

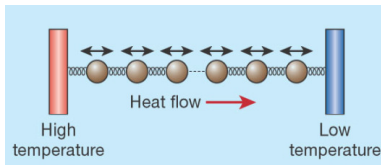
Exactly solvable diffusions and heat conduction

Cristian Giardinà

*Joint work with Jorge Kurchan (ESPCI), Frank Redig (Delft),
Kyamars Vafayi (Eindhoven), Gioia Carinci, Claudio Giberti (Modena)*

Motivations: Fourier law

$$J = \kappa \nabla T$$



(Lepri, Livi, Politi, Phys. Rep. 2003)

- Pure-carbon materials have extremely high thermal conductivity.
- 1D Hamiltonian models:
 - Oscillators chains (Lebowitz, Lieb, Rieder, 1967): $\kappa \sim N$.
 - MD simulations of non-linear oscillators chains: $\kappa \sim N^\alpha$, $0 < \alpha < 1$

- Stochastic models

Kipnis, Marchioro, Presutti (1982):

Observables: Energies at every sites $z = (z_1, \dots, z_N) \in \mathbb{R}_+^N$

Dynamics: Select a bond at random and **uniformly** redistribute the energy under the constraint of conserving the total energy.

$$L^{KMP} f(z) = \sum_{i=1}^N \int_0^1 dp [f(z_1, \dots, p(z_i + z_{i+1}), (1-p)(z_i + z_{i+1}), \dots, z_N) - f(z)]$$

Outline

- 1 Duality.
- 2 Dual processes:
 - Brownian Momentum Process (**BMP**).
 - Symmetric Inclusion Process (**SIP**).
- 3 Heat conduction.
- 4 A larger picture & perspectives.

Duality

Definition

$(\eta_t)_{t \geq 0}$ process on Ω with generator L ,

$(\xi_t)_{t \geq 0}$ process on Ω_{dual} with generator L_{dual}

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ξ_t is **dual** to η_t with duality function $D : \Omega \times \Omega_{dual} \rightarrow \mathbb{R}$ if

$$\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \quad \forall (\eta, \xi) \in \Omega \times \Omega_{dual}$$

η_t is **self-dual** if $L_{dual} = L$.

Duality

Sufficient condition

$$LD(\cdot, \xi)(\eta) = L_{dual}D(\eta, \cdot)(\xi)$$

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$$\begin{aligned}\mathbb{E}_{\eta}(D(\eta_t, \xi)) &= e^{tL}D(\eta, \xi) \\ &= e^{tL_{dual}}D(\eta, \xi)\end{aligned}$$

Duality

Sufficient condition

$$LD(\cdot, \xi)(\eta) = L_{dual}D(\eta, \cdot)(\xi)$$

Indeed

$$\begin{aligned}\mathbb{E}_{\eta}(D(\eta_t, \xi)) &= e^{tL}D(\eta, \xi) \\ &= e^{tL_{dual}}D(\eta, \xi) \\ &= \mathbb{E}_{\xi}(D(\eta, \xi_t))\end{aligned}$$

How to find a dual process?

- 1 Write the generator using **creation and annihilation operators**. Investigate their commutation relations.
- 2 Duality is related to a **change of representation**, i.e. new operators that satisfy the same algebra.
- 3 Self-duality is associated to **symmetries**, i.e. conserved quantities.

The method at work: Energy Diffusion

Brownian momentum process



$SU(1,1)$ algebra



Inclusion process

Brownian momentum process (BMP) on two sites

Given two vertices i and j

$$L_{i,j}^{BMP} f(x_i, x_j) = \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2 f(x_i, x_j)$$

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Think (x_i, x_j) as velocities, then

- polar coordinates

$$L_{i,j}^{BMP} = \frac{\partial^2}{\partial \theta_{ij}^2}$$

- Brownian motion for angle $\theta_{i,j} = \arctan(x_j/x_i)$
- total kinetic energy conserved: $r_{i,j}^2 = x_i^2 + x_j^2$

Brownian momentum process (BMP)

For a graph $G = (V, E)$ let $\Omega = \bigotimes_{i \in V} \Omega_i = \mathbb{R}^{|V|}$.

Configuration $x = (x_1, \dots, x_{|V|}) \in \Omega$

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Generator BMP

$$L^{BMP} = \sum_{(i,j) \in E} L_{i,j}^{BMP} = \sum_{(i,j) \in E} \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2$$

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Stationary *Gaussian* $(0, T)$ product measures.

