

The stationary correlations of the 1D KPZ equation

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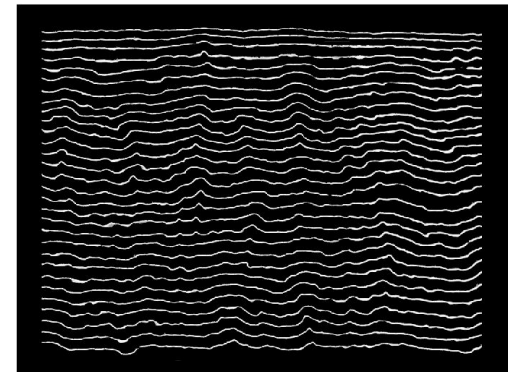
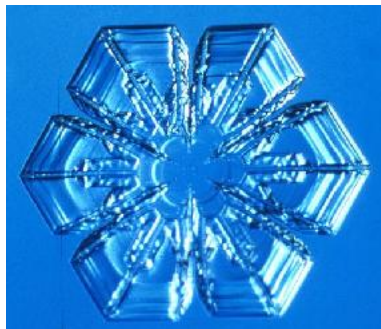
(Based on collaborations with T. Imamura)

18 Dec 2012 @ Rutgers

References: Phys. Rev. Lett. 108,190603(2012); arxiv:1210.4278

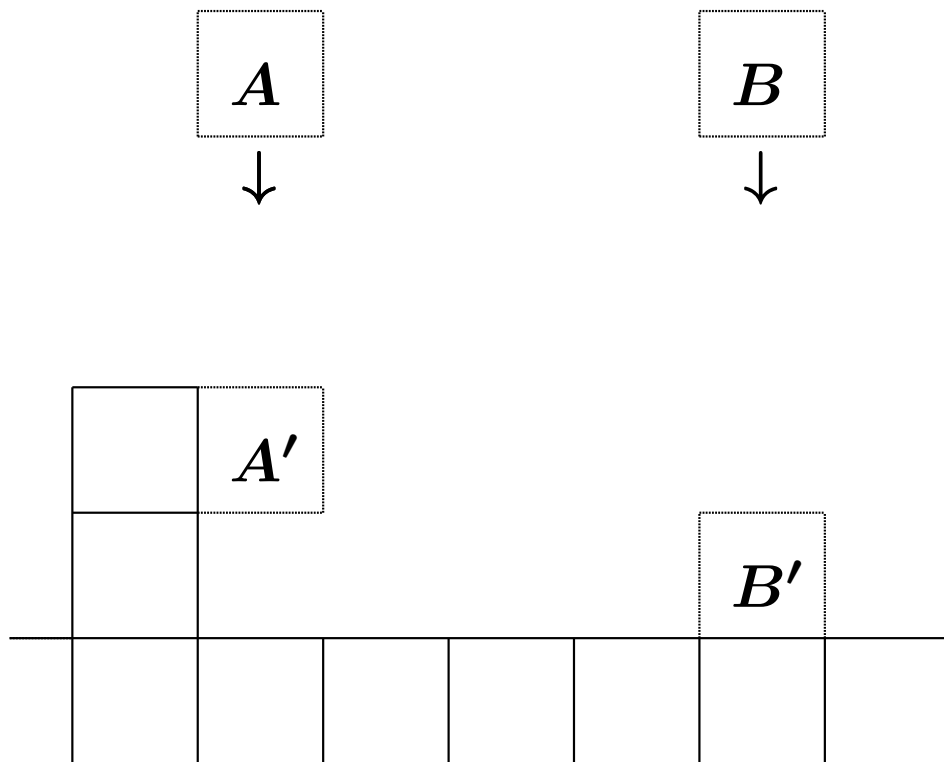
1. Introduction and results: 1D surface growth

- Ubiquitous and interesting physical phenomenon in itself
Kinetic roughening (dynamical), various experiments
- Non-equilibrium statistical mechanical systems
Reaches nonequilibrium steady situation (no detailed balance)
Nonlinearity + Noise + ∞ degrees of freedom

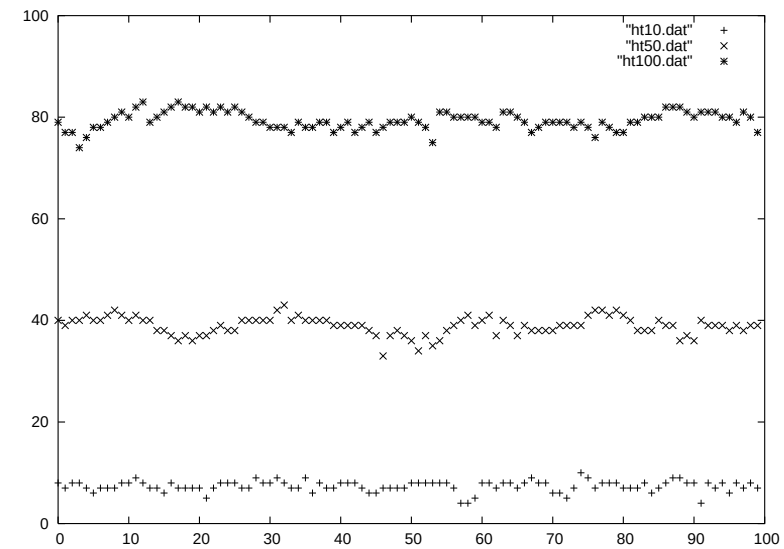


Simulation models

Ex: ballistic deposition model



Kinetic roughening



Scaling

$h(x, t)$: surface height at position x and at time t

Scaling (L : system size)

$$\begin{aligned} W(L, t) &= \langle (h(x, t) - \langle h(x, t) \rangle)^2 \rangle^{1/2} \\ &= L^\alpha \Psi(t/L^z) \end{aligned}$$

For $t \rightarrow \infty$ $W(L, t) \sim L^\alpha$

For $t \sim 0$ $W(L, t) \sim t^\beta$ where $\alpha = \beta z$

In many models, $\alpha = 1/2$ (Gaussian)

$\beta = 1/3$ (Non Gaussian)

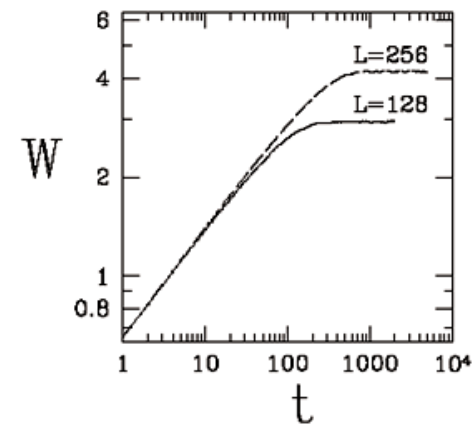
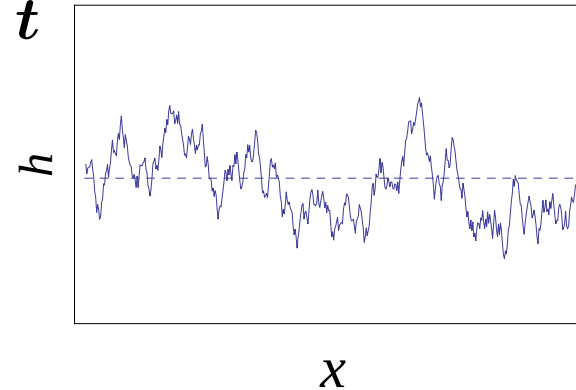


Figure 1. Interface width W versus time t for the RSOS model (Ref. [11]) in 1 + 1 dimensions, in two different lattice lengths.

