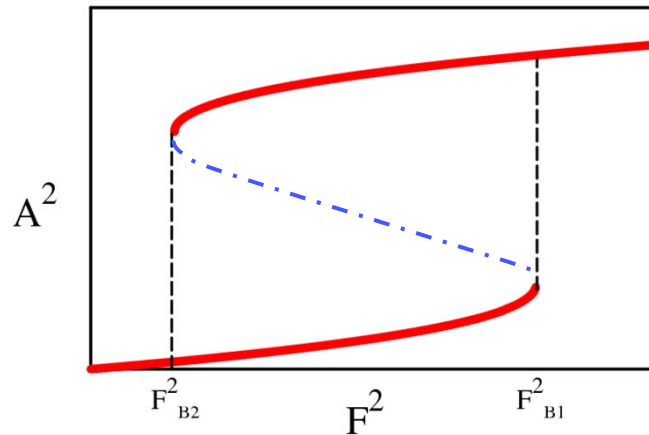
Frequencies 0.1 – 8 GHz

$$Q = \frac{\text{angular frequency}}{\text{decay rate}} \sim 10^3 - 10^7$$



- well-characterized systems for studying classical and quantum physics far from equilibrium

Hysteresis in modulated systems



$$\ddot{q} + 2\Gamma\dot{q} + \omega_0^2 q + \gamma q^3 = F \cos \omega_F t$$

Periodic state: $q = A \cos(\omega_F t + \phi)$

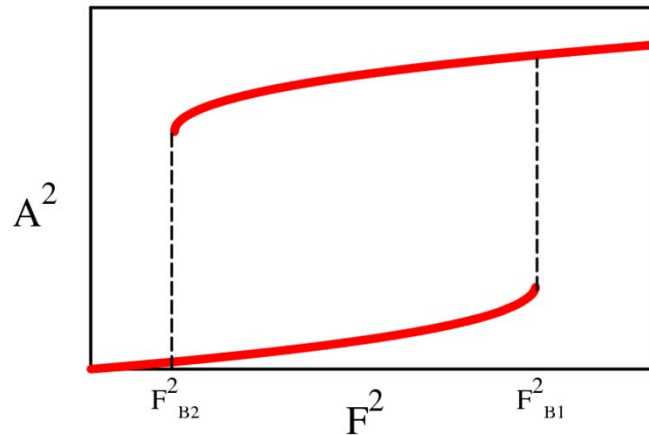
Both A and ϕ display hysteresis

Eigenfrequency depends on vibration amplitude, $\omega_0 \rightarrow \omega_0 + \tilde{\gamma} A^2$, $\tilde{\gamma} \propto \gamma$

Cubic equation for the amplitude of forced vibrations,

$$A^2 = \frac{F^2 / 4\omega_F^2}{(\omega_F - \omega_0 - \tilde{\gamma} A^2)^2 + \Gamma^2}$$

Hysteresis in modulated systems

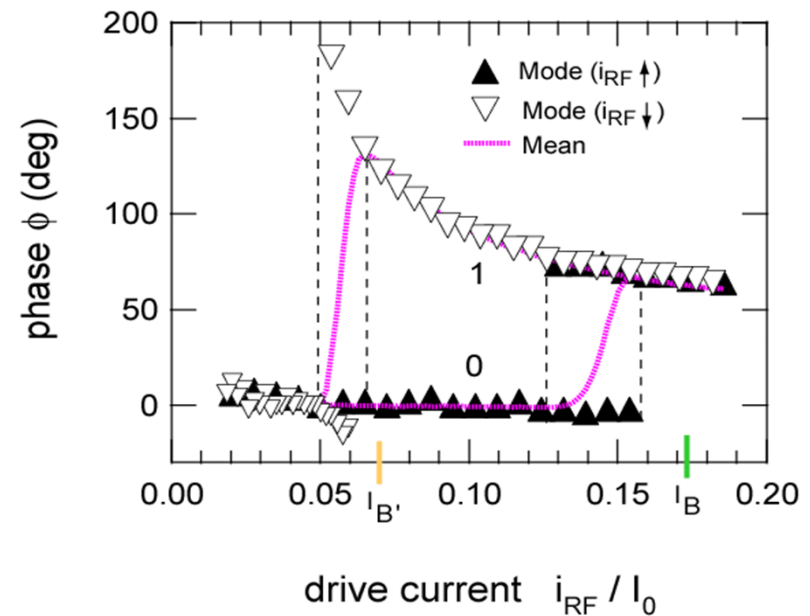
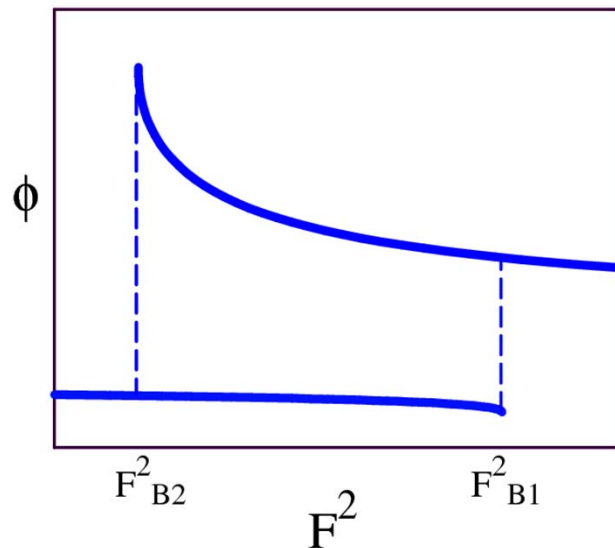