Symmetric Inclusion Process (SIP)

$$\Omega_{dual} = \bigotimes_{i \in V} \Omega_i^{dual} = \{0, 1, 2, \dots\}^{|V|}$$

Configuration $\xi = (\xi_1, \dots, \xi_{|V|}) \in \Omega_{dual}$

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Generator SIP

$$\begin{aligned} L^{SIP} f(\xi) &= \sum_{(i,j) \in E} L_{i,j}^{SIP} f(\xi) \\ &= \sum_{(i,j) \in E} 2\xi_i(2\xi_j + 1)[f(\xi^{i,j}) - f(\xi)] + (2\xi_i + 1)2\xi_j[f(\xi^{j,i}) - f(\xi)] \end{aligned}$$

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Reversible *Negative Binomial* $(\frac{1}{2}, \lambda)$ product measures.

Duality between BMP and SIP

Theorem 1

The process $\{x(t)\}_{t \geq 0}$ with generator $L = L^{BMP}$ and the process $\{\xi(t)\}_{t \geq 0}$ with generator $L_{dual} = L^{SIP}$ are dual on

$$D(x, \xi) = \prod_{i \in V} \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!}$$

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Proof: An explicit computation gives

$$L^{BMP} D(x, \xi) = L^{SIP} D(x, \xi)$$

Duality explained

$SU(1, 1)$ ferromagnetic quantum spin chain

Abstract operator

$$\mathcal{L} = \sum_{(i,j) \in E} \left(K_i^+ K_j^- + K_i^- K_j^+ - 2K_i^o K_j^o + \frac{1}{8} \right)$$

with $\{K_i^+, K_i^-, K_i^o\}_{i \in V}$ satisfying $SU(1, 1)$ commutation relations:

$$[K_i^o, K_j^\pm] = \pm \delta_{i,j} K_i^\pm \qquad [K_i^-, K_j^+] = 2\delta_{i,j} K_i^o$$

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Duality between L^{BMP} e L^{SIP} corresponds to two different representations of the operator $\mathcal{L}$.

$SU(1, 1)$ structure

Continuous representation

$$K_i^+ = \frac{1}{2} x_i^2 \qquad K_i^- = \frac{1}{2} \frac{\partial^2}{\partial x_i^2}$$

$$K_i^o = \frac{1}{4} \left(x_i \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i} x_i \right)$$

satisfy commutation relations of the $SU(1, 1)$ group

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In this representation

$$\mathcal{L} = L^{BMP} = \sum_{(i,j) \in E} \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2$$

$SU(1, 1)$ structure

Discrete representation

$$\mathcal{K}_i^+ |\xi_i\rangle = \left(\xi_i + \frac{1}{2} \right) |\xi_i + 1\rangle$$

$$\mathcal{K}_i^- |\xi_i\rangle = \xi_i |\xi_i - 1\rangle$$

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In a canonical base

$$\mathcal{K}_i^+ = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ \frac{1}{2} & & & \\ & \ddots & & \\ & & \frac{3}{2} & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix} \quad \mathcal{K}_i^- = \begin{pmatrix} 0 & 1 & & \\ & \ddots & & \\ & & 2 & \\ & & & \ddots & \\ & & & & \ddots & \\ & & & & & \ddots \end{pmatrix} \quad \mathcal{K}_i^0 = \begin{pmatrix} \frac{1}{4} & 0 & & \\ & \frac{5}{4} & & \\ & & \ddots & \\ & & & \frac{9}{4} & \\ & & & & \ddots & \\ & & & & & \ddots \end{pmatrix}$$

$SU(1, 1)$ structure

Discrete representation

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Self-duality of the SIP process

Theorem 2

The process with generator L^{SIP} is self-dual on functions

$$D(\eta, \xi) = \prod_{i \in V} \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \xi_i\right)}$$

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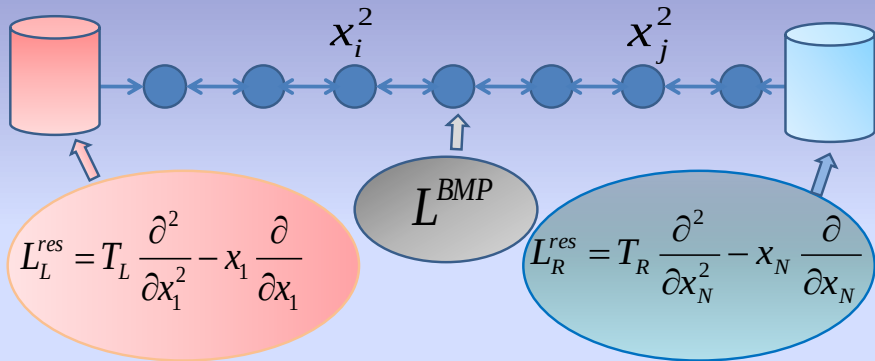
Proof:

$$[L^{SIP}, \sum_i K_i^o] = [L^{SIP}, \sum_i K_i^+] = [L^{SIP}, \sum_i K_i^-] = 0$$

