New limiting theorems for the Möbius function

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The C.-S. probabilistic model

Fix an integer N and let $\Omega(N)$ be the ensemble whose elements are $\xi = \{q_1, \ldots, q_s\}$ with $1 < q_1 < \ldots < q_s < N$, q_j are prime. Define a measure $\widetilde{u_N}$ so that:

$$\widetilde{u_N}(\xi) = \frac{1}{q_1 \ldots q_s}$$

If Z_N – total mass of this measure, we can define a probability measure u_N on $\Omega(N)$:

$$u_N(\xi) = \frac{1}{Z_N}\widetilde{u_N}(\xi)$$

From Mertens product formula it follows that

$$Z_N = \sum_{Q \in \mathcal{Q}:\, q_i \le N} \frac{1}{Q} = \prod_{q \le N} (1 + \frac{1}{q}) = \frac{e^\gamma}{\zeta(2)} \ln N + o(\ln N)$$

as $N \to \infty$.

The C.-S. probabilistic model (continued)

For any random variable $f : \Omega(N) \to \mathbb{R}$

$$\mathbb{E}_N f = \sum_{\xi \in \Omega(N)} f(\xi) u_N(\xi)$$

Characteristic function: $\varphi_f(\lambda) = \mathbb{E}_N e^{i\lambda f}$.
Introduce a random variable:

$$\eta_N(\xi) = \begin{cases} \sum_{j=1}^{s} \frac{\ln q_j}{\ln N}, & \xi = \{q_1, \dots, q_s\}; \\ 0, & \text{otherwise} \end{cases}$$

Reformulating, $\eta_N(Q) = \frac{\ln Q}{\ln N}$ for $Q \in \mathcal{Q}$ and 0 for not square-free Q.

Let $\widetilde{\mathbb{P}_N}$ and $\mathbb{P}_N$ be the distributions on $\mathbb{R}$ corresponding to η_N on $(\Omega(N), \widetilde{u_N})$ and $(\Omega(N), u_N)$ respectively.

Theorem (Cellarosi-Sinai '13)

As $N \to \infty$ the probability distribution $\mathbb{P}_N$ converges weakly to a probability distribution $\mathbb{P}$ whose characteristic function $\varphi(\lambda)$ has the form

$$\varphi(\lambda) = \exp\left\{ \int_0^1 \frac{e^{i\lambda v} - 1}{v} dv \right\}.$$

This distribution is known as the Dickman-de Bruijn distribution and first appeared in the theory of smooth numbers (see, e.g., K. Dickman "On the frequency of numbers containing primes of a certain relative magnitude", 1930)

Implications

Remark that

$$\frac{1}{\ln N}\sum_{n\le N}\frac{\mu^2(n)}{n}=\frac{1}{\ln N}\sum_{Q\le N}\frac{1^{\omega(Q)}}{Q}=\frac{1}{\ln N}\sum_{\substack{0\le\frac{\ln Q}{\ln N}\le 1,\\ q_i\le N}}\frac{1^{\omega(Q)}}{Q}$$

$$=\frac{1}{\ln N}\widetilde{\mathbb{P}_N}[0,1]\xrightarrow{Thm(CS)}\frac{e^{\gamma}}{\zeta(2)}\cdot\mathbb{P}[0,1]=\frac{1}{\zeta(2)}$$

Using Thm(CS) one can study the distribution of square-free numbers on segments of the form $[(\ln N)^{c_1},(\ln N)^{c_2}]$.

Properties of D-B distribution

- The density of the Dickman-de-Bruijn distribution is given by $e^{-\gamma}\rho(t)$ where $\rho(t)$ satisfies the integral equation

$$t\rho(t) = \int_{t-1}^{t} \rho(s)ds, \quad t \in \mathbb{R}$$

with the initial condition

$$\rho(t) = \begin{cases} 0, & t \leq 0; \\ 1, & 0 < t \leq 1. \end{cases}$$

- Define η_k: $\mathbb{P}\{\eta_k = k\} = \frac{1}{k}$ and $\mathbb{P}\{\eta_k = 0\} = 1 - \frac{1}{k}$. Then:

$$\lim_{n\to\infty} \mathbb{P}\left\{ n^{-1}\sum_{j=1}^{n} \eta_j < x \right\} = e^{-\gamma}\int_0^x \rho(t)dt.$$