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A Josephson junction based nonlinear oscillator
(Siddiqi et al. PRL 2004, 2005)



Application: bifurcation amplifier for quantum measurements

Quasienergy spectrum and quantum temperature

Underdamped oscillator in the rotating frame: decay rate $\Gamma \ll |\omega_F - \omega_0|$

$$H(t) = \frac{1}{2}(p^2 + \omega_0^2 q^2) + \frac{1}{4}\gamma q^4 - qF \cos(\omega_F t) \quad i\hbar \dot{\psi} = H(t)\psi, \quad \psi_\varepsilon(t + \tau_F) = \exp(-i\varepsilon\tau_F / \hbar)\psi_\varepsilon(t)$$

↑
quasi-energy

Rotating frame: $q(t) = C(Q \cos \omega_F t + P \sin \omega_F t)$, $p(t) = -\omega_F C(Q \sin \omega_F t - P \cos \omega_F t)$

$$[P, Q] = -i\tilde{\hbar}, \quad \tilde{\hbar} = \frac{3\hbar\gamma}{8\omega_F^2(\omega_F - \omega_0)}$$

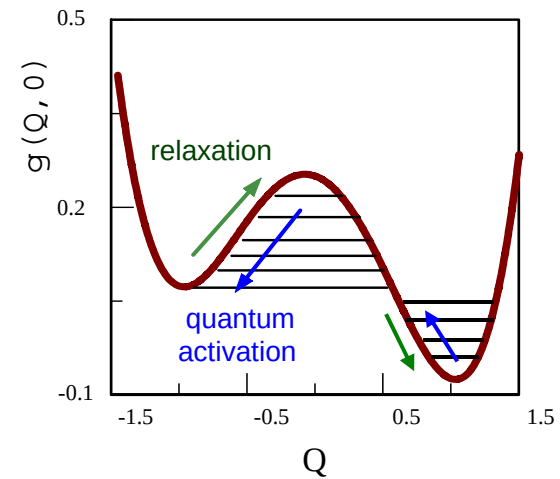
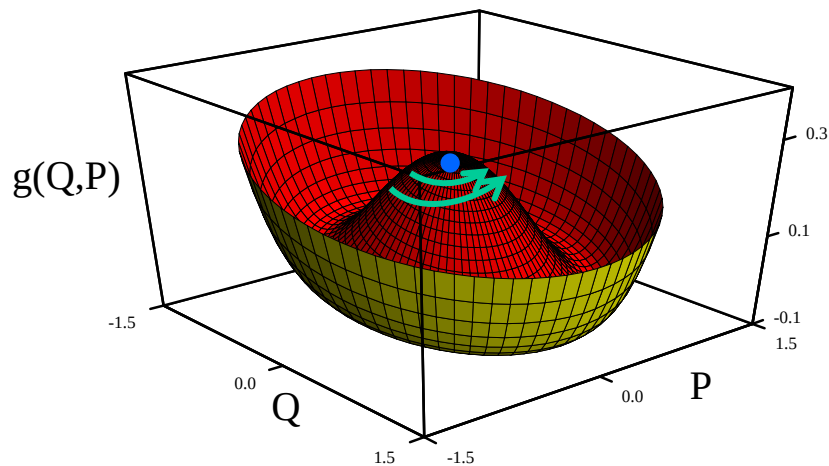
Quasienergy spectrum and quantum temperature