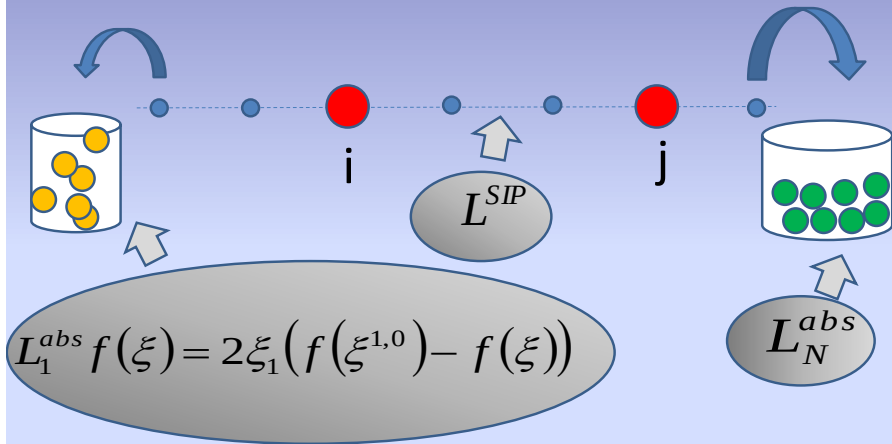
Self-duality fct related to the simmetry $S = e^{K^+}$

Heat conduction models in 1D.

Brownian Momentum Process with reservoirs



Inclusion Process with absorbing reservoirs



Duality between BMP with reservoirs and SIP with absorbing boundaries

Configurations $\bar{\xi} = (\xi_0, \xi_1, \dots, \xi_N, \xi_{N+1}) \in \Omega_{dual} = \mathbb{N}^{N+2}$

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Theorem 3

The process $\{x(t)\}_{t \geq 0}$ with generator $L^{BMP, res}$ is dual to the process $\{\bar{\xi}(t)\}_{t \geq 0}$ with generator $L^{SIP, abs}$ on

$$D(x, \bar{\xi}) = T_L^{\xi_0} \left(\prod_{i=1}^N \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!} \right) T_R^{\xi_{N+1}}$$

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Proof: By inspection

$$L^{BMP, res} D(x, \bar{\xi}) = L^{SIP, abs} D(x, \bar{\xi})$$

CONSEQUENCES OF DUALITY

n -points correlation functions
for the BMP process with reservoirs
(diffusive system of N particles)



absorbing probabilities
of the SIP process with absorbing boundaries
initialized with n inclusion walkers.

Remark: $n \ll N$

Proposition

Let $|\xi| = \sum_{i=1}^N \xi_i$ be the total number of SIP dual walkers

$$\mathbb{E}(D(x, \bar{\xi})) = \sum_{a,b: a+b=|\xi|} T_L^a T_R^b \mathbb{P}_{\bar{\xi}}(\xi_0(\infty) = a, \xi_{N+1}(\infty) = b)$$

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Proof:

$$\mathbb{E}(D(x, \bar{\xi})) = \lim_{t \rightarrow \infty} \int \mathbb{E}_x(D(x_t, \bar{\xi})) d\nu(x)$$

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$$\text{using} \quad D(x, \bar{\xi}) = T_L^{\xi_0} \left(\prod_{i=1}^N \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!} \right) T_R^{\xi_{N+1}}$$

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Temperature profile

$$\begin{aligned} \vec{\xi} = (0, \dots, 0, \underset{\text{site } i \nearrow}{1}, 0, \dots, 0) &\Rightarrow D(x, \vec{\xi}) = x_i^2 \\ &\Rightarrow 1 \text{ SIP walker } (X_t)_{t \geq 0} \text{ with } X_0 = i \end{aligned}$$

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$$\mathbb{E} \left(x_i^2 \right) = T_L \mathbb{P}_i(X_\infty = 0) + T_R \mathbb{P}_i(X_\infty = N + 1)$$

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$$\mathbb{E}(x_i^2) = T_L \mathbb{P}_i(X_\infty = 0) + T_R \mathbb{P}_i(X_\infty = N+1)$$

$$\begin{aligned} T_i &= T_L \left(1 - \frac{i}{N+1}\right) + T_R \left(\frac{i}{N+1}\right) \\ &= T_L + \left(\frac{T_R - T_L}{N+1}\right) i \end{aligned}$$

