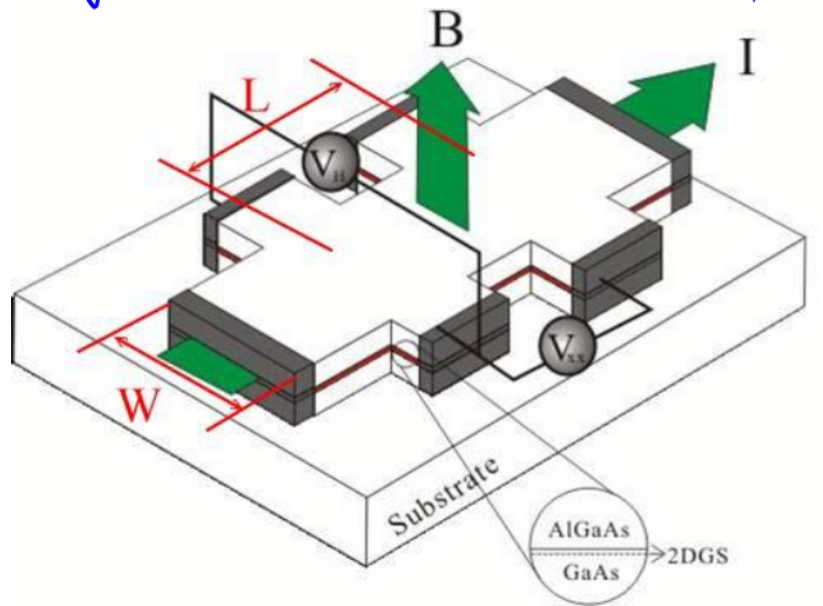


Conformal Scaling Limit of the Network Model for the Integer Quantum Hall Transition

Martin R. Zirnbauer (U of Cologne)
@ Rutgers Stat. Mech. Conference
(May 7, 2017)

Integer Quantum Hall Effect

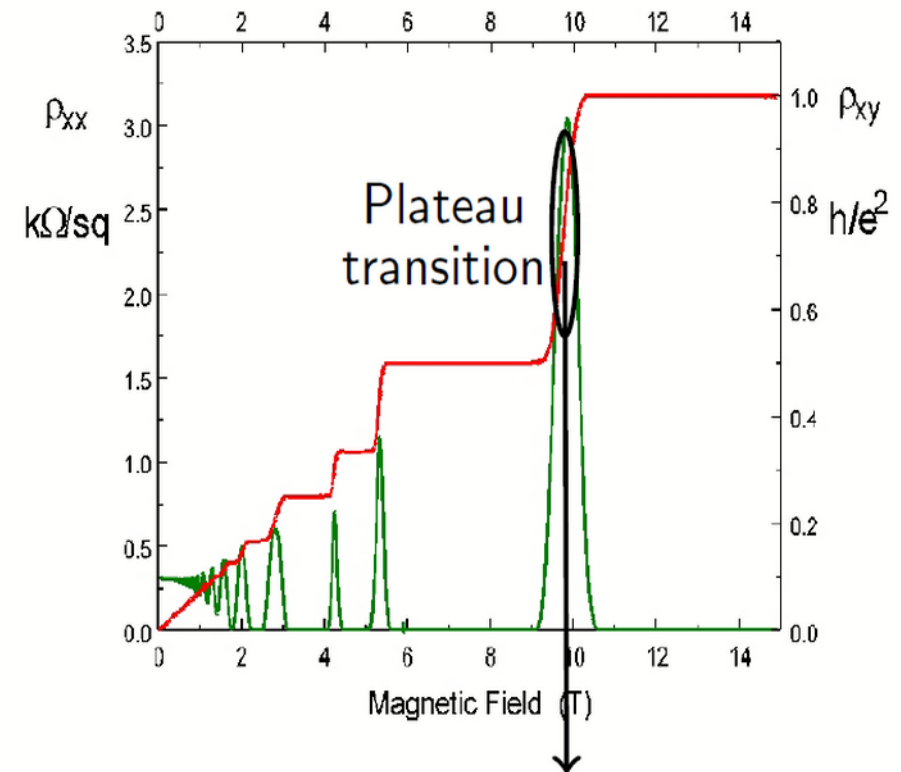


Two-dimensional disordered electron gas at low temperature and in a strong magnetic field.

