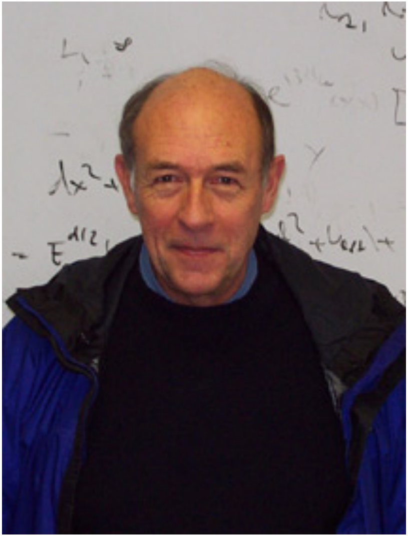
Happy Birthdays!



Fourier law from Hamiltonian dynamics

Domokos Szász
(Budapest University of Technology)
joint w. P. Bálint, IP. Tóth (both BUTE), Th. Gilbert (ULB),
P. Nándori (NYU & Maryland),

Rutgers, December 2015

Derivation of Fourier law

Favourite models for *Microscopic* $\Rightarrow$ *Macroscopic*:

- Oscillators
  ONLY for stochastic models OR models with small noise
  cf. **Basile-Olla, '14**
- Billiard type models
  Proper Hamiltonian dynamics!

Bonetto–Lebowitz–Rey-Bellet, '00
"The absence of interactions between [independent Lorentz] particles makes this system a poor model for heat conduction. In particular there is NO MECHANISM TO ACHIEVEING LTE. It would be necessary to ADD INTERACTIONS between the moving particles, e.g. instead of points make them little balls . . . ."

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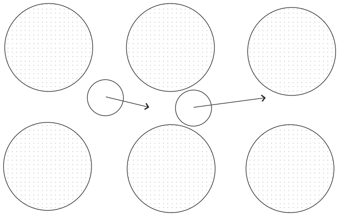
Gaspard-Gilbert, 2008–: model of localized hard disks (balls)



on figure: chain length $N = 2$

periodic scatterers (shaded disks)

confined moving disks (white circles)

NO mass transport

Ergodicity by Bunimovich-Liverani-Pellegrinotti-Sukhov '92

Parameter choice by Gaspard-Gilbert

A two step approach:

1. From the microscopic kinetic equ. of the Hamiltonian model

   derive a mesoscopic master equ. for the energies in the **rare (but strong) interaction limit;**

   It is expected to be a **Markov jump process.**

2. Then from the mesoscopic master equ.

   derive the macroscopic heat equ. for the dynamics of temperature

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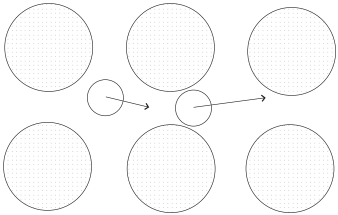

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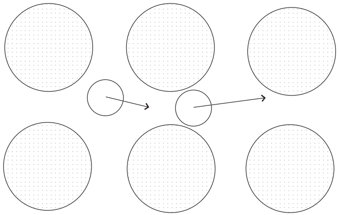

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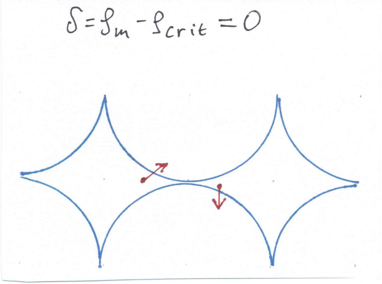
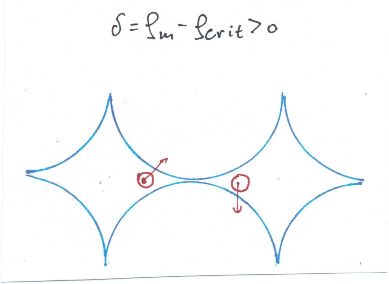




$\rho_m, \quad \rho_f$ radii of moving disks vs. scatterers

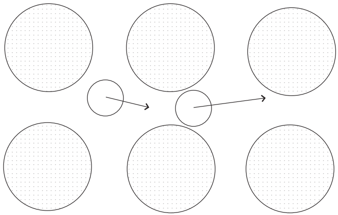
$\rho_m + \rho_f = \text{const.} > 1/2$

$\delta = \rho_m - \rho_{\text{crit}} = 0$

independent billiards

$1 \gg \delta = \rho_m - \rho_{\text{crit}} > 0$

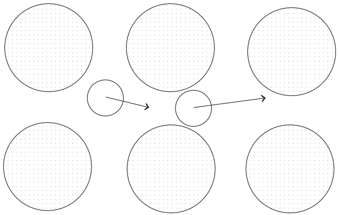
rarely interacting billiards



$N = 2, D = 2$
a 4-dim semi-dispersing billiard
$Q = \{(q_1, q_2) | q_1, q_2 \in \mathbb{R}^2$
$\& \text{ boundary cond.}\}$

!! For billiards of dimension ≥ 3 stochastic properties are only known under *complexity hypothesis* (cf. **P. Bálint-IP. Tóth, '08**).