Kardar-Parisi-Zhang(KPZ) equation

1986 Kardar Parisi Zhang A standard model for surface growth

$$\partial_t h(x, t) = \frac{1}{2} \lambda (\partial_x h(x, t))^2 + \nu \partial_x^2 h(x, t) + \sqrt{D} \eta(x, t)$$

where η is the Gaussian noise with mean 0 and covariance
 $\langle \eta(x, t) \eta(x', t') \rangle = \delta(x - x') \delta(t - t')$

- Dynamical RG analysis: $\alpha = 1/2, \beta = 1/3$

KPZ universality class

- The Brownian motion is stationary (consistent with $\alpha = \frac{1}{2}$).

- Now revival: New experiment 2010 Takeuchi Sano

Exact solutions after 25 years! 2010 S-Spohn, Corwin et al

In their book [1995], Barabási and Stanley write "the KPZ equation cannot be solved in closed form"

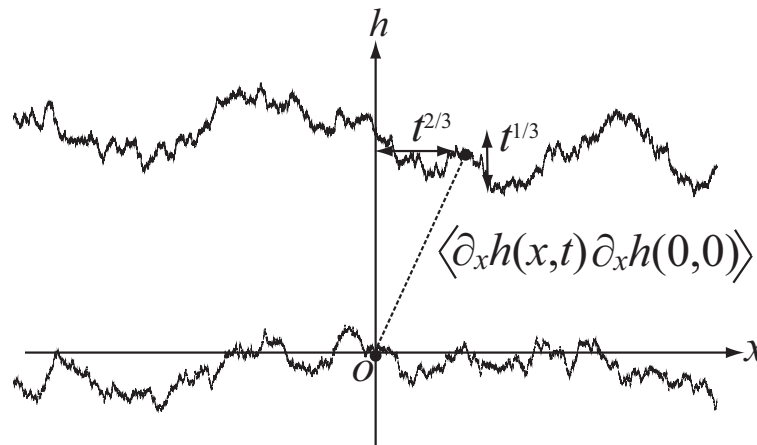
KPZ equation in the stationary state

- By $x \rightarrow \alpha^2 x$, $t \rightarrow 2\nu\alpha^4 t$, $h \rightarrow \frac{\lambda}{2\nu} h$, $\alpha = \frac{\lambda^{1/2}}{(2\nu)^{3/2}}$, we can and will do set $\nu = \frac{1}{2}$, $\lambda = D = 1$.
- For the KPZ equation, the Brownian motion is stationary.

$$h(x, 0) = B(x)$$

where $B(x)$, $x \in \mathbb{R}$ is the two sided BM.

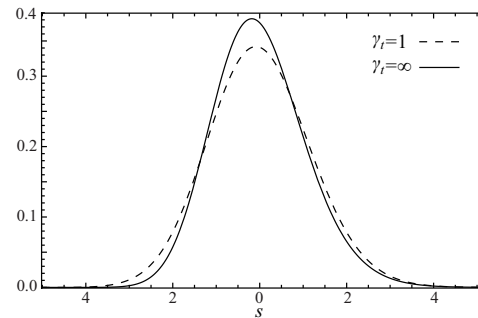
- Height fluctuations and the two point correlation



Main Result 1: Height distribution

$$\begin{aligned} & \text{Prob} \left[h(x, t) + \gamma_t^3 / 12 \leq \gamma_t s \right] \\ &= \frac{1}{\Gamma(1 + \gamma_t^{-1} d/ds)} \int_{-\infty}^{\infty} du \gamma_t e^{\gamma_t(s-u) + e^{-\gamma_t(s-u)}} \nu_{0,t}(u) \end{aligned}$$

where $\nu_{0,t}(u)$ is explicitly written using Fredholm determinants.



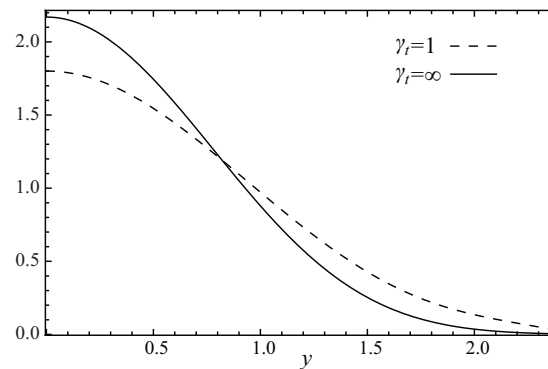
Stationary height distributions for the KPZ equation for $\gamma_t := (\frac{t}{2})^{\frac{1}{3}} = 1$ case. The solid curve is F_0 (scaling limit).

Main result 2: Stationary 2pt correlation function

$$C(x, t) = \langle (h(x, t) - \langle h(x, t) \rangle)^2 \rangle$$

$$(\text{Then } \partial_x^2 C(x, t) = 2 \langle \partial_x h(x, t) \partial_x h(0, 0) \rangle)$$

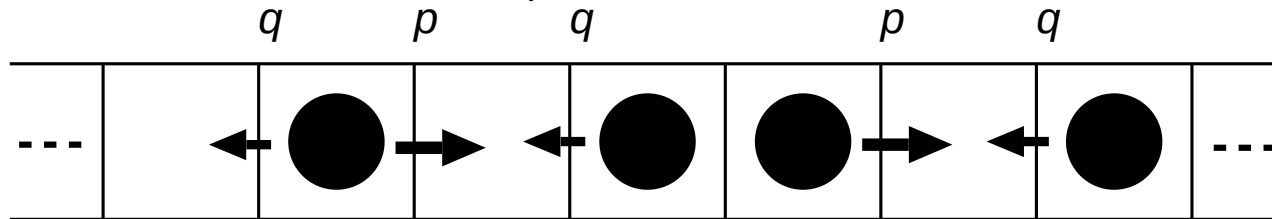
$$g_t(y) = (2t)^{-2/3} C \left((2t)^{2/3} y, t \right)$$



Stationary 2pt correlation function $g_t''(y)$ for $\gamma_t := \left(\frac{t}{2}\right)^{\frac{1}{3}} = 1$.
The solid curve is the scaling limit $g''(y)$.

