Problems in quantum kinetic theory

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Phase transitions

An example of bosons is photons of light.

- Water $\rightarrow$ cooled down $\rightarrow$ Ice
- Bosons $\rightarrow$ cooled down $\rightarrow$?

A Bose-Einstein condensate is a state of matter in which extremely cold atoms clump together and act as if they were a single atom. This state was first predicted, generally, in 1924-25 by Satyendra Nath Bose and Albert Einstein.

Physics History

- Bose and Einstein prediction (1924-25)
- Nordheim (1928), Uehling-Ulenbeck (1933): Quantum Boltzmann equation for dilute Bose gases
- Kirkpatrick - Dorfman kinetic model for BECs (1985), based on the rich body of research carried out in the period 1940-67 by Bogoliubov, Lee and Yang, Pitaevskii, Hugenholtz and Pines, **Beliaev, Hohenberg and Martin**, Gavoret and Nozieres, Kane and Kadanoff and many others.
- Cornell, Wieman, Ketterle (1995): first BECs in the lab $\rightarrow$ Nobel Prize in Physics 2001
- Zoller et. al. (2001): First numerical verification of the MIT experiment (using quantum Boltzmann). Introduced the terminology "Quantum kinetic theory"
- Pomeau et. al. (1999), Zaremba et. al. (1999), (Sir) Burnett et. al. (2000), Zakharov et. al. (2005), Spohn (2010): improvement of quantum kinetic theory
- Reichl et. al. (2012-2015): New missing collision operator