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$$H(t) = \frac{1}{2}(p^2 + \omega_0^2 q^2) + \frac{1}{4}\gamma q^4 - qF \cos(\omega_F t) \xrightarrow{\text{RWA}} C_g g(Q, P), \quad [P, Q] = -i\tilde{\hbar}, \quad \tilde{\hbar} = \frac{3\hbar\gamma}{8\omega_F^2(\omega_F - \omega_0)}$$

$\uparrow$
 time-independent

$$g(Q, P) = \frac{1}{4}(Q^2 + P^2 - 1)^2 - Q\beta^{1/2} \quad (\beta \propto F^2)$$



$$\tilde{\hbar} \ll 1$$

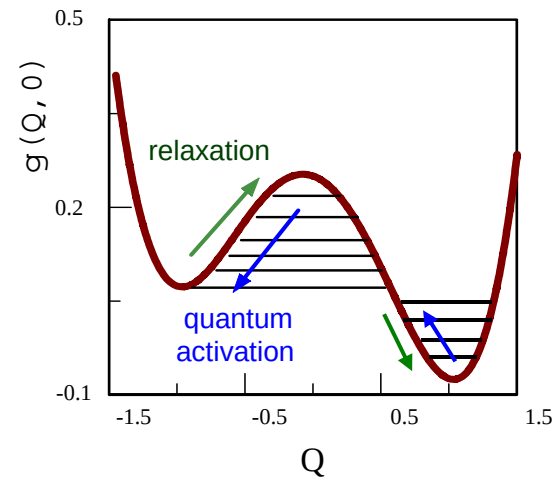
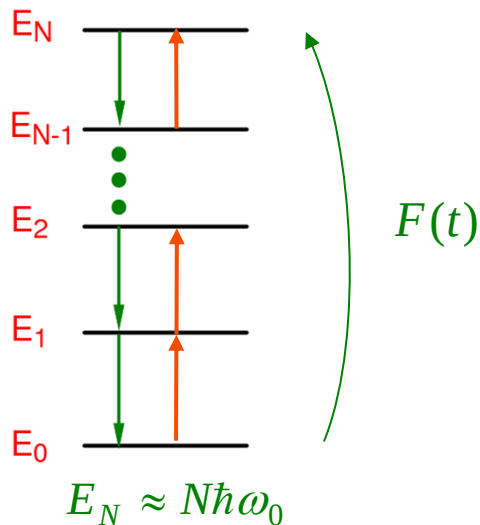
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Quasienergy states are **linear combinations** of Fock states. Inter-level transitions $|N_{\text{Fock}}\rangle \rightarrow |N_{\text{Fock}} - 1\rangle$ correspond to inter-quasi-energy level transitions $|n\rangle \rightarrow |n \pm m\rangle$ ➡

drift and quantum diffusion over quasienergy

The eikonal approximation

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Balance equation for populations ρ_n of quasienergy states

$$\begin{aligned} \partial_t \rho_n &= \sum [W(n-k; k) \rho_{n-k} - W(n; k) \rho_n] \\ &= \sum (e^{-k\partial_n} - 1) W(n; k) \rho_n \end{aligned}$$

Far tail: steep distribution, $\rho_n = \exp(-R_n/\tilde{\hbar})$

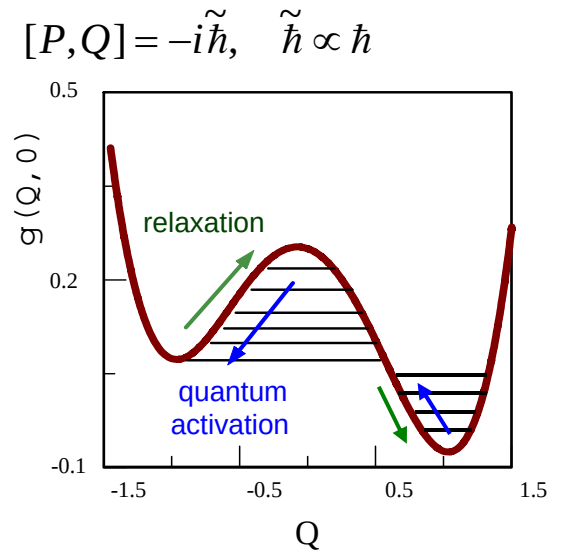
Eikonal approximation: smooth R_n , $R_n \equiv R(g_n) = \int^{g_n} R'(g) dg$