Energy covariance

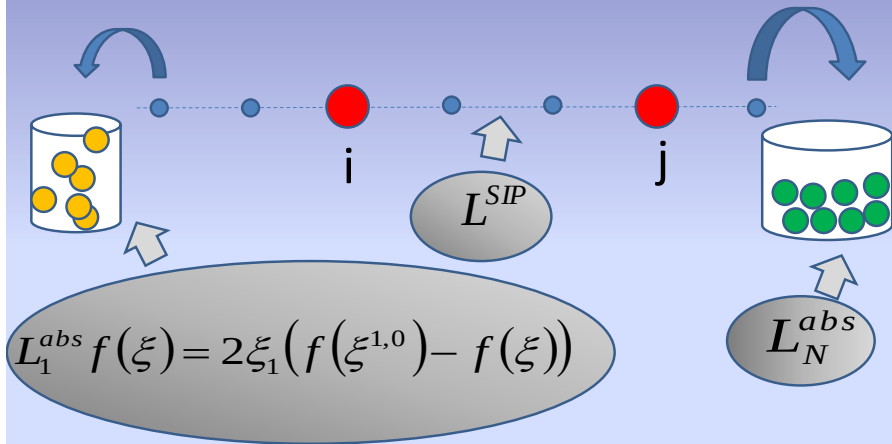
$$\text{If } \vec{\xi} = (0, \dots, 0, \underset{\text{site } i \nearrow}{1}, 0, \dots, 0, \underset{\text{site } j \nearrow}{1}, 0, \dots, 0) \Rightarrow D(x, \vec{\xi}) = x_i^2 x_j^2$$

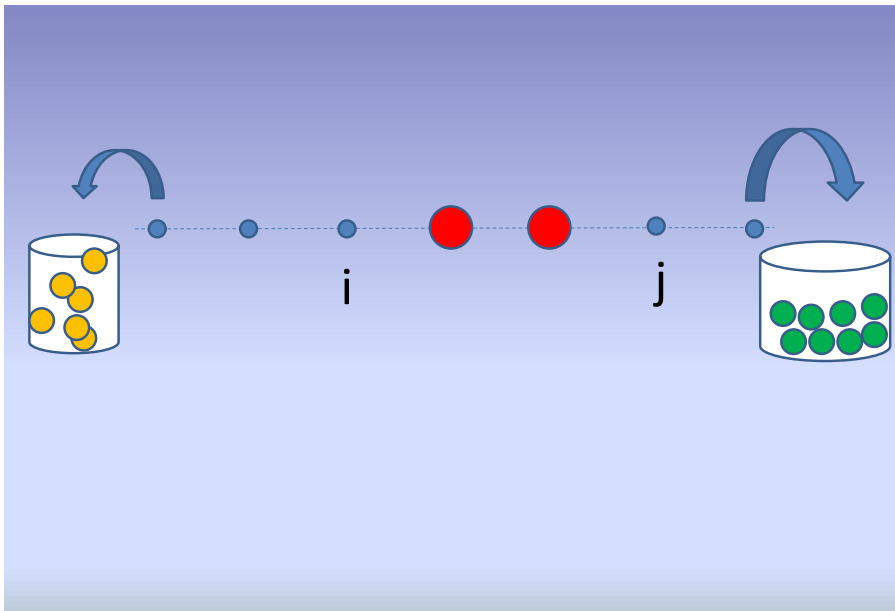
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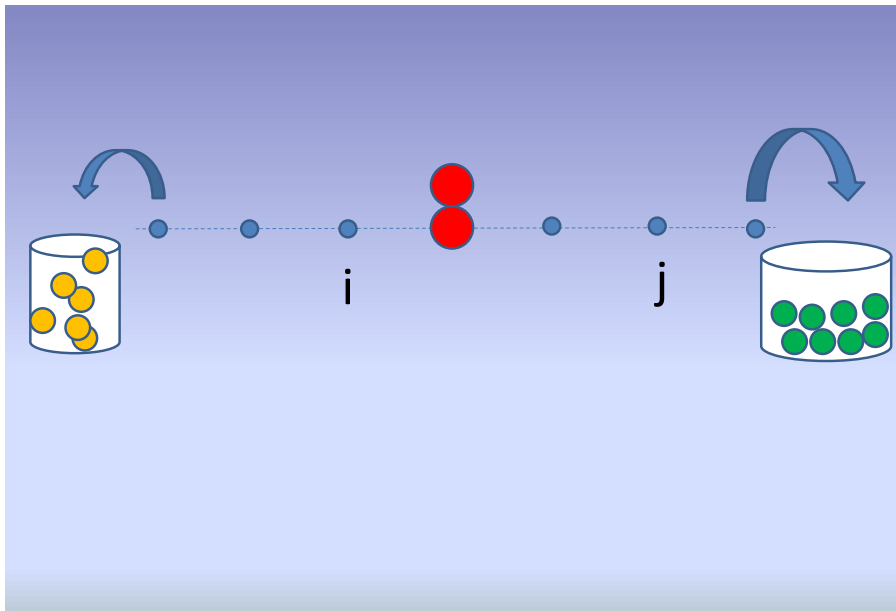
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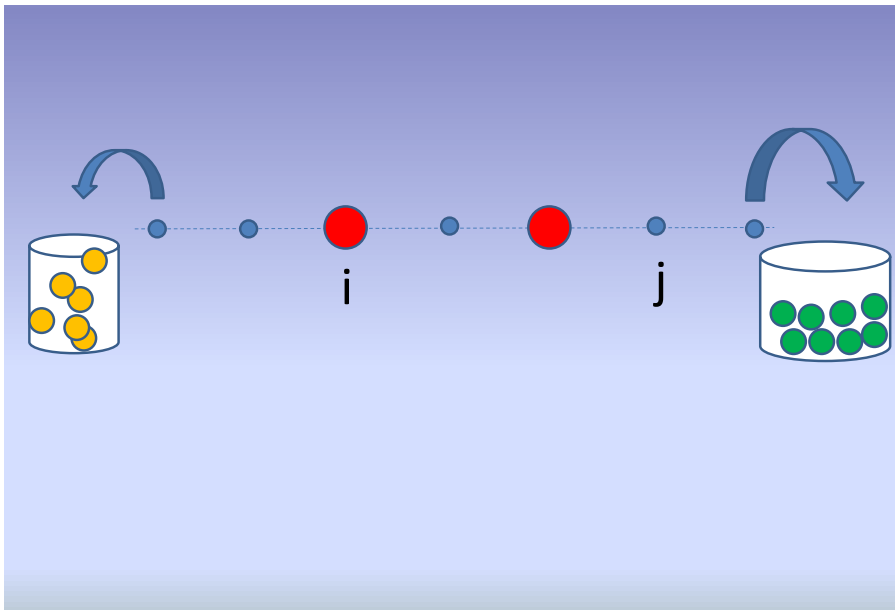
In the dual process we initialize two
SIP walkers $(X_t, Y_t)_{t \geq 0}$ with $(X_0, Y_0) = (i, j)$