Physics History

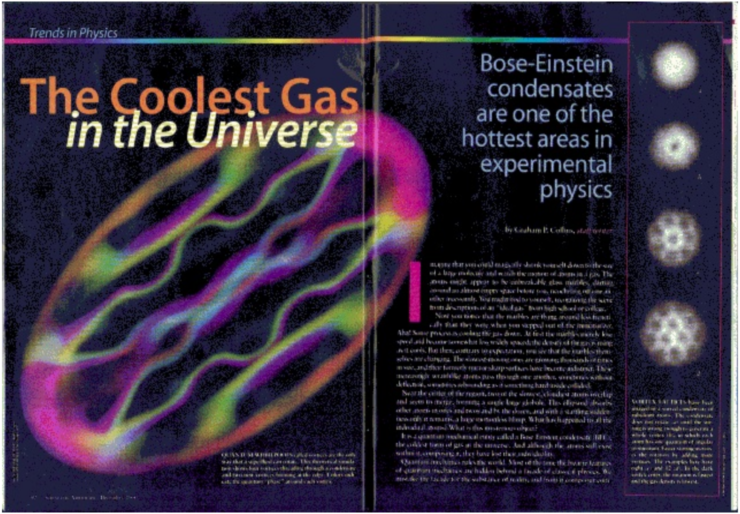


Figure : The cover of the journal Scientific American

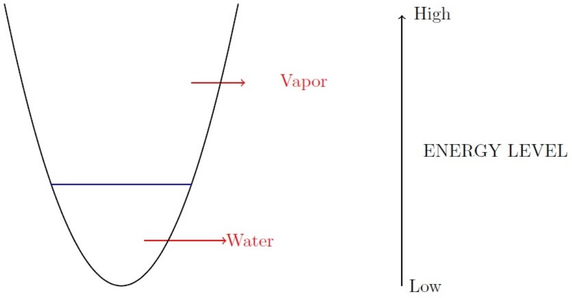
Mathematics History

- First proof of BEC: Lieb-Seiringer (2002)
- Derivation and Bogoliubov excitation spectrum: Benedetto, Castella, Esposito, Pulvirenti (2008), Erdos- Schlein - Yau (2010), Seiringer (2011), Grillakis-Margetis-Machedon (2010, 2011, 2013, 2015), Ben Arous-Kirkpatrick-Schlein (2012), Lewin - Nam - Serfaty - Solovej (2015), Chen-Guo (2015), Bach - Breteaux - Chen - Fröhlich - Sigal (2016), Deckert - Fröhlich- Pickl- Pizzo (2016)
- Simplified models Canizo - Carrillo - Laurencot - Rosado (2016), Levermore-Liu-Pego (2013-2016), Ben Abdallah - Gamba - Toscani(2011), Toscani(2012), Carrillo - Di Francesco - Toscani (2016)
- Non simplified models: Lu (2004,2005,2013), Briant Thesis (2011), Escobedo-Velazquez (2001-2016), Spohn (2010), Arkeryd-Nouri (2009-2016), Escobedo-Tran (2015), Alonso - Gamba - Tran (2016), Soffer - Tran (2016-2 papers), Nguyen-Tran (2016), Craciun-Tran (2016), Jin - Tran (2017)

Motivations of theoretical research

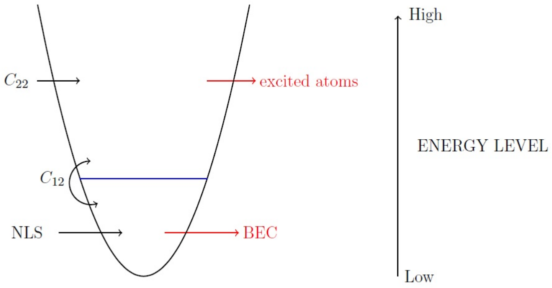
"The behaviour of condensates at finite temperatures is a frontier of many body physics, the experimental exploration of which has so far concentrated mainly on the initial formation of condensates."
"A full theoretical description must include the condensate and its elementary excitations, and the interactions with the cloud of thermal atoms (those not part of the condensate). This quantum kinetics problem has been approached from the perspective of quantum optics, which models the condensation process after lasing; while condensed matter theorists have investigated the same process in terms of symmetry breaking and phase relaxation."
(James R. Anglin - Wolfgang Ketterle, Nature, 2002)

The glass of water



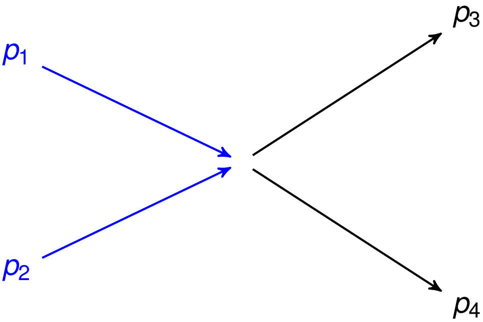
The glass is divided into two parts: water and vapor. The energy of the whole system waver-vapor is increasing from low (water) to high (vapor).

Bose-Einstein Condensate (Kirkpartrick-Dorfman (1985), Zoller et. al. (2001), Spohn (2010))



BEC occupies the lowest quantum state. Wave function Ψ:
$i\partial_t\Psi = -\Delta\Psi + |\Psi|^2\Psi +$ Coupling Terms
Excited atoms occupy higher quantum states. Density f:
$\partial_t f = C_{12}[f, |\Psi|^2] + C_{22}[f]$, where C_{22} excited atoms - excited atoms, C_{12} condensate atoms - excited atoms.

Classical Boltzmann equation

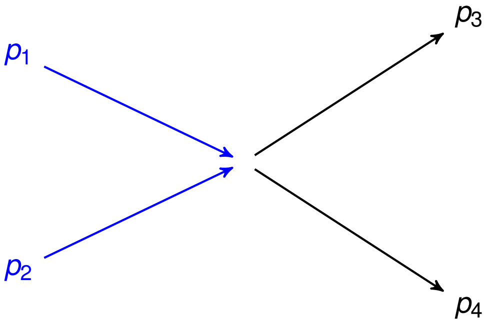


Classical Boltzmann equation

$$\partial_t f(t,p_1) = \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} [f(t,p_3)f(t,p_4) - f(t,p_1)f(t,p_2)]\times$$

$$\times\delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})\delta(p_1 + p_2 - p_3 - p_4)dp_2 dp_3 dp_4$$