“ g -dependent quantum temperature”, $\exp(-g_n/T_e) \rightarrow \exp[-\int^{g_n} T_e^{-1}(g) dg]$, $T_e(g) = \tilde{\hbar} / R'(g)$

Polynomial-type equation for $R'(g)$ in the stationary regime:

$$0 = \sum_k (e^{-k\partial_n} - 1) W(n; k) e^{-R_n/\tilde{\hbar}} \rightarrow \sum_k W(n; k) \{ \exp[kR'(g_n)v_n] - 1 \}$$

$v_n = \tilde{\hbar}^{-1} \partial_n g_n$ is classical vibration frequency in the rotating frame



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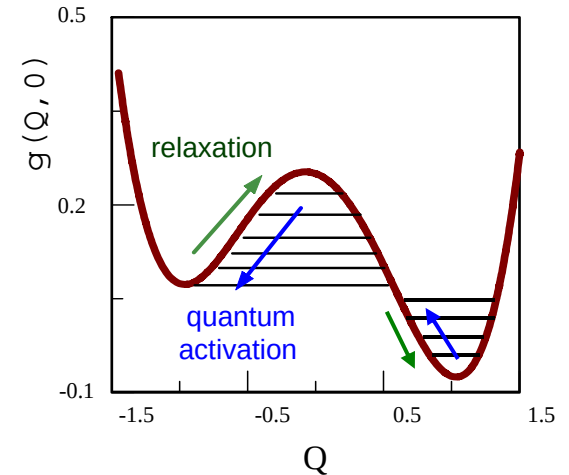
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Real-time instanton, with Hamiltonian $\mathcal{H}(x, p) = \tilde{\hbar} \sum_k W(x/\tilde{\hbar}; k) [\exp(kp) - 1]$, $x = \tilde{\hbar}(n + 1/2)$.

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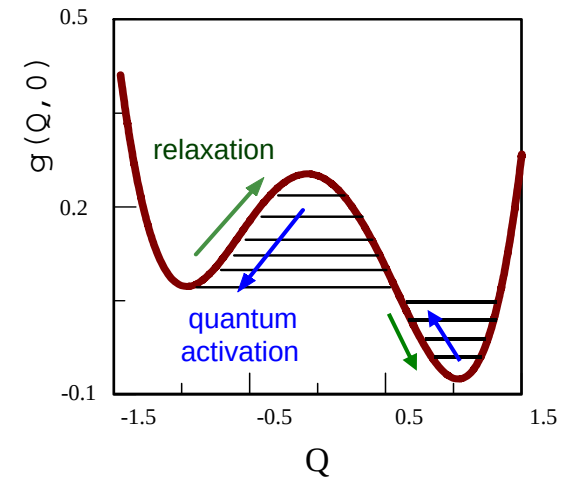
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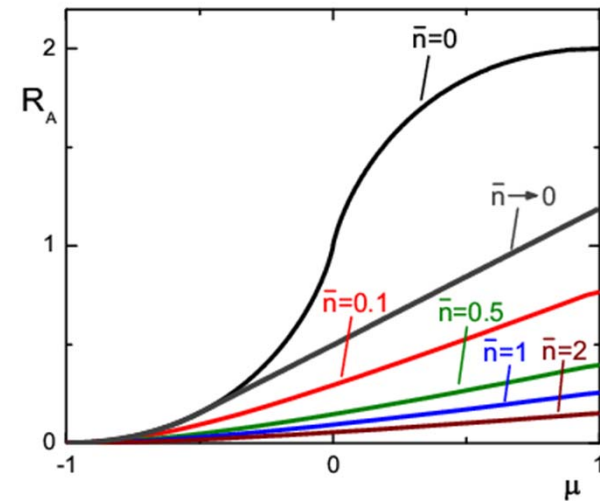
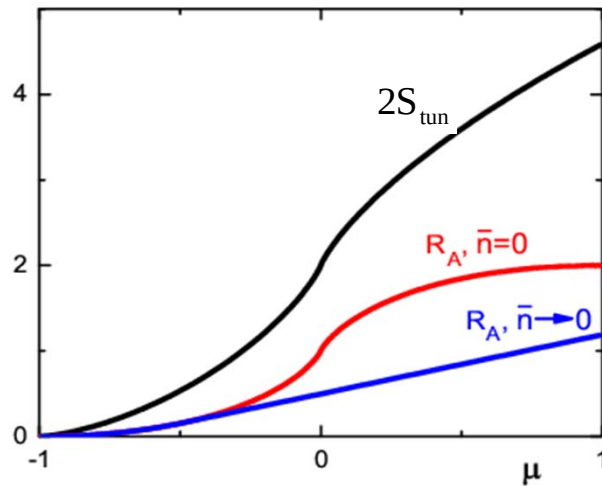