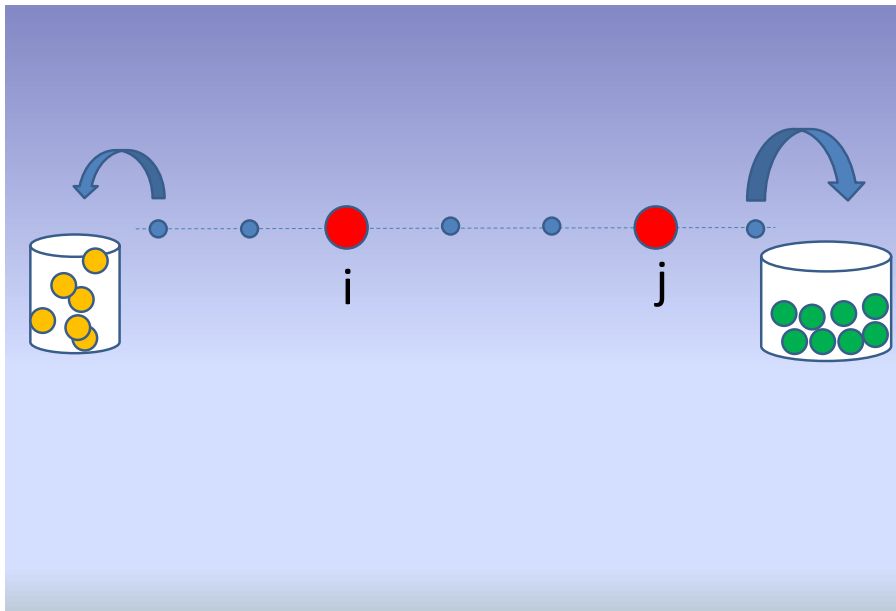
Inclusion Process with absorbing reservoirs