2. Scaling limit fluctuations from discrete models

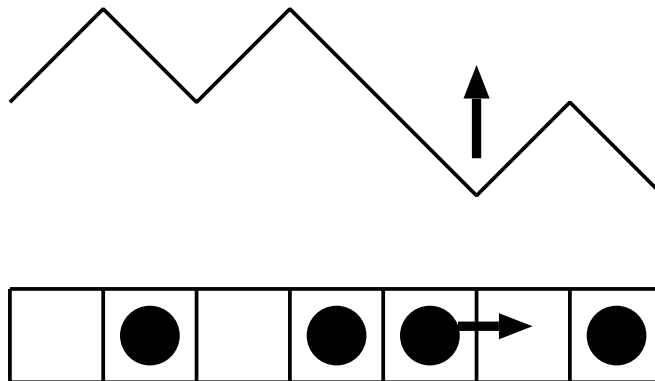
A discrete model: **ASEP** (asymmetric simple exclusion process)



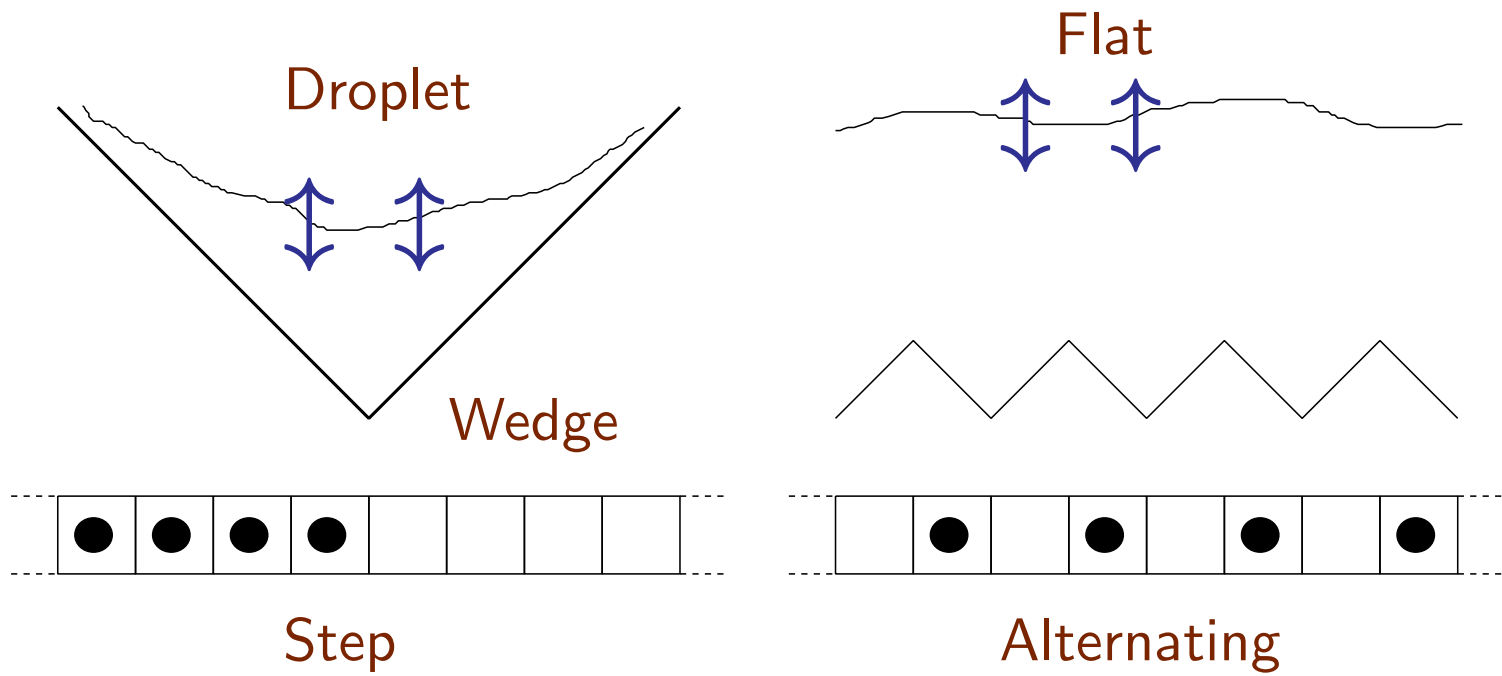
Independent coin toss (random walk) is stationary.

Totally ASEP (TASEP): $p = 0$ or $q = 0$

Mapping to surface growth



Two initial conditions other than stationary



Random matrix

GUE (Gaussian Unitary Ensemble) hermitian matrices

$$A = \begin{bmatrix} u_{11} & u_{12} + iv_{12} & \cdots & u_{1N} + iv_{1N} \\ u_{12} - iv_{12} & u_{22} & \cdots & u_{2N} + iv_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1N} - iv_{1N} & u_{2N} - iv_{2N} & \cdots & u_{NN} \end{bmatrix}$$

$$u_{jj} \sim N(0, 1/2) \quad u_{jk}, v_{jk} \sim N(0, 1/4)$$

GOE (Gaussian Orthogonal Ensemble) real symmetric matrices

Joint eigenvalue distribution ($\beta = 1(\text{GOE}), \beta = 2(\text{GUE})$)

$$P_{N\beta}(x_1, x_2, \dots, x_N) = \frac{1}{Z_{N\beta}} \prod_{1 \leq i < j \leq N} (x_i - x_j)^\beta \prod_{i=1}^N e^{-\frac{\beta}{2} x_i^2}$$

Largest eigenvalue distribution

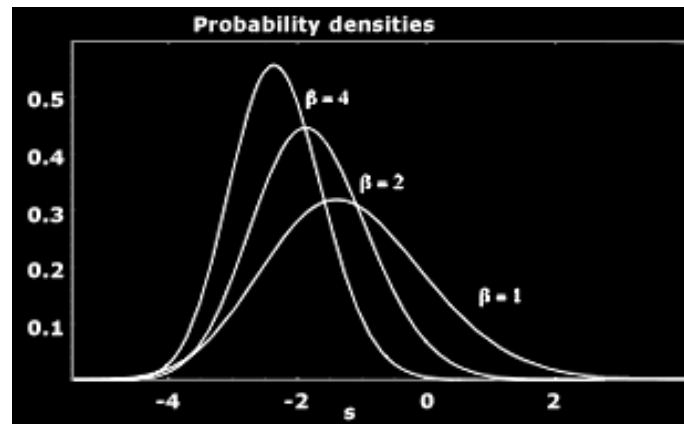
Largest eigenvalue distribution of Gaussian ensembles

$$\mathbb{P}_{N\beta}[x_{\max} \leq s] = \frac{1}{Z_{N\beta}} \int_{(-\infty, s]^N} \prod_{i < j} (x_i - x_j)^\beta \prod_i e^{-\frac{\beta}{2} x_i^2} dx_1 \cdots dx_N$$

Scaling limit As $N \rightarrow \infty$

$$x_{\max} \sim \sqrt{2N} + \frac{1}{\sqrt{2}N^{1/6}} \xi_\beta, \quad \text{Prob}[\xi_\beta \leq s] = F_\beta(s)$$

GOE($\beta = 1$), GUE($\beta = 2$) Tracy-Widom distribution



Scaling limit fluctuations for the 3 important cases

From the studies of discrete models like TASEP

	wedge	flat	stationary
dist.	GUE TW(F_2)	GOE TW(F_1)	F_0

2000 Johansson, Prähofer Spohn For wedge case,

$$\lim_{t \rightarrow \infty} \mathbb{P} \left[\frac{h(0, t) - \frac{1}{4}t}{2^{-4/3}t^{1/3}} \geq -s \right] = F_2(s)$$

where $F_2(s)$ is the GUE Tracy-Widom distribution.