Switching rate $W_{sw} \propto \Gamma \exp(-R_A/\tilde{\hbar})$, $R_A = R(g_S)$

Quantum activation vs tunneling*

$$W_{\text{tun}} \propto \exp(-2S_{\text{tun}} / \hbar)$$

$$W_{\text{sw}} \propto \exp(-R_A / \hbar)$$



$$\bar{n} = [\exp(\hbar\omega_0 / kT) - 1]^{-1}$$

Activation energy < tunneling exponent! ➡ Switching via quantum activation

Fragility: in the WKB limit $R_A(T = 0) \neq R_A(T \rightarrow 0)$

*The plots refer to a parametric oscillator; Marthaler & MD (2006)

Fighting fragility

Quasistationary distribution: $\sum [W(n-k; k)\rho_{n-k} - W(n; k)\rho_n] = 0$

Contribution to transition rates from “photon” emission and absorption:

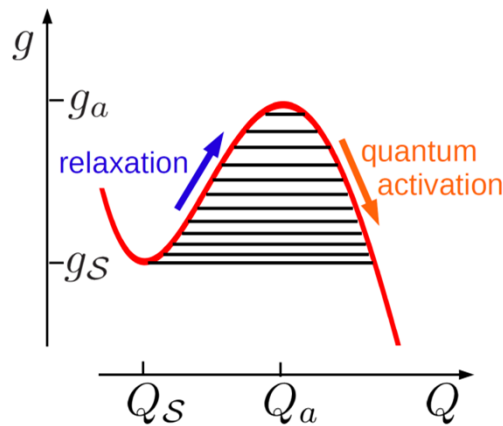
$$W(n; k) = W^{(e)}(n; k) + W^{(abs)}(n; k), \quad W^{(abs)} \propto \bar{n} \propto \exp(-\hbar\omega_0/k_B T)$$

$T = 0$: **detailed balance** $\rightarrow \rho_n^{(0)}$. Thermal perturbation **diverges**,

$\sum_{k>0} W^{(abs)}(n-k; k)\rho_{n-k}^{(0)}/\rho_n^{(0)} \rightarrow \infty$ for $\hbar \rightarrow 0$ $\rightarrow$ “nonlocal” influx from states closer to the stable

state, a geometric series, $W^{(abs)}(n-k; k)\rho_{n-k}^{(0)}/\rho_n^{(0)} \propto \bar{n} v^k, v > 1$ $\rightarrow$ a smoother eikonal

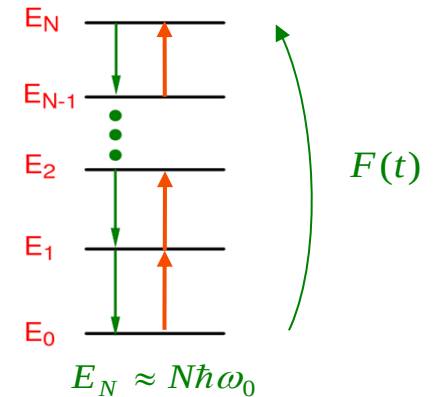
distribution for $\bar{n} \gg \tilde{\hbar}$



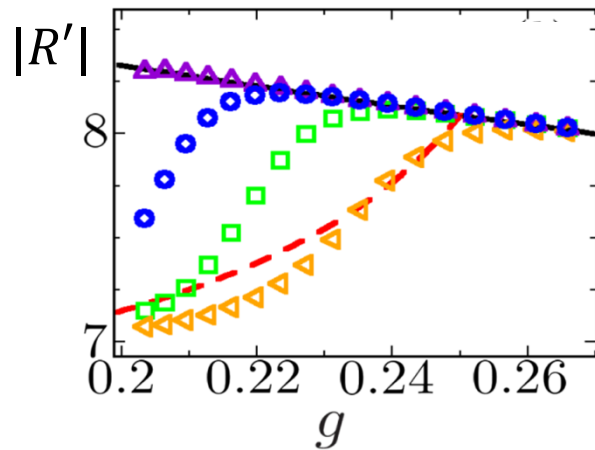
The number $M \propto \tilde{\hbar}^{-1}$ of localized quasienergy states is **finite**.