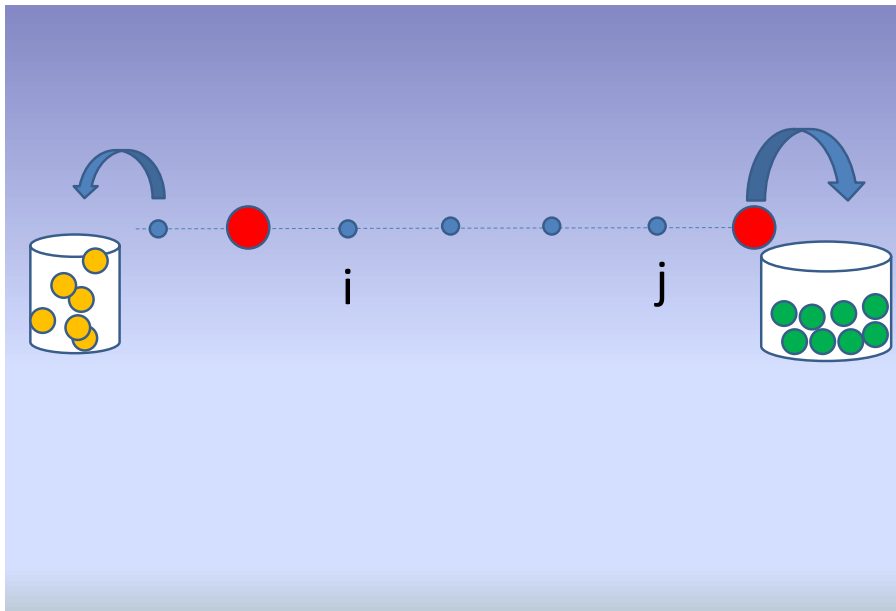


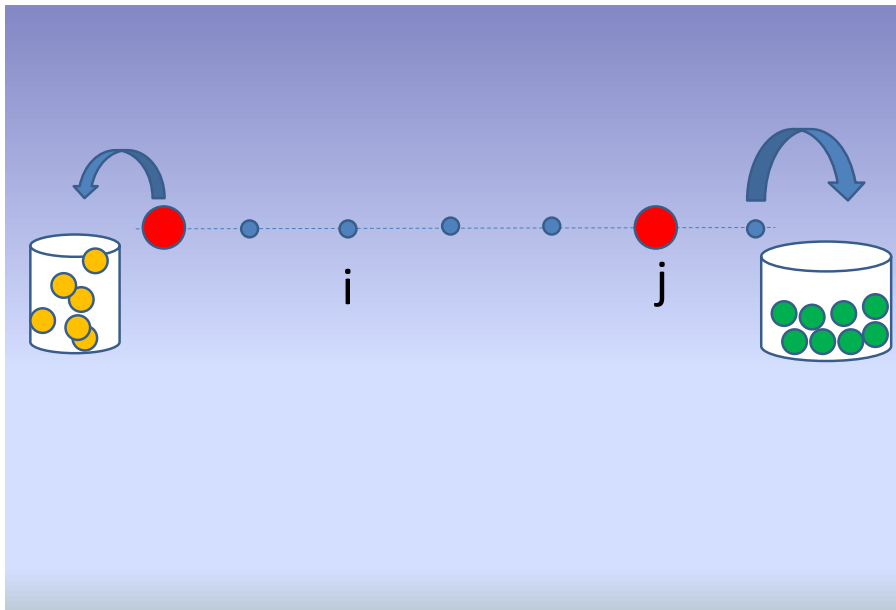


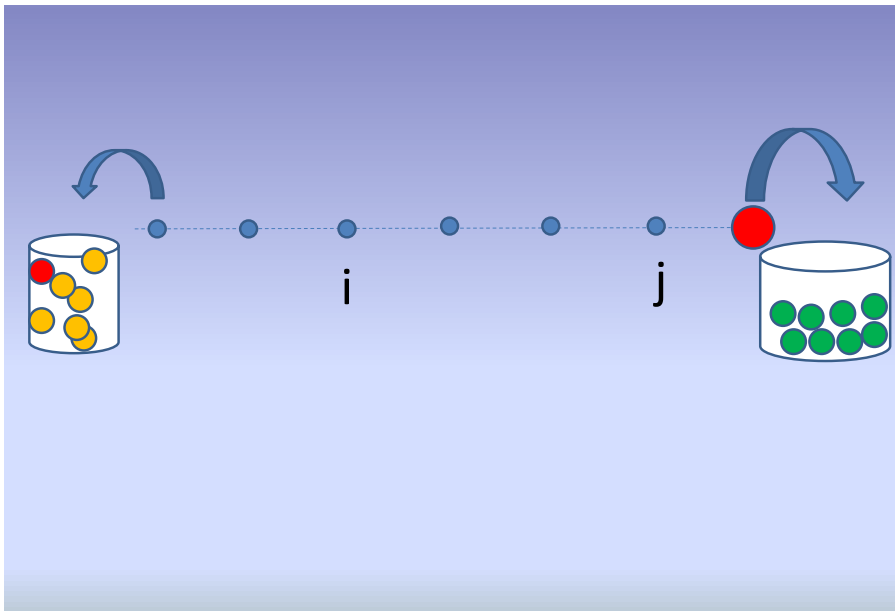


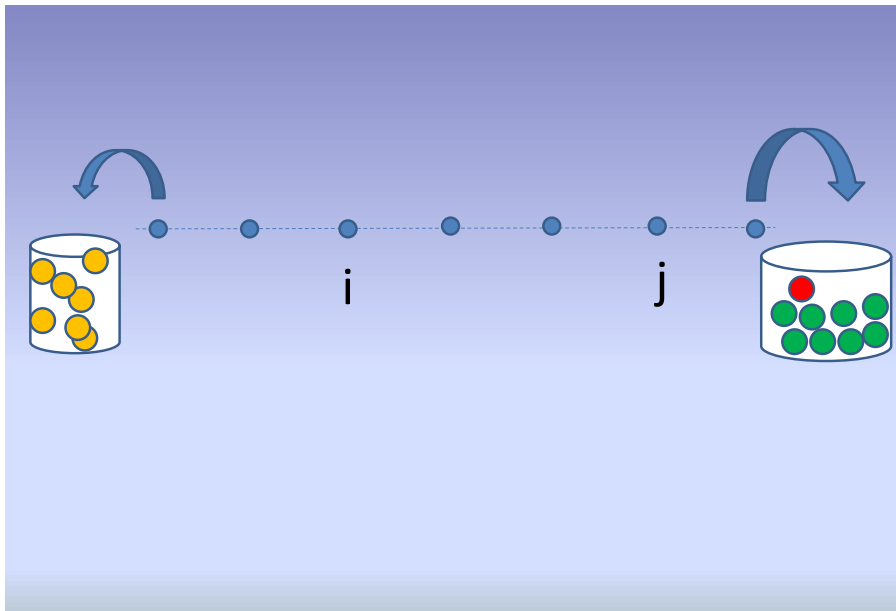


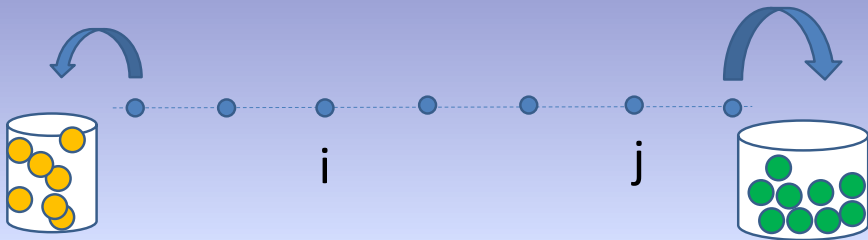












$$\mathbf{E}(x_i^2 x_j^2) = T_L^2 P(\text{yellow}) + T_R^2 P(\text{green}) + T_L T_R (P(\text{yellow-green}) + P(\text{green-yellow}))$$

Energy covariance

$$\mathbb{E} \left(x_i^2 x_j^2 \right) - \mathbb{E} \left(x_i^2 \right) \mathbb{E} \left(x_j^2 \right) = \frac{2i(N+1-j)}{(N+3)(N+1)^2} (T_R - T_L)^2 \geq 0$$

Remark: up to a sign, covariance is the same in the boundary driven Exclusion Process.

A larger picture

1. Brownian Energy Process (BEP)
2. Kipnis-Marchioro-Presutti (KMP)

Brownian Energy Process: BEP

The energies of the Brownian Momentum Process

$$z_i(t) = x_i^2(t)$$

Brownian Energy Process: BEP

The energies of the Brownian Momentum Process

$$z_i(t) = x_i^2(t)$$

evolve with

Generator

$$L^{BEP} = \sum_{(i,j) \in E} z_i z_j \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right)^2 - \frac{1}{2} (z_i - z_j) \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right)$$

Stationary Chi-squared (T) product measures

Generalized Brownian Energy Process: BEP(m)

$$L^{BMP(m)} = \sum_{(i,j) \in E} \sum_{\alpha, \beta=1}^m \left(x_{i,\alpha} \frac{\partial}{\partial x_{j,\beta}} - x_{j,\beta} \frac{\partial}{\partial x_{i,\alpha}} \right)^2$$