Density of D-B distribution

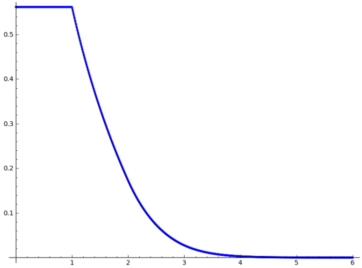
Scaling argument:
We know that

$$\widetilde{\mathbb{P}_N}[0,1] = \sum_{0 \le \frac{\log Q}{\log N} \le 1} \frac{1^{\omega(Q)}}{Q} \sim \log N$$

as $N \to \infty$ hence

$$\widetilde{\mathbb{P}_N}[0,x] = \sum_{0 \le \frac{\log Q}{\log N} \le x} \frac{1^{\omega(Q)}}{Q}$$

$$= \sum_{Q \le N^x} \frac{1^{\omega(Q)}}{Q} \sim x \log N.$$



$y = e^{-\gamma}\rho(x)$

The (new) generalized probabilistic model

For our purposes we have to consider a generalized version of C-S model (old model: $t = 1$).
Fix an integer N and let $\Omega^t(N)$ be the ensemble whose elements are $\xi = \{q_1, \ldots, q_s\}$ with $1 < q_1 < \ldots < q_s < N$, q_j are prime. For any $t \in \mathbb{R}$ define a measure $\widetilde{u_N^t}$ so that:

$$\widetilde{u_N^t}(\xi) = \frac{t^s}{q_1 \ldots q_s}$$

If Z_N^t – total mass of this measure, the normalized distribution u_N^t on $\Omega^t(N)$ has the form:

$$u_N^t(\xi) = \frac{1}{Z_N^t} \widetilde{u_N^t}(\xi)$$

Connections to the Mertens function

Recall Mertens function: $M(N) = \sum_{n \le N} \mu(n)$.
Our analysis is based on the following Euler-type product for $\mu(n)$:

$$\mu(n) = \prod_{\substack{p \ge 2, \\ p \text{ is prime}}} (1 - 2\chi_p(n) + \chi_{p^2}(n))$$

Then:

$$\begin{aligned}\mu(n) &= \sum_{\substack{P,Q \\ P \cap Q = \emptyset}} (-2)^{\omega(Q)} \chi_{P^2 Q}(n) \\ &= 1 + \sum_{Q \neq \emptyset} (-2)^{\omega(Q)} \chi_Q(n) + \sum_{\substack{P \neq \emptyset, \\ P \cap Q \neq \emptyset}} (-2)^{\omega(Q)} \chi_{P^2 Q}(n)\end{aligned}$$

Here P, $Q \in \mathcal{Q}$.

Connections to the Mertens function (continued)

The decomposition for $\mu(n)$ yields the following expression for the Mertens function $M(N) = \sum_{n \le N} \mu(n)$:

$$\frac{M(N)}{N} = \frac{1}{N} \sum_{n \le N} \mu(n) = 1 + \frac{1}{N} \sum_{n \le N} \sum_{Q \ne \emptyset} (-2)^{\omega(Q)} \chi_Q(n) + \text{ terms inv. } P$$

$$= 1 + \frac{1}{N} \sum_{\substack{Q \ne \emptyset, \\ Q \le N}} (-2)^{\omega(Q)} \left[\frac{N}{Q} \right] + \text{ terms involving } P$$

$$= 1 + \sum_{\substack{Q \ne \emptyset, \\ Q \le N}} \frac{(-2)^{\omega(Q)}}{Q} + \text{ other terms}$$

This is where the series $\displaystyle\sum_{\substack{Q \ne \emptyset, \\ Q \le N}} \frac{t^{\omega(Q)}}{Q}$ appears for the special value $t = -2$.

The generalized probabilistic model (continued)

The distribution on square-free numbers: $u_N^t(Q) = \frac{1}{Z_N^t}\frac{t^{\omega(Q)}}{Q}$.

Properties of u_N^t:

- Probability measure for $t > 0$, signed measure with singularities for $t < 0$.
- The partition function Z_N^t is given by

$$\sum_{Q\in\mathcal{Q}:\, q_i\leq N} \frac{t^{\omega(Q)}}{Q} = \prod_{\text{prime } q\leq N} (1+\frac{t}{q}) \sim (\ln N)^t$$

  as $N \to \infty$.