Therefore $\bar{n}v^M$ remains small for very small $\bar{n} \rightarrow T < T_{cr} \propto \hbar^2$

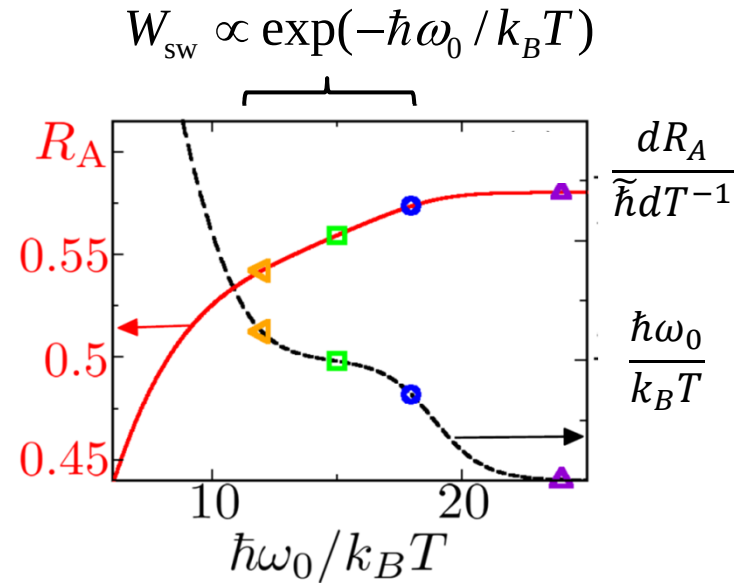
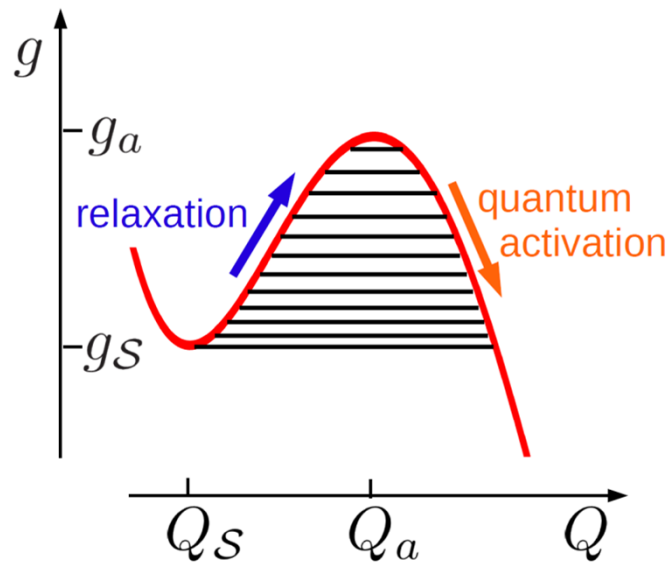
Transition to the eikonal “ $T \rightarrow 0$ ” limit for $T > T_+ \propto \hbar/|\log \tilde{\hbar}|$



inter-instanton kink, which moves with temperature



$$|R'(g)| = \tilde{\hbar}/T_e(g)$$



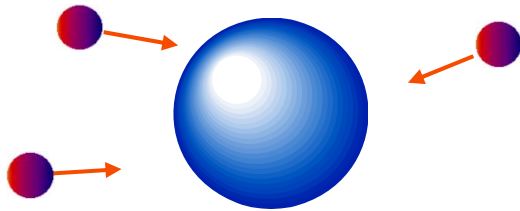
L. Guo, V. Peano, M. Marthaler, and MD, cond-mat 1212...

Quantum noise: a naive view

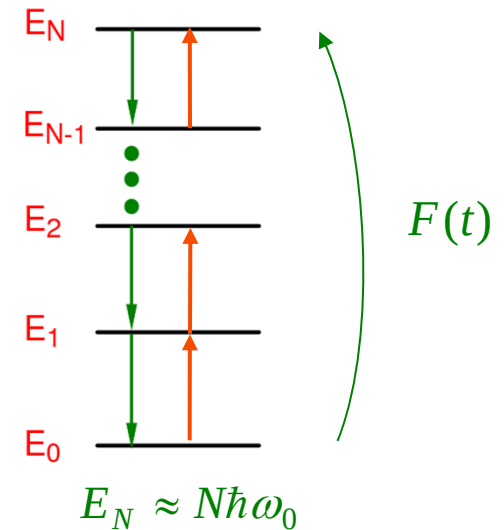
Oscillator Hamiltonian: $H_0 = \frac{1}{2} p^2 + \frac{1}{2} \omega_0^2 q^2 + \frac{1}{4} \gamma q^4 - qF \cos \omega_F t$
 + coupling to a thermal bath