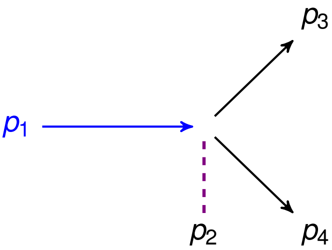
C_{22} excited atoms - excited atoms



Uehling-Ulenbeck (Boltzmann-Nordheim) collision operator C_{22}

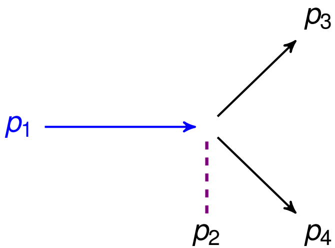
$$C_{22}[f](p_1) = \int_{\mathbb{R}^9} \delta(p_1 + p_2 - p_3 - p_4)\delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})\times$$
$$\times [f(t,p_3)f(t,p_4)(1 + f(t,p_1))(1 + f(t,p_2))$$
$$- f(t,p_1)f(t,p_2)(1 + f(t,p_3))(1 + f(t,p_4))]dp_2 dp_3 dp_4.$$

C_{12} condensate atoms - excited atoms



$$
\begin{aligned}
C_{12}[f](p_1) \backsim \int_{\mathbb{R}^9} & \delta(p_1 + p_2 - p_3 - p_4)\delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) \times \\
& \times [f(t,p_3)f(t,p_4)(1+f(t,p_1))(1+f(t,p_2)) \\
& - f(t,p_1)f(t,p_2)(1+f(t,p_3))(1+f(t,p_4))]dp_2 dp_3 dp_4 .
\end{aligned}
$$

C_{12} condensate atoms - excited atoms



$$
\begin{aligned}
C_{12}[f](p_1) \backsim \int_{\mathbb{R}^9} & \delta(p_1 - p_3 - p_4)\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) \times \\
& \times [f(t,p_3)f(t,p_4)(1 + f(t,p_1)) \\
& - f(t,p_1)(1 + f(t,p_3))(1 + f(t,p_4))]dp_2 dp_3 dp_4 .
\end{aligned}
$$

Different temperature regimes

- $T > T_{BEC}$:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = C_{22}[f]$$

- $0.6T_{BEC} < T < T_{BEC}$:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = n_c C_{12}[f] + C_{22}[f]$$

- $T < 0.4T_{BEC}$:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = n_c C_{12}[f]$$

Note that n_c is the density distribution of the BEC, whose wave function follows the cubic NLS.

Results

$T < 0.4T_{BEC}$: Very hard sphere collision kernel

- Spohn'10, Arkeryd-Nouri'12-'13-'15 results: truncated kernel
- Escobedo-Tran'15: Convergence to equilibrium of the linearized equation
- Alonso-Gamba-Tran'16 L^1 global existence, propagation and creation of exponential and polynomial moments (kernel is different from the Soffer-Tran case)

$$\frac{\mathrm{d}f}{\mathrm{d}t} = n_c C_{12}[f]$$

- Nguyen-Tran'16 for any time $T > 0$, there exist positive constants θ_0, θ_1 such that

$$f(t,p) \geq \theta_1 \exp(-\theta_0 |p|^2), \qquad \forall\, t \geq T.$$

Results

- $T > T_{BEC}$: Escobedo-Velazquez'15 L^∞ blow-up result

$$\frac{\mathrm{d}f}{\mathrm{d}t} = C_{22}[f]$$

Results: Difficulty 1

- Classical Boltzmann equations:

$$p_1 + p_2 = p_3 + p_4, \quad |p_1|^2 + |p_2|^2 = |p_3|^2 + |p_4|^2.$$

Then p_1, p_2, p_3, p_4 are on a sphere centered at $\frac{p_1+p_2}{2}$ with radius $\frac{|p_1-p_2|}{2}$. The Boltzmann collision operator can be expressed as an integral on a sphere (Carleman Representation) or a Radon transform (Wennberg'07, Mouhot-Villani'09).