N.B.: for integer $t \leq 0$, Z_N^t might hit zero. In this case we neglect some subset of primes: e.g., for $t = -2$ consider only odd square-free integers.

The generalized probabilistic model (continued)

Recall, $\eta_N(Q) = \frac{\ln Q}{\ln N}$ for $Q \in \mathcal{Q}$ and 0 for non-squarefree Q.

Let μ_N^t be the distribution on $\mathbb{R}$ induced by η_N on $(\Omega^t(N), u_N^t)$.

Note, for $x \geq 0$,

$$\mu_N^t[0,x] = \frac{1}{Z_N^t} \sum_{\substack{q_i \leq N, \\ 0 \leq \frac{\ln Q}{\ln N} \leq x}} \frac{t^{\omega(Q)}}{Q} = \frac{1}{Z_N^t} \sum_{\substack{q_i \leq N, \\ Q \leq N^x}} \frac{t^{\omega(Q)}}{Q}$$

We will be particularly interested in the latter expression for negative integer t's and its asymptotics in the limit $N \to \infty$.

Main problem

Let $t < 0$. For the sake of simplicity we will mostly consider $t = -2$.
Questions.

- Do $\{\mu_N^t\}$ converge as $N \to \infty$?
- If they do, what is the limiting distribution?

Obstacle: Since $Z_N^t \to 0$ good control of the error terms is required.
Preliminary answer. Formal computation using characteristic functions suggests that the limiting distribution is the "generalized Dickman-de Bruijn" distribution, i.e.

$$\varphi^{(t)}(\lambda) = \exp\left\{ t \int_0^1 \frac{e^{i\lambda v} - 1}{v} dv \right\}.$$

Obstacle: this is a tempered distribution; $\varphi^{(-2)}(\lambda) \sim |\lambda|^2$ as $\lambda \to \infty$.

Smooth sums over k-free integers

In his recent preprint *"Smooth sums over k-free integers and statistical mechanics"*, F. Cellarosi studied a modification of our problem.
Define

$$S_{\Omega,f}(k,z;N) = \sum_{\substack{n \text{ is } k-free,\\ p_i \le N}} f\left(\frac{\log n}{\log N}\right) \frac{z^{\Omega(n)}}{n};$$

$$S_\eta(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} : \exists C > 0 \text{ s.t. } \left|\widehat{f}(\lambda)\right| \le \frac{C}{1+|\lambda|^\eta} \forall \lambda \in \mathbb{R} \right\}.$$

E.g., $S \subset S_\eta$ for every η.

Note that $\mu_N^t[0,x]$ is closely connected to $S_{\Omega,\chi_{[0,x]}}(2,t;N)$.

Smooth sums over k-free integers (continued)

Theorem (Cellarosi)

Let $f \in S_\eta$ with $\eta > |z| - \Re z + 1$. Then there exists a non-zero constant $C = C(k,z) \in \mathbb{C}$ such that, for every $R = R(N)$ satisfying

$$\frac{R}{\log N} \to 0 \text{ and } \frac{(\log N)^{|z| - \Re z}}{R^{\eta - 1}} \to 0 \text{ as } N \to \infty,$$

we have

$$S_{\Omega,f}(k,z;N) = C \cdot (\log N)^z \left(\int_{|\lambda| \le R} \varphi^{(z)}(\lambda) \widehat{f}(\lambda) d\lambda + \varepsilon_N \right),$$

where $\varphi^{(z)}(\lambda)$ is a z-convolution of D-B distribution on $\mathbb{R}_{\ge 0}$ and $f \in S_\eta(\mathbb{R})$.

Convergence of μ_N^t on finite intervals

Theorem

Consider the sum $S_N(x) = \sum_{\substack{n<N^x, \\ p|n \Rightarrow p \le N}} \frac{(-2)^{\omega(n)}}{n}\mu^2(n)$. *Then, for* $N \to \infty$,

$$S_N(x) = \begin{cases} O\left(e^{-c\sqrt{x \log N}}\right), & \text{for } 0 < x \le 1, \\ \frac{1}{(\log N)^2}\left(G(x) + o_N(1)\right), & \text{for } x > 1. \end{cases}$$

Here $c > 0$ *is some sufficiently small constant and* $G(x)$ *is a function with singularities at* $x = 1, 2$.