Relaxation: emission of excitations in the bath

An elementary transition is **short**, $\sim \omega_0^{-1}$. It happens **at random**.
 Classically, a “kick” to the oscillator coordinate and momentum



à la Brownian particle colliding with molecules

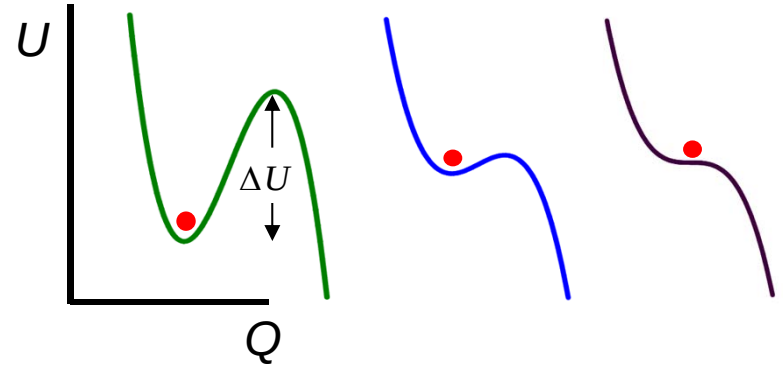


Naively: “noise intensity” for a quantum oscillator is the total rate of emission / absorption:

$$k_B T \Rightarrow k_B T_{\text{eff}} = \hbar\omega_0(2\bar{n} + 1)/2, \quad \bar{n} = [\exp(\hbar\omega_0/k_B T) - 1]^{-1}$$

Switching rate near bifurcation points

Near bifurcation points one of the motions is **slow**, a “soft mode” $\rightarrow$ *universal behavior of the escape rate*

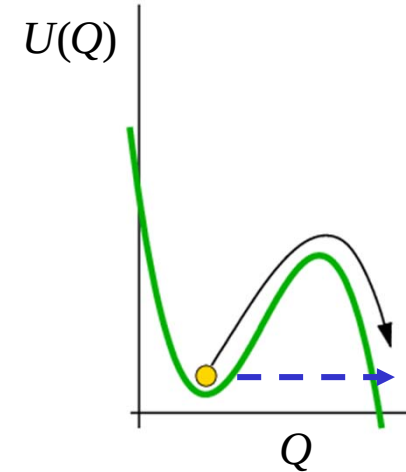
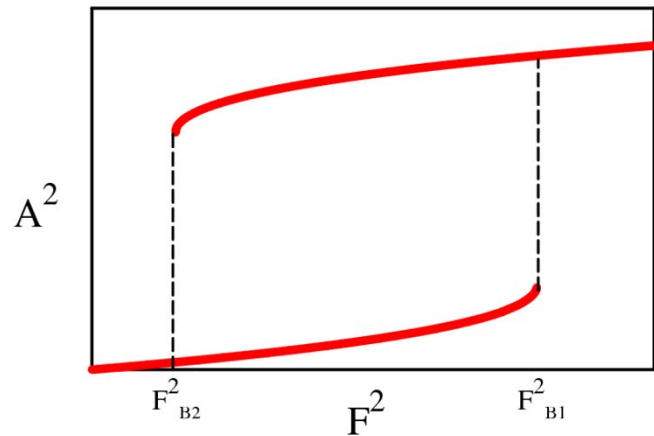


Technically: separation of fast and slow variables in the master equation **or** the **adiabatic approximation** in the quantum Langevin equation $\rightarrow$ mapping on an overdamped Brownian particle with quantum noise

$$\dot{Q} = -\partial_Q U + f(t), \quad U(Q) = -\frac{1}{3}Q^3 + \eta Q, \quad \Delta U = \frac{4}{3}\eta^{3/2}$$