Sz.-IP. Tóth, '09: *standard pairs* of Chernov-Dolgopyat
(= a sophisticated, efficient variant of earlier Markov tools.)
Relies on the fact that *between binary collisions of the two disks there are long time intervals* and thus there is *averaging* for the motion of each disk in its cell.
Still technical problems!



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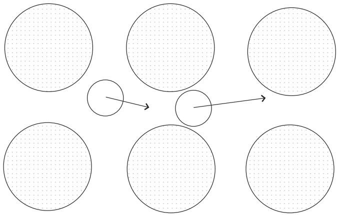

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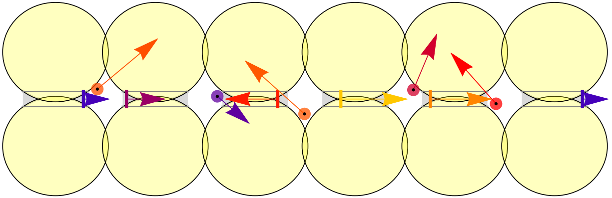
Disk-piston model



The arrows' lengths and colours reflect the magnitude of kinetic energy of the corresponding disk or piston (blends of blue for low energy values, red and yellow for high energy values).

Ergodic (à la Krámli-Simányi-Sz., '91 or Bunimovich-Liverani-Pellegrinotti-Sukhov, '92), however no statistical properties are known.

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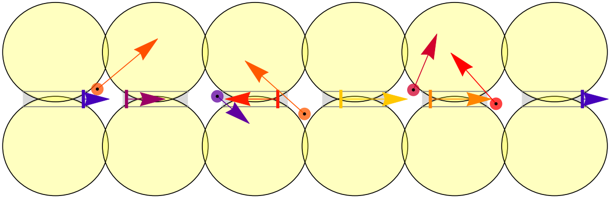

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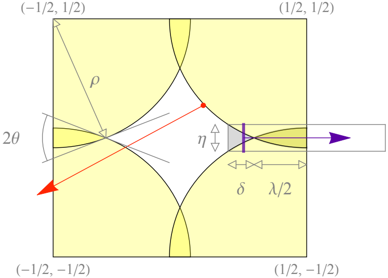
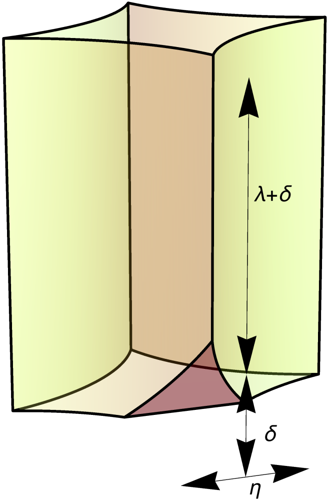
Minimal version: one disk and one piston particle



NB: blank diamond =
config. space of center of disk
Rare interaction limit ∼
δ = penetration length $\searrow 0$,
$\lambda + 2\delta$ = piston length



isomorphic 3-dim.
semi-dispersing billiard

$\mathcal{M}^\delta = \{(q_x, q_y, v_x, v_y; z, v) \in \mathcal{M} \mid q_x + \delta \le z\}$
$\Phi^{\delta,t} : \mathcal{M} \to \mathcal{M}$ – flow describing the mechanical process

$E_L = v_x^2 + v_y^2 : \mathcal{M} \to \mathbb{R}_+$ – energy of left (disk) particle
$\tilde{X}^\delta(t) = \tilde{X}^\delta(t, w) = E_L(\Phi^{\delta,t}(w))$ – left particle energy process
initial state $w \in M$
$X^\delta(t) = X^\delta(t, w) = \tilde{X}^\delta(\frac{t}{\delta^2}, w)$ – properly rescaled energy process

Theorem (Bálint-Nándori-Sz.-IP. Tóth, in progress)

Assume ν is the initial law for the energies of the billiard particle for both processes. For a wide class of ν, as $\delta \to 0$,

$$X^\delta(t) \overset{D[0,\infty)}{\Longrightarrow} X(t)$$

$X(t)$ – Markov jump process w. kernel $K(E_L, E_L^+)$ (and rate $\Lambda(E_L)$)

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Calculation yields: (for brevity $E = E_L, E^+ = E_L^+$)
Transition kernel:

$$K(E, E^+) = \frac{\tan\beta}{2\pi L|Q|} \frac{\sqrt{1 - \min\{E, E^+\}}}{\sqrt{1 - E^+}\sqrt{E + E^+ - 1}} \mathbb{1}_{\{E+E^+>1\}}$$

Transition rate:

$$\Lambda(E) = \begin{cases} \frac{\tan\beta}{2L|Q|}\sqrt{1-E} & \text{if } E < 1/2 \\ \frac{\tan\beta}{\pi L|Q|}\left[\sqrt{2E-1} + \frac{\sqrt{1-E}}{2}\left(\frac{\pi}{2} - \arcsin\left(3 - \frac{2}{E}\right)\right)\right] & \text{if } E > 1/2 \end{cases}$$

$\beta = 1/2$ diamond's angle

B-**Gilbert**-*N-Sz.-T, '15:*

1. Assertive statistical test whether the rare interaction limit is a Markov jump process of energies.
2. Transition kernel of limiting Markov jump process
3. Mean free times (MFT), coll. frequencies (CF) and rates
4. Conditional MFT, CF, rates, too.