Hall resistance exhibits plateaus: $R_H = \frac{h}{ne^2}$

Transition between plateaus is a critical phenomenon (of Anderson-localization type).

Could/should be a paradigm, but is not understood in quantitative detail ...



Nonlinear sigma model (Pruisken et al., 1983)

weak localization

$$\mathcal{L} = \frac{\sigma_{xx}}{8} \text{Str} \partial_\mu Q \partial_\mu Q + \frac{\sigma_{xy}}{8} \epsilon_{\mu\nu} \text{Str} Q \partial_\mu Q \partial_\nu Q$$

θ -term

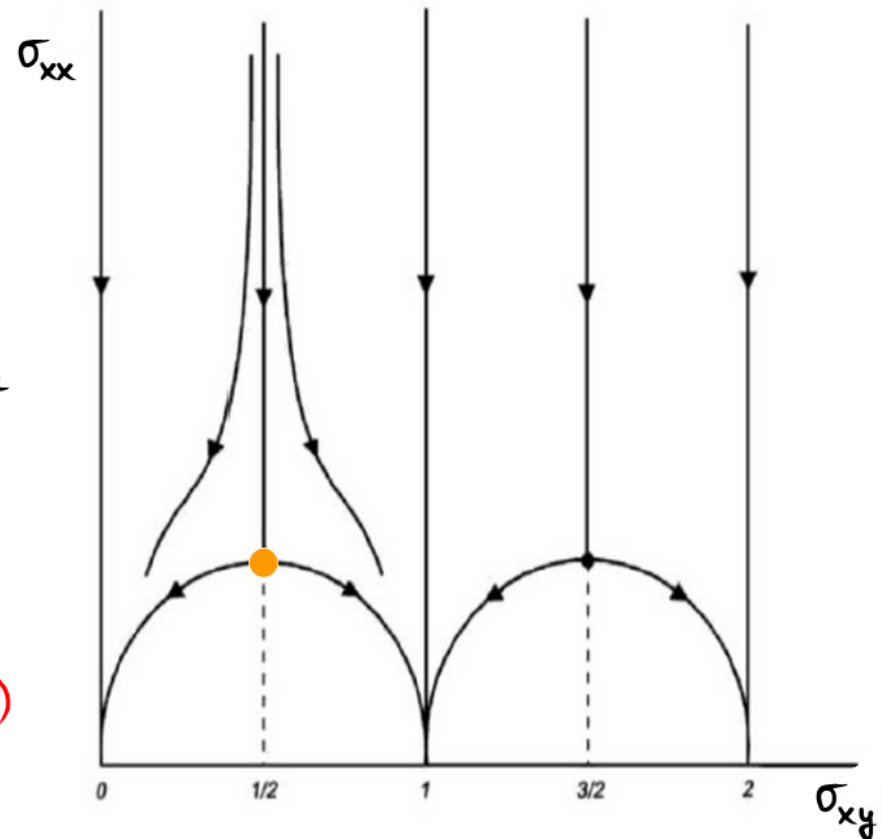
Wegner-Efeto SUSY method $\leadsto$ target space

$$Q = u \Sigma_3 u^{-1}, \quad \Sigma_3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$$

is complex Grassmann manifold U/K

with global symmetry group $U = U(r, r | 2r)$

CFT = ?