Generalized Brownian Energy Process: BEP(m)

$$L^{BMP(m)} = \sum_{(i,j) \in E} \sum_{\alpha, \beta=1}^m \left(x_{i,\alpha} \frac{\partial}{\partial x_{j,\beta}} - x_{j,\beta} \frac{\partial}{\partial x_{i,\alpha}} \right)^2$$

The energies $z_i(t) = \sum_{\alpha=1}^m x_{i,\alpha}^2(t)$ evolve as

Generator

$$L^{BEP(m)} = \sum_{(i,j) \in E} z_i z_j \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right)^2 - \frac{m}{2} (z_i - z_j) \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right)$$

Stationary measures: $\nu_{m,T} = \bigotimes_{i=1}^{|V|} \text{Gamma}_i \left(\frac{m}{2}, T \right)$

Adding-up $SU(1, 1)$ spins

$$\mathcal{L}^{(m)} = \sum_{(i,j) \in E} \left(\mathcal{K}_i^+ \mathcal{K}_j^- + \mathcal{K}_i^- \mathcal{K}_j^+ - 2\mathcal{K}_i^o \mathcal{K}_j^o + \frac{m^2}{8} \right)$$

$$\{\mathcal{K}_i^+, \mathcal{K}_i^-, \mathcal{K}_i^o\}_{i \in V} \quad \text{satisfy } SU(1, 1)$$

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$$\{\mathcal{K}_i^+, \mathcal{K}_i^-, \mathcal{K}_i^o\}_{i \in V} \quad \text{satisfy } SU(1, 1)$$

$$\begin{cases} \mathcal{K}_i^+ = z_i \\ \mathcal{K}_i^- = z_i \partial_{z_i}^2 + \frac{m}{2} \partial_{z_i} \\ \mathcal{K}_i^o = z_i \partial_{z_i} + \frac{m}{4} \end{cases}$$

$$\begin{cases} \mathcal{K}_i^+ |\xi_i\rangle = \left(\xi_i + \frac{m}{2}\right) |\xi_i + \mathbf{1}\rangle \\ \mathcal{K}_i^- |\xi_i\rangle = \xi_i |\xi_i - \mathbf{1}\rangle \\ \mathcal{K}_i^o |\xi_i\rangle = (\xi_i + m) |\xi_i\rangle \end{cases}$$

Generalized Symmetric Inclusion Process: SIP(m)

Generator

$$L^{SIP(m)} f(\xi) = \sum_{(i,j) \in E} 2\xi_i(2\xi_j + m)[f(\xi^{i,j}) - f(\xi)] + 2\xi_j(2\xi_i + m)[f(\xi^{j,i}) - f(\xi)]$$

Reversible product measure *Negative Binomial* $(\frac{m}{2}, \lambda)$

Duality between BEP(m) and SIP(m)

Theorem 4

The process $\{z(t)\}_{t \geq 0}$ with generator $L^{BEP(m)}$ and the process $\{\xi(t)\}_{t \geq 0}$ with generator $L^{SIP(m)}$ are dual on

$$D(z, \xi) = \prod_{i \in V} z_i^{\xi_i} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2} + \xi_i)}$$

Kipnis-Marchioro-Presutti model

Generator

$$L^{KMP} f(z) = \sum_i \int_0^1 dp [f(z_1, \dots, p(z_i + z_{i+1}), (1-p)(z_i + z_{i+1}), \dots, z_N) - f(z)]$$

KMP model is an instantaneous thermalization limit of BEP(2).

Kipnis-Marchioro-Presutti model

Generator

$$L^{KMP} f(z) = \sum_i \int_0^1 dp [f(z_1, \dots, p(z_i + z_{i+1}), (1-p)(z_i + z_{i+1}), \dots, z_N) - f(z)]$$

KMP model is an instantaneous thermalization limit of BEP(2).

Conditioning two i.i.d. $\text{Gamma}(\frac{m}{2}, \theta)$ random variables (Z_i, Z_{i+1}) to $Z_i + Z_{i+1} = E$, one has a conditional density

$$g_{Z_i}(y|Z_i + Z_{i+1} = E) = c_m(E) (y(E - y))^{\frac{m}{2}-1}$$

Perspectives

Diffusion dual to $SEP(n)$.

$ASIP_q(m)$ and $ABEP_q(m)$ asymmetric version $\rightarrow$
Deformed quantum group $SU_q(1, 1)$.

Condensation in the SIP/BEP model.

Explicit matrix solution for models with reservoirs.