2000 Baik Rains GOE TW for flat

2000 Baik Rains, Prähofer Ferrari Spohn F_0 for stationary

Experiments using liquid crystal turbulence

2010-2012 Takeuchi Sano

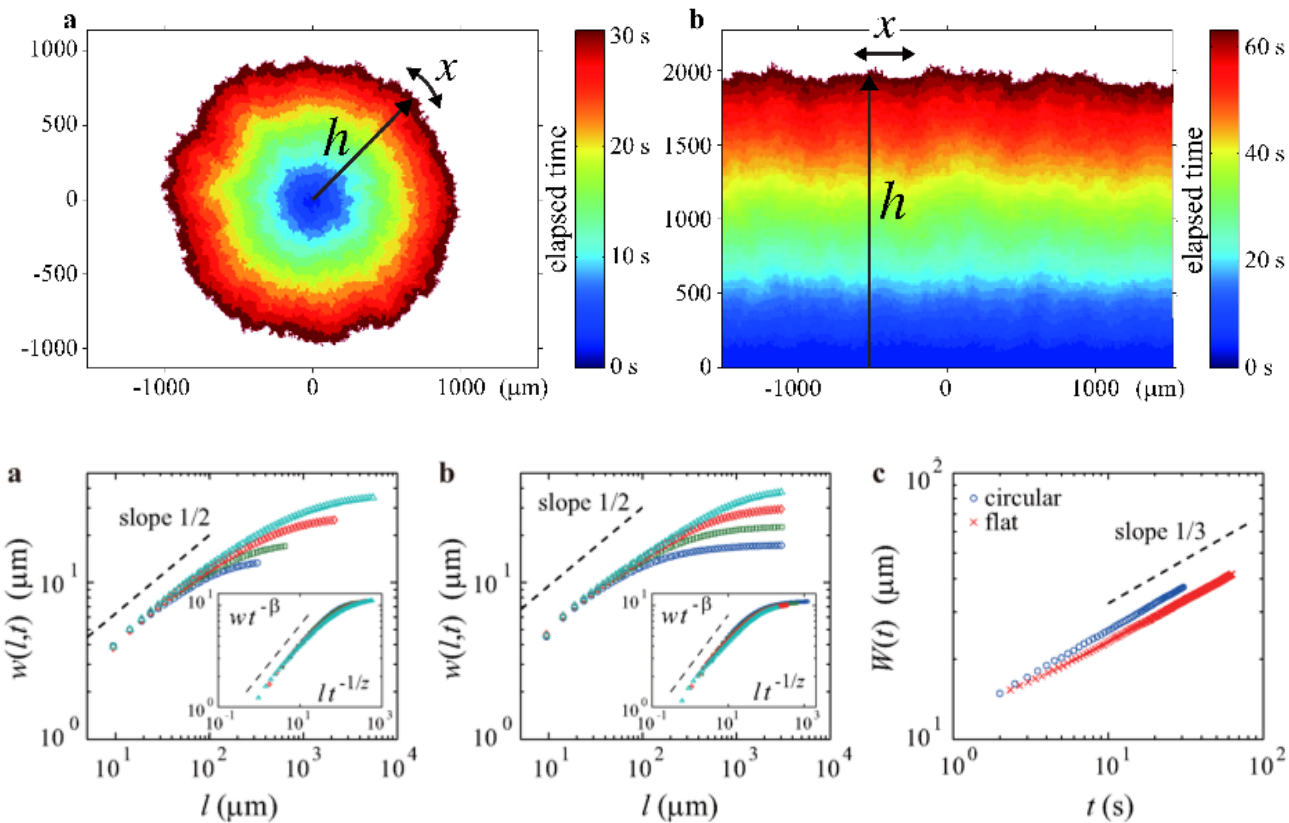
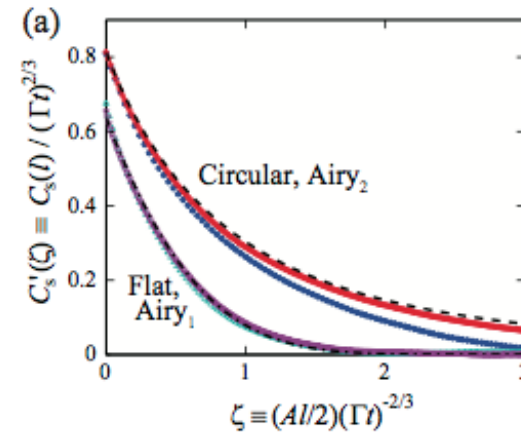
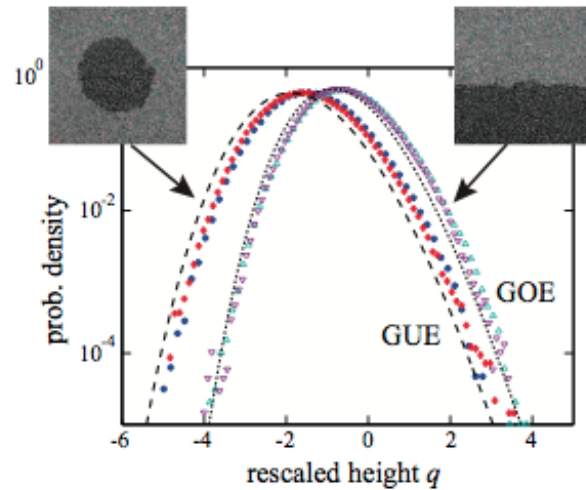


Figure 2 | Family-Vicsek scaling. a,b, Interface width $w(l, t)$ against the length scale l at different times t for the circular (a) and flat (b) interfaces. The four data correspond, from bottom to top, to $t = 2.0$ s, 4.0 s, 12.0 s and 30.0 s for the panel a and to $t = 4.0$ s, 10.0 s, 25.0 s and 60.0 s for the panel b. The insets show the same data with the rescaled axes. c, Growth of the overall width $W(t) \equiv \sqrt{\langle [h(x, t) - \langle h \rangle]^2 \rangle}$. The dashed lines are guides for the eyes showing the exponent values of the KPZ class.

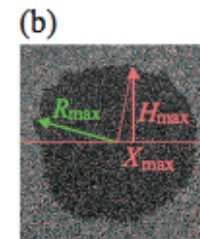
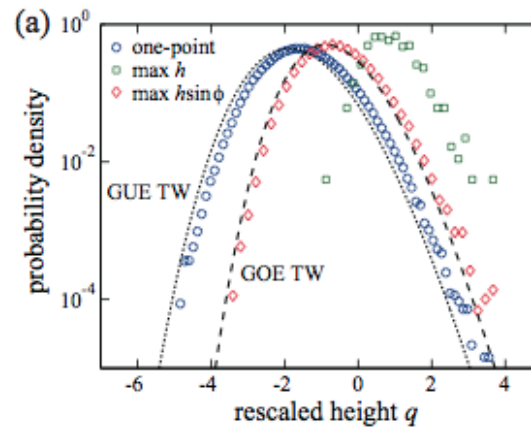
Fluctuations in experiments

1pt



$\text{Airy}_{1,2}$

$\text{Max}[\text{Airy}_2 - x^2]$



Stationary case has not been observed.

4. KPZ equation: Cole-Hopf transformation

What about the KPZ equation?