B-N-Sz-T, '15: Chernov-Dolgopyat standard pair method permits to rely solely on statistical properties of the projected 2-dim. billiard (in this case of a finite horizon Sinai-billiard). Some arising problems:

1. need control of correlation decay for the flow
2. these properties should be strong: *for an initial law prescribed by a standard pair, i. e. by a singular measure*

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Chernov, '07: *Stretched exp. corr. decay for billiard flows*

For a bounded $f : D \to \mathbb{R}$ let

$$(osc_r f)(x) := \sup_{y \in B_r(x)} f(y) - \inf_{y \in B_r(x)} f(y).$$

and for $0 < \alpha \leq 1$ let

$$||f||_\alpha := \sup_{r>0} \frac{1}{r^\alpha} \int_D (osc_r f)(x) \mathrm{d}\, x.$$

$$var_\alpha(f) := ||f||_\alpha + \sup_M f - \inf_M f$$

If $||f||_\alpha < \infty$, then f is generalized Hölder continuous.

Theorem (Chernov, '07)

Planar finite horizon Sinai billiard, no corner points. Let $F, G : M \to \mathbb{R}$ generalized Hölder continuous with $\int_M F d\mu = 0$. Then

$$\left| \int_M (F \circ \Phi^t) G d\mu \right| \leq c\, var_\alpha(F) var_\alpha(G) e^{-a\sqrt{|t|}},$$

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(W^u, φ) *is a standard pair and let* $\tilde{\varphi}$ *be the measure on* M *given by it.*
Let $F : M \to \mathbb{R}$ *generalized Hölder continuous with* $\int_M F d\mu = 0$.
Then

$$\left| \int_{W^u} F \circ \Phi^t d\tilde{\varphi} \right| \leq C_\varphi C_F e^{-\alpha\sqrt{t}},$$

with

- C_φ, C_F *explicitly given in terms of the regularity constants of* φ *and global constants depending only on the billiard table,*
- α *depending only on the dynamical Hölder exponent of* φ, *the generalized Hölder exponent of* F *and global constants depending only on the billiard table,.*

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Oscillation bound

For $0 < \alpha \leq 1$ the generalized α-Hölder norm is

$$||f||_\alpha := \sup_{r>0} \frac{1}{r^\alpha} \int_D (osc_r f)(x) \mathrm{d}\, x.$$

Theorem

For any convex $D \subset \mathbb{R}^d$, any bounded $f : D \to \mathbb{R}$, any $r > 0$ and any $0 < \alpha \leq 1$

$$||osc_r f||_\alpha \leq 2(\sup_D f - \inf_D f) Leb(D) \left(\frac{2d+1}{r}\right)^\alpha .$$

Approach map

Let $\emptyset \neq H \subset \mathbb{R}^d, 0 < \Delta \in \mathbb{R}$. For any $x \in \mathbb{R}^d$ let $\pi(x)$ be the point in $\bar{H}$ which is closest to x. (If there is more than one such y, then let $\pi(x)$ be any of them. So $d(x, \pi(x)) = d(x, H)$.) The *approach map* $T_\Delta : \mathbb{R}^d \to \mathbb{R}^d$ is

$$T_\Delta x := \begin{cases} x + \Delta \frac{\pi(x) - x}{|\pi(x) - x|} & \text{if } d(x, H) > \Delta \\ \pi(x) & \text{if } d(x, H) \leq \Delta. \end{cases}$$

Proposition

If $\emptyset \neq H \subset \mathbb{R}^d$, $A \subset \mathbb{R}^d$ is Lebesgue measurable and $d(H, A) \geq R \geq \Delta \geq 0$, then

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Conclusions

1. Introduced a model, where rare interaction limit is mathematically feasible
2. Proofs: Rare interaction limit: work in progress
3. Task 1: Treatment of N-chain is simple
4. Task 2: Correlation decay rate and asymptotic equilibrium for planar billiards with corner points
5. Task 3: Spectral gap for ball-piston model
6. Task 4: HDL is hard
   Dolgopyat-Liverani, '11: Limit of *weakly coupled* Anosov systems (here HDL is problematic, unlike in GG model, cf. Grigo-Khanin-Sz. '12, Sasada, '15)
7. Task 5: Treatment of original GG model