Results: Difficulty 1

- Quantum Boltzmann equations:

$$p_1 + p_2 = p_3 + p_4, \quad \mathcal{E}_{p_1} + \mathcal{E}_{p_2} = \mathcal{E}_{p_3} + \mathcal{E}_{p_4}.$$

$$p_1 = p_2 + p_3, \quad \mathcal{E}_{p_1} = \mathcal{E}_{p_2} + \mathcal{E}_{p_3}.$$

$\longrightarrow$ Collision operators are integrals on complicated manifolds.

Results: Difficulty 1

- Quantum Boltzmann equations:

$$p_1 + p_2 = p_3 + p_4, \quad \mathcal{E}_{p_1} + \mathcal{E}_{p_2} = \mathcal{E}_{p_3} + \mathcal{E}_{p_4}$$

$$p_1 = p_2 + p_3, \quad \mathcal{E}_{p_1} = \mathcal{E}_{p_2} + \mathcal{E}_{p_3}$$

$$\mathcal{E}_p = \sqrt{C_1|p|^2 + C_2|p|^4}$$

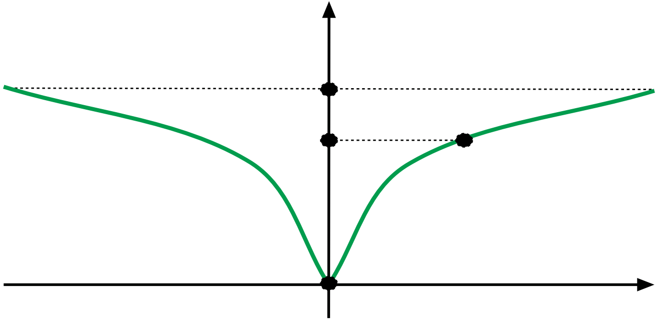
$\longrightarrow$ Collision operators are integrals on complicated manifolds.
$\longrightarrow$ Resonance Manifold Problem (Zakharov Book '92, Nazarenko Book '04)

Some dispersion laws

- Quantum kinetic equations
  - Bogoliubov dispersion law $\sqrt{C_1|p|^2 + C_2|p|^4}$
- Wave turbulence equations
  - Capillary waves $C|p|^{\frac{3}{2}}$
  - Gravity capillary waves $\sqrt{C_1|p|^2 + C_2|p|^3}$
  - Alfvén waves in solar wind $Cp_{||}, \quad p = (p_{||}, p_{\perp})$
  - Whislers (helicones) in collisionless magnetoactive plasma $Cp_{||}|p_{\perp}|, \quad p = (p_{||}, p_{\perp})$
  - Langmuir waves in Isotropic Plasma $C_1 + C_2|p|^2$
  - Inertial waves in rotating fluids $C\frac{p_{||}}{|p|}, \quad p = (p_{||}, p_{\perp})$
  - Stratified internal waves in open oceans $\sqrt{C_1 + C_2|p|^2}$
  - Weak turbulence waves in nonlinear optics $C|p|^2$
  - Phonon interaction in anharmonic crystal lattices in solid state physics $C|p|$

$\Longrightarrow$ **Due to the complexity of these dispersion laws, the Resonance Manifolds are not spheres!**

Results: Difficulty 1



The resonance manifold for $p_1 = p_2 + p_3, \quad \mathcal{E}_{p_1} = \mathcal{E}_{p_2} + \mathcal{E}_{p_3}$, $\mathcal{E}_p = |p|^{\frac{3}{2}}$.

Results: Difficulty 2

- Boltzmann collision operator: order 2

$$f_3 f_4 - f_1 f_2$$

- Quantum Boltzmann collision operator: orders $1, 2, 3$

$$f_3 f_4 (1 + f_1)(1 + f_2) - f_1 f_2 (1 + f_3)(1 + f_4)$$

and

$$f_2 f_3 (1 + f_1) - f_1 (1 + f_2)(1 + f_3)$$

Results: Difficulty 3

- Boltzmann: conservation of mass, momentum and energy

$$\frac{d}{dt}\int_{\mathbb{R}^3} f dp = \frac{d}{dt}\int_{\mathbb{R}^3} p f dp = \frac{d}{dt}\int_{\mathbb{R}^3} |p|^2 f dp = 0$$

