

Formal perturbation analysis of a non equilibrium stationary state

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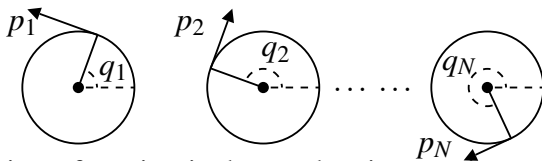
Non-equilibrium: statistics is often shown to exist:
“compactness methods”: very unsatisfactory.

Dissatisfaction: when needed: “no answers”.

E.g. for hard spheres systems regularity of correlation of a stationary nonequilibrium distribution is not known $\Rightarrow$ no answer to simple questions like $\langle \partial_q \rho(q - q') \rangle$.

Unfortunately very few exact, most involve stochastic forces.

Even so the study is very difficult. Here the question of “computing” correlations for a “trivial” system. The $N = 1$ case



The equation of motion is the stochastic equation

$$\dot{q} = \frac{p}{J} \quad \dot{p} = -\partial U - \tau - \frac{\xi}{J}p + \sqrt{\frac{2\xi}{J}}\dot{w}$$

$\dot{w}$ w.n. of width $(\frac{J}{\beta_0 dt})^{\frac{1}{2}}$,

$U = -gV \cos(q)$ conserv. force, τ torque, ξ friction, J inertia

β_0 noise inverse temperature

Problem: find the stationary state distribution for $N = 1$

It is hard to believe that this is not known ???.

$F(p, q)dpdq$: the stochastic equation yields the PDE $\mathcal{L}^*F = 0$

$$\mathcal{L}^*F = - \left\{ \left(\frac{p}{J} \partial_q F(q, p) - (\partial_q U(q) + \tau) \partial_p F(q, p) \right) \right. \\ \left. - \xi \left(\beta_0^{-1} \partial_p^2 F(q, p) + \frac{1}{J} \partial_p (p F(q, p)) \right) \right\}$$

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Gen. results: most interesting are simulations, ILOS1: [1]

but “no” theory but

(1) There **exists** a **smooth** solution (Hormander)

(2) it is **positive** $F(q, p) = \frac{e^{-\frac{\beta}{2J}p^2} \rho(p, q)}{\sqrt{2J\beta^{-1}}} = G(p) \rho(p, q)$ (MS)

(3) is **exponentially** approached by initial δ (Mattingly-Stuart)

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Guess if : $p^n := \left(\frac{J\beta^{-1}}{2} \right)^{\frac{n}{2}} H_n \left(\frac{p}{\sqrt{2J\beta^{-1}}} \right)$ (Wick, Hermite poly.)

$\rho(p, q) = G(p) (\sum_{n=0}^{\infty} \rho_n(q) : p^n :)$, **Problem:** Find it

By algebra $\mathcal{L}^*F = 0$ becomes: $n \geq 0$ ($\rho_{<0} \equiv 0$)

$$n\beta^{-1}\partial\rho_n(q) + \left[\frac{1}{J}\partial\rho_{n-2}(q) + \frac{\beta}{J}(\partial U(q) + \tau)\rho_{n-2}(q) + (n-1)\frac{\xi}{J}\rho_{n-1}(q) \right] = 0$$

Go dimensionless: $\sigma_n(q) \stackrel{\text{def}}{=} \rho_n(q)\xi^n n!, \eta \stackrel{\text{def}}{=} \beta\xi^2/J, \beta\tau, \quad \beta V$

(1) Recursion is **linear**: take the F.T. $\sigma_{n,k}$

(2) Recursion is second degree: take $\mathbf{Z}_{n,k} \stackrel{\text{def}}{=} \begin{pmatrix} \sigma_{n,k} \\ \sigma_{n-1,k}(q) \end{pmatrix}$

$$\begin{aligned} \mathbf{Z}_{n,k}^{[r]} &= M_{n,k} \mathbf{Z}_{n-1,k}^{[r]} + \mathbf{X}_{n,k}^{[r]}, & \mathbf{Z}_{2,k}^{[r]} &= \mathbf{Y}_k^{[r]} \\ u_n^{[r]} &= -\beta\tau u_{n-1}^{[r]} + v_n^{[r]}, & u_2^{[r]} &= u^{[r]} \end{aligned}$$

Conclusion: $\sigma_{0,k}^{[r']}, r' < r$ and $\sigma_{0,k}^{[r]}$ determines everything at given order: (at order r it must be $|k| \leq r$.)

Conclusion: So infinitely many solutions!

However $\rho(q,p)$ must be $L_2(G)$ (Olla)

Requirement (at least): $|\sigma_n(q)| \leq \frac{c^n}{\sqrt{n!}}$. **Formal solution**

$$\mathbf{Z}_n = \lambda M_{n+1}^{-1} \dots M_N^{-1} \dots + \sum_{h=n+1}^{\infty} M_{n+1}^{-1} \dots M_h^{-1} \mathbf{X}_h$$

Infinite matrix product: for η^{-1} small enough, let $|2\rangle \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Key: *Given k , if η is large enough there is a sequence $\Lambda_k(n_1, N)$ such that for all $n_1 \geq 2$ and all $k \neq 0$ the*

$$\zeta_k(n_1, N) \stackrel{\text{def}}{=} \begin{pmatrix} \zeta_k(n_1, N)_1 \\ 1 \end{pmatrix} = \Lambda_k(n_1, N) M_{n_1+1}^{-1} \dots M_N^{-1} |2\rangle$$

is s.t. $|\zeta_k(n_1, N)_1| \leq |k|$ and $\zeta_k(n_1) \stackrel{\text{def}}{=} \lim_{N \rightarrow \infty} \zeta_k(n_1, N)$ exists.