If we set

$$Z(x, t) = \exp(h(x, t))$$

$Z(x, t)$ satisfies the stochastic heat equation,

$$\frac{\partial Z(x, t)}{\partial t} = \frac{1}{2} \frac{\partial^2 Z(x, t)}{\partial x^2} + \eta(x, t) Z(x, t).$$

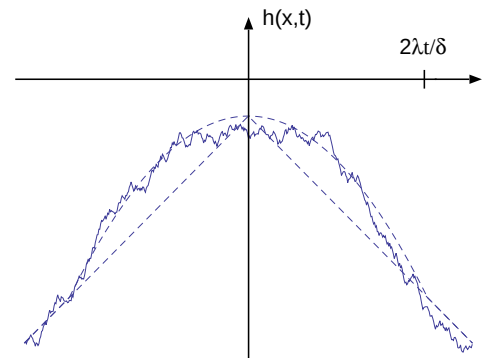
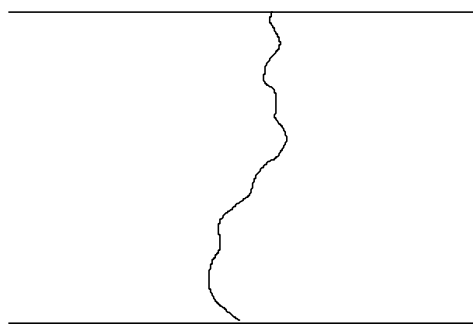
$Z(x, t)$ is regarded as the partition function of the directed polymer in random potential η .

Point-to-Point polymer/Narrow wedge KPZ

The initial condition

$$Z(x, 0) = \delta(x)$$

corresponds to the polymer starting from the origin and ending at x . (For KPZ corresponds to the wedge initial condition.)



Distribution for wedge/pt-to-pt in 1D

Thm(2010 Sasamoto Spohn, Amir Corwin Quastel)

$$h(x, t) = \log Z(x, t) = -x^2/2t - \frac{1}{12}\gamma_t^3 + \gamma_t\xi_t$$

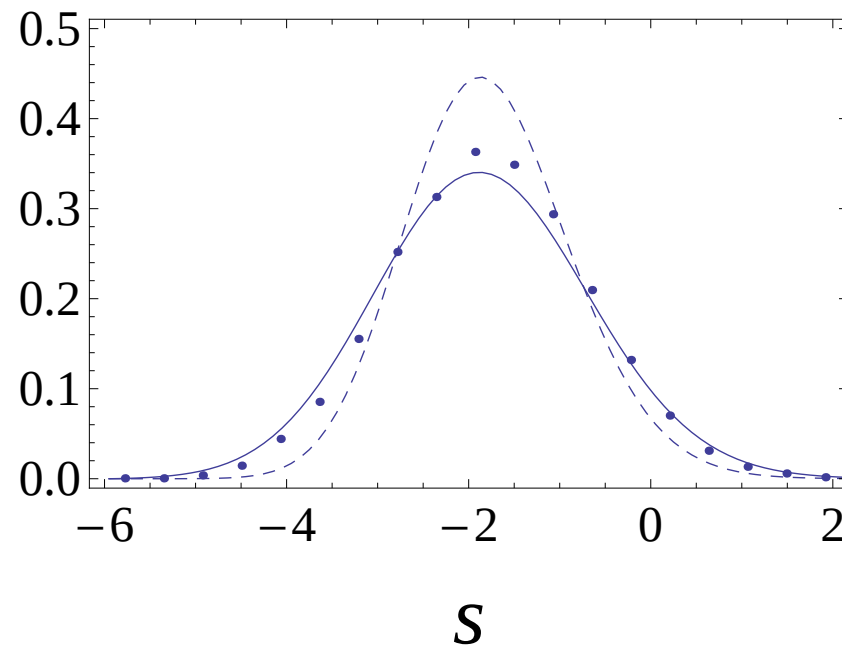
where $\gamma_t = (t/2)^{1/3}$. The distribution function of ξ_t is

$$F_t(s) = \mathbb{P}[\xi_t \leq s] = 1 - \int_{-\infty}^{\infty} \exp \left[-e^{\gamma_t(s-u)} \right] \\ \times \left(\det(1 - P_u(B_t - P_{\text{Ai}})P_u) - \det(1 - P_u B_t P_u) \right) du$$

where $P_{\text{Ai}}(x, y) = \text{Ai}(x)\text{Ai}(y)$, P_u is the projection onto $[u, \infty)$ and the kernel B_t is

$$B_t(x, y) = \int_{-\infty}^{\infty} d\lambda \frac{\text{Ai}(x + \lambda)\text{Ai}(y + \lambda)}{e^{\gamma_t \lambda} - 1}$$

Finite time KPZ distribution and TW



—: exact KPZ density $F'_t(s)$ at $\gamma_t = 0.94$

--: Tracy-Widom density ($t \rightarrow \infty$ limit)

●: ASEP Monte Carlo at $q = 0.6$, $t = 1024$ MC steps

Comments and developments

- The result was first obtained by considering a weakly asymmetric limit of asymmetric simple exclusion process (ASEP), using the works of Bertini, Gianomin and Tracy, Widom .
- The result (for the pt-to-pt) was rederived by using the replica method (Dotsenko, Calabrese, le Doussal, Rosso). The multi-point statistics (Prolhac, Spohn), the point-to-line case (Calabrese, le Doussal), stationary case (Imamura, Sasamoto) and recently a half-line case have been studied.
- Algebraic understanding of solvability has been developing, in particular for the wedge/pt-to-pt case (O'Connell, Seppäläinen, COSZ, Borodin-Corwin...)

Replica method

For a system with randomness, e.g. for random Ising model,

$$H = \sum_{\langle ij \rangle} J_{ij} s_i s_j$$

where i is site, $s_i = \pm 1$ is Ising spin, J_{ij} is iid random variable (e.g. Bernoulli), we are often interested in the averaged free energy $\langle \log Z \rangle$, $Z = \sum_{s_i = \pm 1} e^{-H}$.

In some cases, computing $\langle \log Z \rangle$ seems hopeless but the calculation of N th replica partition function $\langle Z^N \rangle$ is easier.

In replica method, one often resorts to the following identity

$$\langle \log Z \rangle = \lim_{N \rightarrow 0} \frac{\langle Z^N \rangle - 1}{N}.$$

Replica method for 1D KPZ equation

Feynmann-Kac expression for the partition function,

$$Z(x, t) = \mathbb{E}_x \left(e^{\int_0^t \eta(b(s), t-s) ds} Z(b(t), 0) \right)$$

Because η is a Gaussian variable, one can take the average over the noise η to see that the replica partition function can be written as (Kardar 1987)

$$\langle Z^N(x, t) \rangle = \langle x | e^{-H_N t} | \Phi \rangle$$

where H_N is the Hamiltonian of the δ -Bose gas,

$$H_N = -\frac{1}{2} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} - \frac{1}{2} \sum_{j \neq k}^N \delta(x_j - x_k).$$

$|\Phi\rangle$ represents the state corresponding to the initial condition.

Recovering the distribution from $\langle Z^N \rangle$

Remember $h = \log Z$. We are interested not only in the average $\langle h \rangle$ but the full distribution of h . Here we compute the generating function $G_t(s)$ of the replica partition function,

$$G_t(s) = \sum_{N=0}^{\infty} \frac{(-e^{-\gamma_t s})^N}{N!} \langle Z^N(0, t) \rangle e^{N \frac{\gamma_t^3}{12}}$$

with $\gamma_t = (t/2)^{1/3}$. This turns out to be written as a Fredholm determinant. We apply the inversion formula to recover the p.d.f for h . But for the KPZ, $\langle Z^N \rangle \sim e^{N^3}$.

Computation of $\langle Z^N(x, t) \rangle$

We compute $\langle Z^N(x, t) \rangle$ by expanding in terms of the eigenstates of H_N ,

$$\langle Z^N(x, t) \rangle = \sum_z \langle x | \Psi_z \rangle \langle \Psi_z | x \rangle e^{-E_z t}$$

where E_z and $|\Psi_z\rangle$ are the eigenvalue and the eigenfunction of H_N : $H_N |\Psi_z\rangle = E_z |\Psi_z\rangle$.