- Quantum Boltzmann: conservation of momentum and energy (since the particles are going in and out of the condensate, then the mass is not preserved)

$$\frac{d}{dt}\int_{\mathbb{R}^3} p f dp = \frac{d}{dt}\int_{\mathbb{R}^3} \mathcal{E}_p f dp = 0$$

Results: Difficulty 4

- The coupling Schrödinger-Boltzmann

$$
\begin{aligned}
i\partial_t \Phi(t,r) &= \Big(-\Delta_r + |\Phi(t,r)|^2 + \rho(t,r) \Big) \Phi(t,r), \\
&\qquad \text{on } \mathbb{R}_+ \times \mathbb{R}^3, \\
\Phi(0,r) &= \Phi_0(r), \forall r \in \mathbb{R}^3, \\
\rho[f](t,r) &= \int_{\mathbb{R}^3} f(t,r,p')dp'.
\end{aligned}
$$

Results: Some Lemmas

Lemma

Let S_p be defined as

$$S_p := \left\{ p_* \in \mathbb{R}^3 \ : \ \mathcal{E}(p + p_*) = \mathcal{E}(p) + \mathcal{E}(p_*) \right\}$$

There are positive constants C_0, C_1 independent of p such that

$$C_0 \int_{\mathbb{R}^3} \frac{F(u)}{|u|} du \leq \int_{S_p} F(w)\, d\sigma(w) \leq C_1 \int_{\mathbb{R}^3} \frac{F(u)}{|u|} du.$$

Results: Some Lemmas

Lemma

Let $[0, T]$ *be a time interval,* $E := (E, \|\cdot\|)$ *be a Banach space,* $\mathcal{S}$ *be a bounded, convex and closed subset of* E*, and* $Q : \mathcal{S} \to E$ *be an operator satisfying the following properties:*

$(\mathfrak{A})$ *Let* $\|\cdot\|_*$ *be a different norm of* E*, satisfying* $\|\cdot\|_* \le C_E\|\cdot\|$ *for some universal constant* C_E*, and the function* $u \longrightarrow |u|_*$ *satisfying*

$$|u+v|_* \le |u|_* + |v|_*, \text{ and } \quad |\alpha u|_* = \alpha|u|_*$$

for all u, v *in* E *and* $\alpha \in \mathbb{R}_+$*. Moreover,*

$$|u|_* = \|u\|_*, \forall u \in \mathcal{S}, |u|_* \le \|u\|_* \le C_E\|u\|, \forall u \in E,$$

$$|Q(u)|_* \le C_*(1 + |u|_*), \forall u \in \mathcal{S},$$

then

$$\mathcal{S} \subset \overline{B_*\left(O, (2R_* + 1)e^{(C_*+1)T}\right)} := \overline{\left\{u \in E \middle| \|u\|_* \le (2R_* + 1)e^{(C_*+1)T}\right\}},$$

for some positive constant $R_* \ge 1$*.*

Results: Some Lemmas

$(\mathfrak{B})$ Sub-tangent condition

$$\liminf_{h\to 0^+} h^{-1}\mathrm{dist}\big(u+hQ[u],\, \mathcal{S}\big) = 0, \qquad \forall\, u \in \mathcal{S}\cap B_*\Big(O, (2R_*+1)e^{(C_*+1)T}\Big),$$

$(\mathfrak{C})$ Hölder continuity condition

$$\big\|Q[u]-Q[v]\big\| \le C\|u-v\|^{\beta}, \quad \beta\in(0,1), \quad \forall\, u, v \in \mathcal{S},$$

$(\mathfrak{D})$ one-side Lipschitz condition

$$\big[Q[u]-Q[v], u-v\big] \le C\|u-v\|, \qquad \forall\, u, v \in \mathcal{S},$$

where

$$\big[\varphi,\phi\big] := \lim_{h\to 0^-} h^{-1}\big(\|\phi+h\varphi\| - \|\phi\|\big).$$

Then the equation

$$\partial_t u = Q[u] \text{ on } [0,T]\times E,\ \ u(0) = u_0 \in \mathcal{S}\cap B_*(O, R_*) \tag{1}$$

has a unique solution in $C^1((0,T),E)\cap C([0,T],\mathcal{S})$.