Once more this is a problem on a **Ising system** controlled by γ

$$\gamma = -\frac{4ia_k k^2}{m\eta}, \quad \lambda_{m,k,\sigma} \stackrel{\text{def}}{=} -\frac{1 + \sigma \sqrt{1 + \gamma}}{2ka_k}, \quad \sigma = \pm 1$$

$a_k \stackrel{\text{def}}{=} i(1 - \frac{i\beta\tau}{k})$ and spectral decomposition of M_n^{-1} :

$$M_{n,k}^{-1} = \sum_{\sigma=\pm 1} \lambda_{n,k,\sigma} \frac{|v_{n,\sigma}\rangle \langle w_{n,\sigma}|}{\langle w_{n,\sigma}|v_{n,\sigma}\rangle},$$

$$|v_{n,\sigma}\rangle = \begin{pmatrix} \lambda_{n,\sigma}^{-1} \\ 1 \end{pmatrix},$$

$$\langle w_{n,\sigma}| = \left(\varepsilon_n, \lambda_{n,\sigma} \right), \quad \varepsilon_n = \frac{\gamma}{4k^2 a_k^2}$$

The Ising model arises because $M_n^{-1} \cdots M_N^{-1} |2\rangle$ is

writing explicitly the matrix product $M_n^{-1} \cdots M_N^{-1} |2\rangle$ as

$$\sum_{\sigma_3, \dots, \sigma_N} |v_{3, \sigma_3}\rangle \cdot \left(\prod_{j=3}^N \lambda_{j, \sigma_j} \right) \prod_{j=3}^N \frac{\langle w_{j, \sigma_j} | v_{j+1, \sigma_{j+1}} \rangle}{\langle w_{j, \sigma_j} | v_{j, \sigma_j} \rangle}$$

“expectation” of $|v_{3, \sigma_3}\rangle$ in a Gibbs state. Gibbs factor?

Spin configuration = intervals of $-$ separated $+$, then

$$\bar{\Lambda}_N \prod_{i=1}^p I_{J_i} \quad \text{is the Gibbs weight}$$

$$\rho(J) \stackrel{\text{def}}{=} \frac{\sum_{p \geq 1} \sum_{J_1 < \dots < J_p}^* \left(\prod_{s=1}^p I_{J_s} \right)}{\Omega_2(N)}$$

$$P_- \stackrel{\text{def}}{=} \sum_{\{3\} \in J} \rho(J), \quad \zeta(2) = \lambda_{3,-}^{-1} P_- + \lambda_{3,+}^{-1} P_+$$

$$\begin{aligned}
\bar{\Lambda}_N^{[2]} &\stackrel{\text{def}}{=} \left(\prod_{j=3}^N \lambda_{j,+} \right) \left(\prod_{j=3}^N \frac{\langle w_{j,+} | v_{j+1,+} \rangle}{\langle w_{j,+} | v_{j,+} \rangle} \right) \\
I_J &\stackrel{\text{def}}{=} \left(\prod_{j=k}^{k'} \frac{\lambda_{j,-}}{\lambda_{j,+}} \right) \left(\prod_{j=k}^{k'} \frac{\langle w_{j,-} | v_{j+1,-} \rangle}{\langle w_{j,-} | v_{j,-} \rangle} \frac{\langle w_{j,+} | v_{j,+} \rangle}{\langle w_{j,+} | v_{j+1,+} \rangle} \right) \\
&\quad \cdot \frac{\langle w_{k-1,+} | v_{k,-} \rangle}{\langle w_{k-1,+} | v_{k,+} \rangle} \frac{\langle w_{k',-} | v_{k'+1,+} \rangle}{\langle w_{k',+} | v_{k'+1,+} \rangle} \frac{\langle w_{k',+} | v_{k',+} \rangle}{\langle w_{k',-} | v_{k',-} \rangle} \\
|I_J| &\leq W_J \stackrel{\text{def}}{=} \left(\frac{\sqrt{2}}{\eta} \right)^{|J|} \left(\prod_{j \in J} \frac{1}{j-1} \right) \frac{2k^4}{\eta^2 (k'-1)^3}
\end{aligned}$$

Non transl. inv. but very small already for J close to the origin.

Hence the $P_{\pm}$ can be computed as convergent series in I_J 's as well as the partition function $\exp(\Omega_N)$ which admits a limit as $N \rightarrow \infty$ (no logarithm necessary).

Fixed R then if viscosity ξ is large enough and $0 \leq r \leq R$

Theorem: The equation $\mathcal{L}^*F(p, q) = 0$

(1) admits (at most) a 1-que C^r (in g) solution nonnegative and in $L_1(dp dq)$ for g small.

(2) the coefficients $F^{[r]}(p, q)$ of its Taylor expansion at $g = 0$ are uniquely determined, analytic in q, p ,

(3) are explicitly computable.

Conjecture: The coefficients exist for all r

Question: is F analytic in g because $\exists A, C, c$ s.t.

$$|\rho_{n,k}^{[r]}| \leq A^r C^n e^{-c|k|} \frac{1}{n!} \quad ? \quad C = \overline{C}|k|?$$

The idea should be applicable to the evaluation of the Lyapunov exponents of infinite products of matrices close to a hyperbolic matrix.

In particular could yield analyticity of the leading exponent in terms of parameters defining the matrices (a very special case of a result by Ruelle).



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Negative thermal conductivity of chains of rotors with mechanical forcing.

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