$\nwarrow$ quantum noise with intensity $\propto T_{eff}$

Scaling in quantum activation



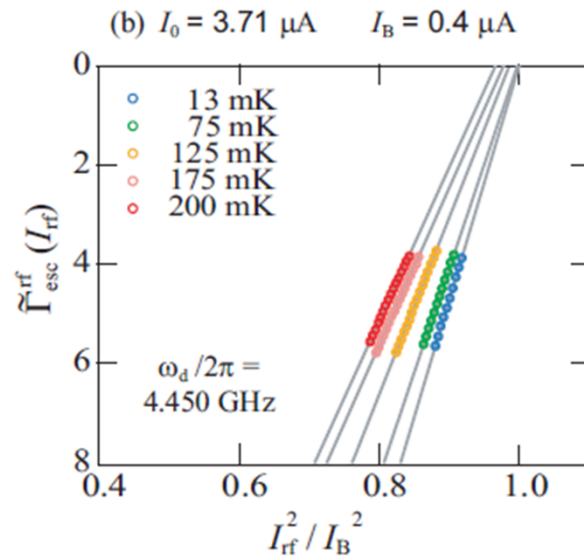
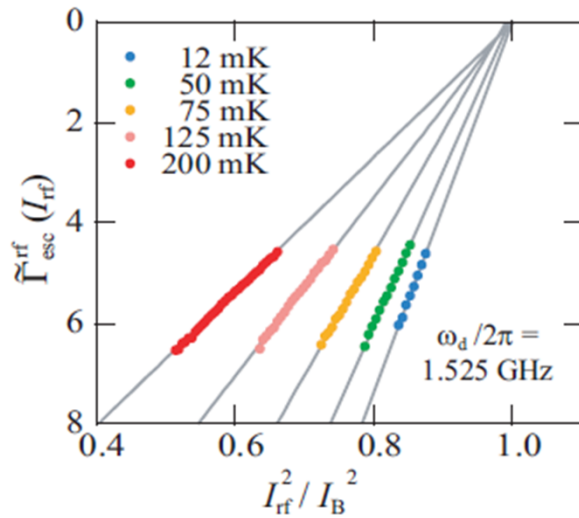
Quantum noise leads to switching via an “overbarrier transition” even for $T \rightarrow 0$ $\rightarrow$
“*quantum activation*”

$$W_{\text{sw}} \propto \exp(-\Delta U / k_B T_{\text{eff}}), \quad \Delta U \propto |F - F_B|^{3/2}$$

$$T_{\text{eff}} \propto \hbar \quad \text{for } T \rightarrow 0$$

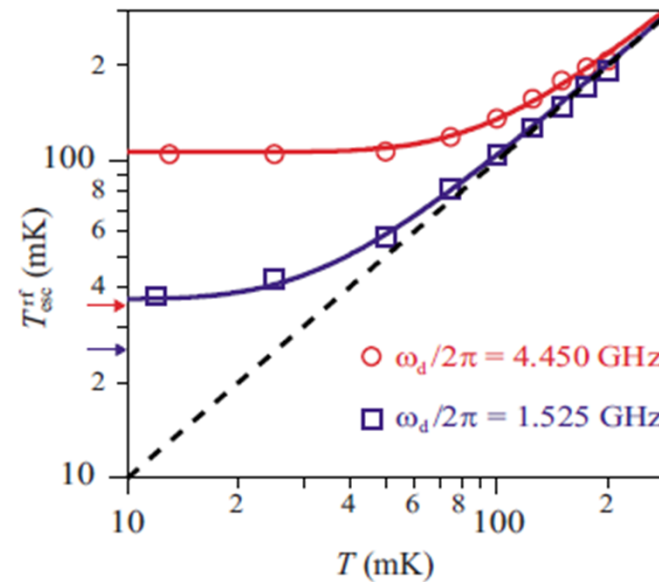
Tunneling exponent is **parametrically larger**

Quantum activation: JBA experiment



$$W_{\text{sw}} \propto \exp(-\Delta U / k_B T_{\text{eff}}), \quad \Delta U \propto |F - F_B|^{3/2}$$

$$k_B T_{\text{eff}} = \hbar \omega_0 (\bar{n} + \frac{1}{2}), \quad \bar{n} = [\exp(\hbar \omega_0 / k_B T) - 1]^{-1}$$



$$\tilde{\Gamma}_{\text{esc}}^{\text{rf}} = [\log(\omega_{\text{attempt}} / W_A)]^{2/3}$$

Vijay et al. (2009, 2012)

Conclusions

- *Quantum relaxation comes with noise.* Escape of modulated quantum oscillators occurs via **quantum activation** – a restriction on JBA-based quantum measurements
- Switching rates in systems lacking detailed balance display **fragility**: a *parametrically large* change of the rate exponent occurs in a *parametrically narrow* parameter range – the beauty of **real-time instantons**, but also a **limitation** on the technique
- The exponents and prefactors of switching rates **scale** as a power of the distance to the bifurcation point, both for thermal noise and for quantum activation