Results: Normal form transformation

Let us consider

$$\begin{aligned} i\partial_t\Psi(t,r) &= \Big(-\Delta_r + |\Psi(t,r)|^2 + U(t,r)\Big)\Psi(r,t), \\ \Psi(0,r) &= \Psi_0(r), \forall r \in \mathbb{R}^3, \end{aligned}$$

where

$$\begin{aligned} \|V(t,\cdot)\|_{H^1_r(\mathbb{R}^3)} &= \|U(t,\cdot)+1\|_{H^1_r(\mathbb{R}^3)} \le \mathfrak{C}_1 e^{-\mathfrak{C}_2 t}, \\ \|V(t,\cdot)\|_{L^{3/2}_r(\mathbb{R}^3)} &= \|U(t,\cdot)+1\|_{L^{3/2}_r(\mathbb{R}^3)} \le \mathfrak{C}_3 e^{-\mathfrak{C}_4 t}, \end{aligned}$$

and

$$\|V(t,\cdot)\|_{L^3_r(\mathbb{R}^3)} = \|U(t,\cdot)+1\|_{L^3_r(\mathbb{R}^3)} \le \mathfrak{C}_5 e^{-\mathfrak{C}_6 t},$$

with some positive constants $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5, \mathfrak{C}_6$.

Results: Normal form transformation

Define

$$u = \Psi - 1 = u_1 - iu_2,$$

we obtain the following system of equations whose solution is (u_1, u_2)

$$\begin{aligned} \dot{u}_1 &= -\Delta u_2 + 2u_1 u_2 + |u|^2 u_2 + V u_2, \\ \dot{u}_2 &= -(2-\Delta)u_1 - 3u_1^2 - u_2^2 - |u|^2 u_1 - V(u_1 + 1). \end{aligned}$$

We define

$$v = u_1 + iUu_2, \;\; U = \sqrt{-\Delta(2 - \Delta)^{-1}},$$

and obtain the following equation for v,

$$i\partial_t v - Hv = U(3u_1^2 + u_2^2 + |u|^2 u_1) + i(2u_1 u_2 + |u|^2 u_2),$$

where

$$H = \sqrt{-\Delta(2 - \Delta)}.$$

Results

$0.6T_{BEC} < T < T_{BEC}$: Soffer-Tran'16 L^1 global existence

$$\frac{\mathbf{d}f}{\mathbf{d}t} = n_c C_{12}[f] + C_{22}[f]$$

Theorem

Suppose that $f_0(p) = f_0(|p|) \geq 0$, and

$$\int_{\mathbb{R}^3} (1 + \mathcal{E}_p) f_0(p) dp < \infty.$$

For any time interval $[0, T]$, let n, n^* be two positive integers, $n > 1$, $n^* > n + 4$, and $\mathfrak{c}_{n^*}$ is some positive constant.

Results

We assume that

$$\int_{\mathbb{R}^3} \mathcal{E}_p^{n^*} f_0(p) < \mathfrak{c}_{n^*}.$$

Then there exists a unique classical positive radial solution

$$f(t,p) = f(t,|p|) \in C^0([0,T], \mathbb{L}^1_n(\mathbb{R}^3)) \cap C^1((0,T), \mathbb{L}^1_n(\mathbb{R}^3)).$$

Moreover, there exists a constant $C_{n^*,T} = \mathfrak{c}_{n^*} e^{\mathfrak{C}T}$ such that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} \mathcal{E}_p^{n^*} f(t,p)dp < C_{n^*,T}.$$

Results

Coupling kinetic and Schrödinger equations Soffer-Tran'16:
Convergence to equilibrium for the linear kinetic equation and scattering theory for the cubic nonlinear Schrödinger equation.

THANK YOU FOR YOUR ATTENTION