Bethe states

By the Bethe ansatz, the eigenfunction is given as

$$\langle x_1, \dots, x_N | \Psi_z \rangle = C_z \sum_{P \in S_N} \text{sgn} P \times \prod_{1 \leq j < k \leq N} (z_{P(j)} - z_{P(k)} + i \text{sgn}(x_j - x_k)) \exp \left(i \sum_{l=1}^N z_{P(l)} x_l \right)$$

N momenta z_j ($1 \leq j \leq N$) are parametrized as

$$z_j = q_\alpha - \frac{i}{2} (n_\alpha + 1 - 2r_\alpha), \quad \text{for } j = \sum_{\beta=1}^{\alpha-1} n_\beta + r_\alpha.$$

($1 \leq \alpha \leq M$ and $1 \leq r_\alpha \leq n_\alpha$). They are divided into M groups where $1 \leq M \leq N$ and the α th group consists of n_α quasimomenta z'_j s which shares the common real part q_α .

$$C_z = \left(\frac{\prod_{\alpha=1}^M n_{\alpha}}{N!} \prod_{1 \leq j < k \leq N} \frac{1}{|z_j - z_k - i|^2} \right)^{1/2}$$

$$E_z = \frac{1}{2} \sum_{j=1}^N z_j^2 = \frac{1}{2} \sum_{\alpha=1}^M n_{\alpha} q_{\alpha}^2 - \frac{1}{24} \sum_{\alpha=1}^M (n_{\alpha}^3 - n_{\alpha}) .$$

Expanding the moment in terms of the Bethe states, we have

$$\begin{aligned} & \langle Z^N(x, t) \rangle \\ &= \sum_{M=1}^N \frac{N!}{M!} \prod_{j=1}^N \left(\int_{-\infty}^{\infty} \prod_{\alpha=1}^M \frac{dq_{\alpha}}{2\pi} \sum_{n_{\alpha}=1}^{\infty} \right) \delta_{\sum_{\beta=1}^M n_{\beta}, N} \\ & \quad \times e^{-E_z t} \langle x | \Psi_z \rangle \langle \Psi_z | x \rangle \end{aligned}$$

The completeness of Bethe states is known.

Generating function

$$\begin{aligned}
 G_t(s) &= \sum_{N=0}^{\infty} \sum_{M=1}^N \frac{(-e^{-\gamma_t s})^N}{M!} \prod_{\alpha=1}^M \left(\int_0^{\infty} d\omega_{\alpha} \sum_{n_{\alpha}=1}^{\infty} \right) \delta_{\sum_{\beta=1}^M n_{\beta}, N} \\
 &\quad \times \det \left(\int_{\mathbb{R}} \frac{dq}{\pi} e^{-\gamma_t^3 n_j q^2 + \frac{\gamma_t^3}{12} n_j^3 - n_j(\omega_j + \omega_k) - 2iq(\omega_j - \omega_k)} \right) \\
 &= \det(1 - P_0 K_{t,s} P_0)
 \end{aligned}$$

$$K_{t,s}(\xi_j, \xi_k) = \int_{-\infty}^{\infty} dy \text{Ai}(\xi_j + y) \text{Ai}(\xi_k + y) \frac{e^{\gamma_t y}}{e^{\gamma_t y} + e^{\gamma_t s}}$$

5. Stationary case

- **Narrow wedge** is technically the simplest (transient).
- **Flat** case is a well-studied case in surface growth (transient)
- **Stationary** case is important for stochastic process and nonequilibrium statistical mechanics
 - Two-point correlation function
 - Experiments: Scattering, direct observation
 - A lot of approximate methods (renormalization, mode-coupling, etc.) have been applied to this case.
 - Nonequilibrium steady state(NESS): No principle. Dynamics is even harder.

Modification of initial condition

Original: two sided BM

$$h(x, 0) = \begin{cases} B_-(-x), & x < 0, \\ B_+(x), & x > 0, \end{cases}$$

where $B_{\pm}(x)$ are two independent standard BMs is stationary.

Modification: we consider a generalized initial condition

$$h(x, 0) = \begin{cases} \tilde{B}(-x) + v_-x, & x < 0, \\ B(x) - v_+x, & x > 0, \end{cases}$$

where $B(x), \tilde{B}(x)$ are independent standard BMs and $v_{\pm}$ are the strength of the drifts.

Result

For the generalized initial condition with $v_{\pm}$

$$F_{v_{\pm},t}(s) := \text{Prob} \left[h(x, t) + \gamma_t^3 / 12 \leq \gamma_t s \right]$$

$$= \frac{\Gamma(v_+ + v_-)}{\Gamma(v_+ + v_- + \gamma_t^{-1} d/ds)} \left[1 - \int_{-\infty}^{\infty} du e^{-e^{\gamma_t(s-u)}} \nu_{v_{\pm},t}(u) \right]$$

Here $\nu_{v_{\pm},t}(u)$ is expressed as a difference of two Fredholm determinants,

$$\nu_{v_{\pm},t}(u) = \det \left(1 - P_u (B_t^{\Gamma} - P_{\text{Ai}}^{\Gamma}) P_u \right) - \det \left(1 - P_u B_t^{\Gamma} P_u \right),$$

where P_s represents the projection onto (s, ∞) ,

$$P_{\text{Ai}}^{\Gamma}(\xi_1, \xi_2) = \text{Ai}_{\Gamma}^{\Gamma} \left(\xi_1, \frac{1}{\gamma_t}, v_-, v_+ \right) \text{Ai}_{\Gamma}^{\Gamma} \left(\xi_2, \frac{1}{\gamma_t}, v_+, v_- \right)$$

$$B_t^\Gamma(\xi_1, \xi_2) = \int_{-\infty}^{\infty} dy \frac{1}{1 - e^{-\gamma_t y}} \text{Ai}_\Gamma^\Gamma \left(\xi_1 + y, \frac{1}{\gamma_t}, v_-, v_+ \right) \\ \times \text{Ai}_\Gamma^\Gamma \left(\xi_2 + y, \frac{1}{\gamma_t}, v_+, v_- \right),$$

and

$$\text{Ai}_\Gamma^\Gamma(a, b, c, d) = \frac{1}{2\pi} \int_{\Gamma_{i\frac{d}{b}}} dz e^{iza + i\frac{z^3}{3}} \frac{\Gamma(ibz + d)}{\Gamma(-ibz + c)},$$

where Γ_{z_p} represents the contour from $-\infty$ to ∞ and, along the way, passing below the pole at $z = id/b$.

Height distribution for the stationary KPZ equation

$$F_{0,t}(s) = \frac{1}{\Gamma(1 + \gamma_t^{-1} d/ds)} \int_{-\infty}^{\infty} du \gamma_t e^{\gamma_t(s-u) + e^{-\gamma_t(s-u)}} \nu_{0,t}(u)$$

where $\nu_{0,t}(u)$ is obtained from $\nu_{v_{\pm},t}(u)$ by taking $v_{\pm} \rightarrow 0$ limit.