Structure theorem for generalized DB distribution ($t \in \mathbb{Z}^-$)

Let $\rho^t_{DB}(x)$ be the density of μ^t_{DB}, i.e.

$$\rho^t_{DB}(x) = \frac{1}{2\pi} \lim_{R\to\infty} \int_{-R}^{R} \exp\left(t \int_0^1 \left(e^{i\lambda s} - 1\right) \frac{ds}{s} \right) e^{-i\lambda x} d\lambda.$$

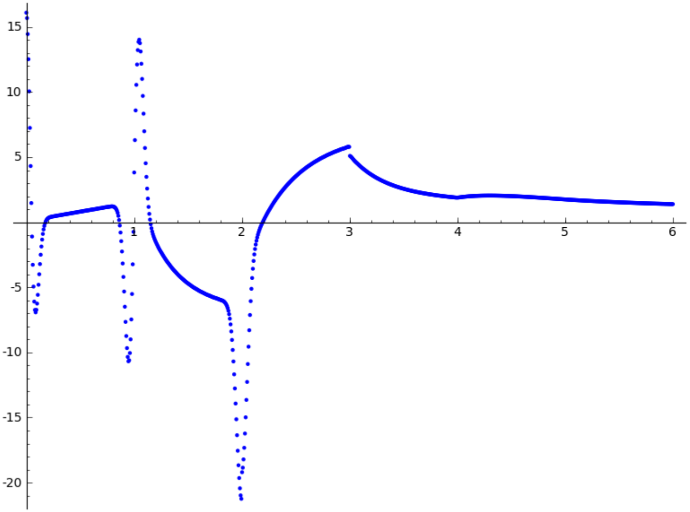
Theorem

The density ρ^t_{DB} is a distribution on $\mathbb{R}$ which admits the following representation: $\rho^t_{DB} = \rho^t_c + \psi^t$, where $\rho^t_c \in C(\mathbb{R})$ and

$$\psi^t(x) = \sum_{l=0}^{|t|+1} \sum_{k=0}^{l} C_{k,l}(t)(sgn(x-k))^{(|t|+1-l)}$$

Here $sgn(x)$ is the usual sign function and $C_{k,l}(t)$ are independent of x.

Distribution for $t = -2$



$$\rho_{DB}^{(-2)}(x) = -\delta''(x) - \delta'(x-1) + \delta(x-1) + \mathrm{sgn}(x-1) - \delta(x-2) + \mathrm{sgn}(x-2) - \frac{1}{3}\mathrm{sgn}(x-3) + \rho_c^{(-2)}(x)$$

What is $G(x)$?

In the Theorem

$$G(x) = G_1(x) - H(x-1) + H(x-2) + G_2(x)$$

where

$$H(x) = \begin{cases} 1, & \text{for } x > 0; \\ 0, & \text{for } x \leq 0; \end{cases}$$

$$G_1(x) = \frac{a_1}{2\pi i} \cdot \int_\gamma e^{-2\int_0^{-s} \frac{e^t - 1}{t} dt + xs} \frac{ds}{s} + c.c;$$

$$G_2(x) = \int_1^\infty g_2(\lambda) e^{-i\lambda x} d\lambda + c.c.$$

Here γ is the right half unit circle on $\mathbb{C}$ and $g_2(\lambda) = O\left(\lambda^{-2}\right)$.

Dickman-de Bruijn in disguise

$\mu_N^t\,[0,x] \to \mu_\infty^t\,[0,x]$

- One can show

$$\rho_\infty^{(-2)} = \delta(x-1)+\mathrm{sgn}(x-1)-\delta(x-2)+\mathrm{sgn}(x-2)-\frac{1}{3}\mathrm{sgn}(x-3)+\rho_c^{(-2)}$$

- In comparison,

$$\rho_{DB}^{(-2)} = -\delta''(x) - \delta'(x-1)+$$
$$\delta(x-1)+\mathrm{sgn}(x-1)-\delta(x-2)+\mathrm{sgn}(x-2)-\frac{1}{3}\mathrm{sgn}(x-3)+\rho_c^{(-2)}$$

- Unlike $f = \chi_{[0,x]}$, a smooth test function f satisfies $\mathbb{E}_{\mu_N^t} f \to \mathbb{E}_{\mu_{DB}^t} f$, i.e. picks up $-\delta''(x) - \delta'(x-1)$ singularities.

Thank you!