Pruisken-Khmelnitskii RG flow diagram
(conjectured, 1983)

The Conundrum

Apparent Conflict

- Symmetry group $U = U(r, r|2r)$ must act (by conjugation) on target X .
- All known CFTs with continuous symmetries are Wess-Zumino-Witten models or Goddard-Kent-Olive coset theories (= gauged WZW models).
- WZW models (with non-compact target X) are **ruled out** by
 - RG-instability against infinity of relevant U -inv. perturbations
 - exact results for related critical point (class C) Mudry, Chamon & Wen, 1995
Read & Saleur, 2001
- Coset theories X/H with $H \subset U$ greater than center of U are **ruled out** by U -symmetry.
- Gauging by center $H = U(1)$ does not suffice to remove RG instability.

WZW model - operator formalism

WZW field $M: \Sigma \rightarrow X$.

Left and right translations $M \mapsto g_L M g_R^{-1}$ ($g_L, g_R \in G$) give rise to

Currents $J^A = k \operatorname{Str}(A \partial M \cdot M^{-1})$ holomorphic

$\bar{J}^A = k \operatorname{Str}(A M^{-1} \bar{\partial} M)$ anti-holom.

Current algebra $\hat{\mathfrak{g}}$. Operator product expansion:

$$J^A(z) J^B(w) \sim -\operatorname{Str}(AB) \frac{k}{(z-w)^2} + \frac{J^{[A,B]}(w)}{z-w}$$

Energy-momentum tensor (Sugawara) $\mathcal{T} = \frac{1/2}{k + c_V} (J^{e_i} J^{e_i})$

is Virasoro: $\mathcal{T}(z) \mathcal{T}(w) \sim \frac{c/2}{(z-w)^4} + \frac{2\mathcal{T}(w)}{(z-w)^2} + \frac{\partial \mathcal{T}(w)}{z-w}$

due to $\mathcal{T}(z) J^A(w) \sim \frac{J^A(w)}{(z-w)^2} + \frac{\partial J^A(w)}{z-w}$

GKO coset CFT (gauged WZW model)

Subgroup $H \subset G$ with anomaly-free action $M \mapsto h_L M h_R^{-1}$

Current subalgebra $\hat{\mathfrak{h}} \subset \hat{\mathfrak{g}}$.

Energy-momentum tensor $\tau^{\hat{\mathfrak{g}}/\hat{\mathfrak{h}}} := \tau^{\hat{\mathfrak{g}}} - \tau^{\hat{\mathfrak{h}}}$ generates

Virasoro algebra with central charge $c_{\mathfrak{g}/\mathfrak{h}} = c_{\mathfrak{g}} - c_{\mathfrak{h}}$.

NOTE: gauging by H -action is incompatible
with action $M \mapsto u M u^{-1}$ by symmetry group $\mathcal{U}$.

How to get around this difficulty?
Totally new ideas needed ??

"Internal" coset construction

Observation. Geometrically, current subalgebras are associated with distributions (i.e. smooth subbundles) of the tangent bundle TX .

$\mathcal{U}$ acts on $X \equiv X_{r,r|2r}$, hence on TX .

To gauge the WZW model without ruining the $\mathcal{U}$ -symmetry, one needs a $\mathcal{U}$ -invariant distribution in TX .

Fact. Such distributions do exist.

Hint. View the WZW target space as an associated bundle over the nonlinear sigma model target space:

$$X_{r,r|2r} = \mathcal{U} \times_K (X_{r|r}^+ \times X_{r|r}^-) \mapsto \mathcal{U}/K,$$

$$M = u \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix} u^{-1} = u h^{-1} \begin{pmatrix} h_+ M_+ h_+^{-1} & 0 \\ 0 & h_- M_- h_-^{-1} \end{pmatrix} h u^{-1}, \quad h \in K.$$

superseding Pruisken's $u \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} u^{-1}$

Gauge by K !