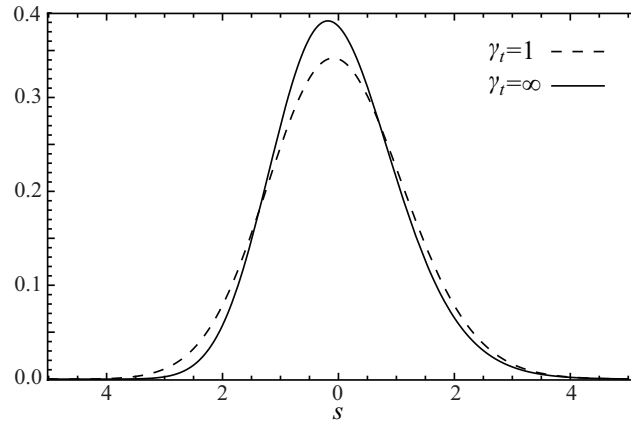


Figure 1: Stationary height distributions for the KPZ equation for $\gamma_t = 1$ case. The solid curve is F_0 .

Stationary 2pt correlation function

$$C(x, t) = \langle (h(x, t) - \langle h(x, t) \rangle)^2 \rangle$$

$$g_t(y) = (2t)^{-2/3} C \left((2t)^{2/3} y, t \right)$$

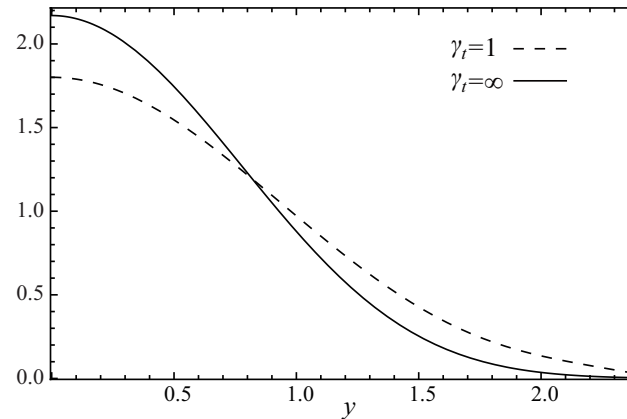


Figure 2: Stationary 2pt correlation function $g''_t(y)$ for $\gamma_t = 1$. The solid curve is the corresponding quantity in the scaling limit $g''(y)$.

Summary

- The 1D KPZ equation has been solved exactly for a few geometries. In particular the stationary two point correlation functions has been computed using the replica method.
- For the continuum models, there are some tricky points in the replica computations. For a few discrete models, it can be made rigorous (so far for the wedge/pt-to-pt case).
- Renewed interests in the KPZ systems.
Universality: various models and methods, 1D fluids?...theory, simulation, experiments
Integrable structures.

Tracy-Widom distributions

GUE Tracy-Widom distribution

$$F_2(s) = \det(1 - P_s K_2 P_s)$$

where P_s : projection onto $[s, \infty)$ and K_2 is the Airy kernel

$$K_2(x, y) = \int_0^\infty d\lambda \text{Ai}(x + \lambda) \text{Ai}(y + \lambda)$$

Painlevé II representation

$$F_2(s) = \exp \left[- \int_s^\infty (x - s) u(x)^2 dx \right]$$

where $u(x)$ is the solution of the Painlevé II equation

$$\frac{\partial^2}{\partial x^2} u = 2u^3 + xu, \quad u(x) \sim \text{Ai}(x) \quad x \rightarrow \infty$$

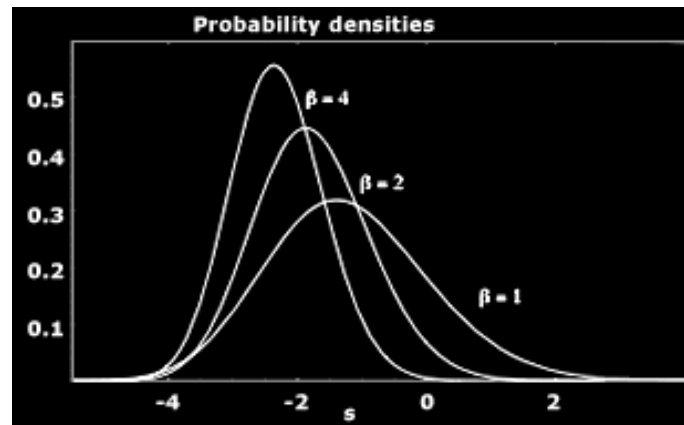
GOE Tracy-Widom distribution

$$F_1(s) = \exp \left[-\frac{1}{2} \int_s^\infty u(x) dx \right] (F_2(s))^{1/2}$$

GSE Tracy-Widom distribution

$$F_4(s) = \cosh \left[-\frac{1}{2} \int_s^\infty u(x) dx \right] (F_2(s))^{1/2}$$

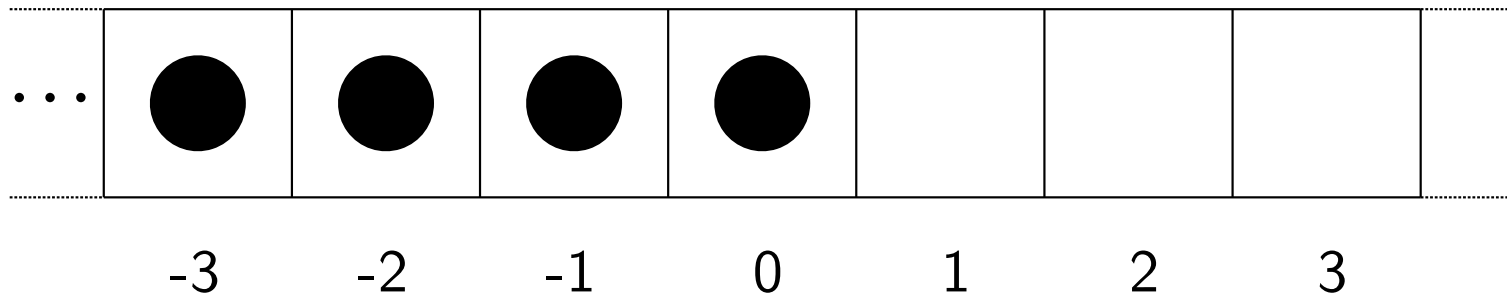
Figures for Tracy-Widom distributions



Connection to random matrix: Johansson

TASEP(Totally ASEP, hop only in one direction)

Step initial condition ($t = 0$)



$N(t)$: Number of particles which crossed $(0,1)$ up to time t

LUE formula

$$\mathbb{P}[N(t) \geq N] = \frac{1}{Z_N} \int_{[0,t]^N} \prod_{i < j} (x_i - x_j)^2 \prod_i e^{-x_i} dx_1 \cdots dx_N$$

Painlevé II representation (Baik Rains)

$u(s)$: Solution to the Painlevé II equation

$$u_{xx} = 2u^3 + xu$$

with the asymptotics $u(x) \sim -\text{Ai}(x)$, $x \rightarrow \infty$.

$$v(x) = \int_{\infty}^x u(s)^2 ds = u(x)^4 + xu(x)^2 - (u'(x))^2$$

In terms of u and v

$$F_2(s) = \exp \left(- \int_s^{\infty} (x - s) u(x)^2 dx \right)$$

$$F_1(s) = F_2(s)^{1/2} e^{\frac{1}{2} \int_s^{\infty} u(x) dx}$$

$$F_0(s) = \{1 - (s + 2u'(s) + 2u(s)^2)v(s)\} e^{2 \int_s^{\infty} u(x) dx} F_2(s)$$

Fredholm determinant representation

$$F_2(s) = \det(1 - P_s K_2 P_s)$$

$$K_2(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}(y)\text{Ai}'(x)}{x - y}$$

$$F_1(s) = \text{Pf}(1 - P_s K_1 P_s) = \det(1 - P_s K_{\text{flat}} P_s)$$

$$K_1(x, y) = 2 \times 2 \text{ matrix kernel}$$

$$K_{\text{flat}}(x, y) = \text{Ai}(x + y)$$

F_0 (Imamura TS, Ferrari Spohn)

From the analysis of the PNG model with external sources and TASEP with two Bernoulli initial conditions, one gets an distribution function with two parameters w_+, w_- .

$$F_{w_+, w_-}(s) = \left(1 + \frac{1}{w_+ + w_-} \frac{d}{ds}\right) \det(1 - P_s K_{w_+, w_-} P_s)$$

$$K_{w_+, w_-}(x, y) = K_2(x, y) + (w_+ + w_-) B(x, w_+) B(y, w_-)$$

where

$$B_A(x, w) = \int_0^\infty e^{wx} \text{Ai}(x + \lambda) d\lambda$$

$$B(x, w) = e^{w^3/3 - xw} - B_A(x, w)$$

Stationary case corresponds to taking $w_+ = -w_- (:= w)$.

$$\begin{aligned}
F_w(s) = & \frac{d}{ds} \left(\left\{ s - w^2 - 1 + \int_s^\infty dx \left(e^{-w^3/3+wx} B_A(x, w) \right. \right. \right. \\
& \left. \left. \left. + e^{w^3/3-wx} B_A(x, -w) \right) - \int_s^\infty B_A(x, w) B_A(x, -w) dx \right\} F_2(s) \right. \\
& \left. + \det(1 - P_s \tilde{K}_2 P_s) \right) \\
\tilde{K}_2(x, y) = & K_2(x, y) + \int_s^\infty dz K_2(x, z) B(z, w) B(y, -w)
\end{aligned}$$

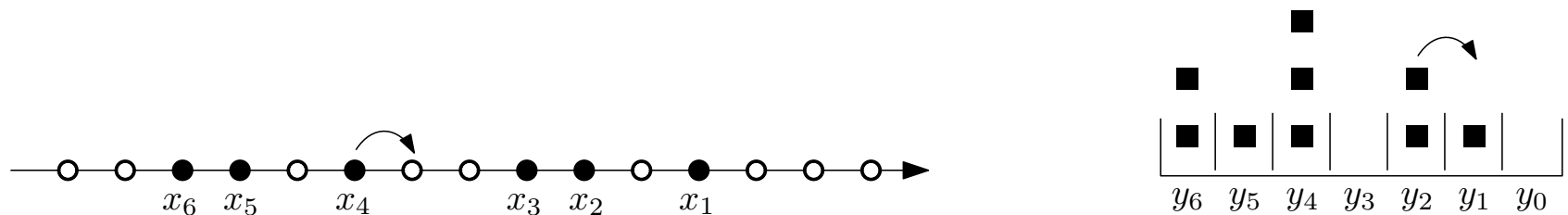
- The height distribution in the stationary situation is obtained by setting $w = 0$: $F_0(s) = F_{w=0}(s)$.
- The two point correlation can be obtained by computing

$$g(y) = \int_{-\infty}^{\infty} s^2 F'_y(s) ds$$

5. Discrete models

For the wedge case, the above computations can be regarded as a "shadow" of a rigorous version at a higher level (Borodin-Corwin, Borodin-Corwin-Sasamoto).

q -TASEP particle i hops with rate $1 - q^{x_{i-1} - x_i - 1}$.



The dynamics of the gaps $y_i = x_{i-1} - x_i - 1$ is a version of the zero range process in which a particle hops to the right site with rate $1 - q^{y_i}$. The generator of the process can be written in terms of " q -deformed boson". (1998 Sasamoto, Wadati)

Moments

Set $Z_q(n, t) = q^{x_n + n}$. Then the quantity

$$\left\langle \prod_{i=1}^N Z_q(n_i, t) \right\rangle$$

satisfies (∇_i is a discrete derivative)

$$\frac{d}{dt} \left\langle \prod_{i=1}^N Z_q(n_i, t) \right\rangle = H_{N,q} \left\langle \prod_{i=1}^N Z_q(n_i, t) \right\rangle$$

$$H_{N,q} = (1 - q) \left[\sum_{i=1}^N \nabla_i + (1 - q^{-1}) \sum_{i < j}^N \delta_{n_i = n_j} q^{j-i} \nabla_i \right]$$

This is regarded as a q -deformed discrete Bose gas (self-duality).

Contour integral formula

For the step initial condition $x_i(t = 0) = -i$, the solution can be written as a contour integral.

Prop. For any $N \geq 1$ and $n_1 \geq n_2 \geq \cdots \geq n_N > 0$,

$$\left\langle \prod_{i=1}^N q^{x_{n_i}(t) + n_i} \right\rangle = \frac{(-1)^N q^{\frac{N(N-1)}{2}}}{(2\pi\iota)^N} \int \cdots \int \prod_{1 \leq A < B \leq N} \frac{z_A - z_B}{z_A - qz_B} \\ \times \prod_{j=1}^N (1 - z_j)^{-n_j} e^{(q-1)tz_j} \frac{dz_j}{z_j},$$

where the integration contour for z_A contains $\{qz_B\}_{B>A}$ and 1 but not 0.

Setting $n_i = n, \forall i$ gives a formula for the N th moment.

Thm. We can rewrite the generating function of the moment in the form of Fredholm determinant

$$\sum_{N=0}^{\infty} \frac{\zeta^N \langle q^{Nx_n(t)} \rangle}{(q; q)_N} = \det(I + K_\zeta)$$

where the kernel is

$$K_\zeta(w, w') = \frac{1}{2\pi i} \int_{C_{1,2,\dots}} \Gamma(-s) \Gamma(1+s) (-(1-q)\zeta)^s \\ \times \frac{g(w)}{g(q^s w)} \frac{1}{q^s w - w'} ds.$$

$$g(w) = \frac{e^{-ntw}}{(w; q)_\infty^n}$$

$$(a; q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1})$$

There is a similar formula for ASEP as well.

Lemma For all functions f which satisfy the conditions below, we have the identity that for $\zeta \in \{\zeta : |\zeta| < 1, \zeta \notin \mathbb{R}_+\}$:

$$\sum_{n=1}^{\infty} f(q^n) \zeta^n = \frac{1}{2\pi\iota} \int_{C_{1,2,\dots}} \Gamma(-s) \Gamma(1+s) (-\zeta)^s f(q^s) ds,$$

where the infinite contour $C_{1,2,\dots}$ is a negatively oriented contour which encloses $1, 2, \dots$ and no poles of $f(q^s)$ (e.g.

$C_{1,2,\dots} = \frac{1}{2} + \iota\mathbb{R}$ oriented from $\frac{1}{2} - \iota\infty$ to $\frac{1}{2} + \iota\infty$), and z^s is defined with respect to a branch cut along $z \in \mathbb{R}^-$,

One can invert the formula to recover the probability distribution for the height. In a certain limit, the q -TASEP goes to the δ -Bose gas. The above computations are free of nasty divergence and is regarded as a rigorous version of the replica computations.