Gauged WZW action functional:

notation: $d = \partial + \bar{\partial}$, $\partial = dz \wedge \frac{\partial}{\partial z}$

$$S_k^{\text{WZW}}[M; A = A^{10} + A^{01}] = \frac{ik}{4\pi} \int_{\Sigma} \left(\text{STr } M^{-1} \partial M \wedge M^{-1} \bar{\partial} M + \frac{1}{3} d^{-1} \text{STr } (M^{-1} dM)^{\wedge 3} \right) \\ + \frac{ik}{2\pi} \int_{\Sigma} \text{STr} \left(\partial M \cdot M^{-1} \wedge A^{01} - A^{10} \wedge M^{-1} \bar{\partial} M - A^{10} \wedge M^{-1} A^{01} M + A^{10} \wedge A^{01} \right).$$

Let there be two copies of M for retarded sector (M_+)
and advanced sector (M_-).

Riemannian symmetric
superspace of type A/A

Proposed fixed-point action = $S_{k=4}^{\text{WZW}}[M_+, A_+] + S_{k=4}^{\text{WZW}}[M_-, A_-]$

$$+ \frac{ik}{2\pi} \int_{\Sigma} \text{STr} \left((u^{-1} \partial u)_{+-} \wedge (u^{-1} \bar{\partial} u)_{-+} - (u^{-1} \partial u)_{+-} \wedge M_-^{-1} (u^{-1} \bar{\partial} u)_{-+} M_+ \right. \\ \left. + (u^{-1} \partial u)_{-+} \wedge (u^{-1} \bar{\partial} u)_{+-} - (u^{-1} \partial u)_{-+} \wedge M_+^{-1} (u^{-1} \bar{\partial} u)_{+-} M_- \right).$$

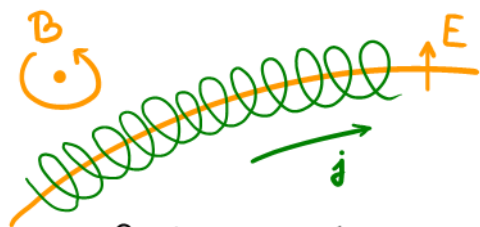
Invariance under gauge transformations

$$M_s \mapsto h_s M_s h_s^{-1}, \quad A_s \mapsto h_s A_s h_s^{-1} - dh_s \cdot h_s^{-1}, \quad u \mapsto u \text{diag}(h_+, h_-)^{-1}$$

Interpretation: nonlinear sigma model coupled to maximally gauged WZW models

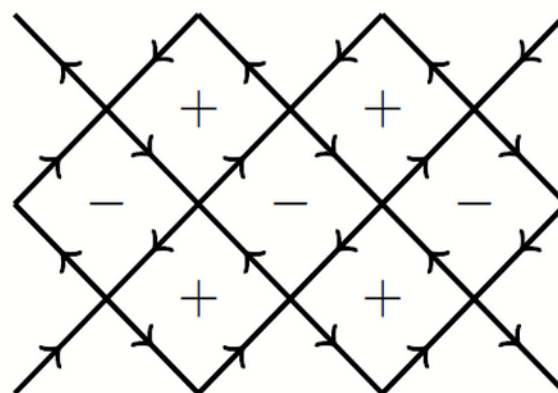
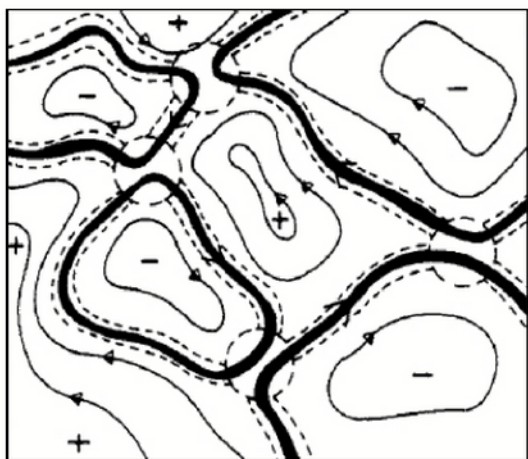
Network Model

(Chalker & Coddington)



Guiding center (of rapid cyclotron motion)

drifts slowly along equipotential contours



Hilbert space = $\mathbb{C}^{\# \text{links}}$

discrete time evolution $\psi_{t+1} = U \psi_t$

$$U = U_r U_s$$

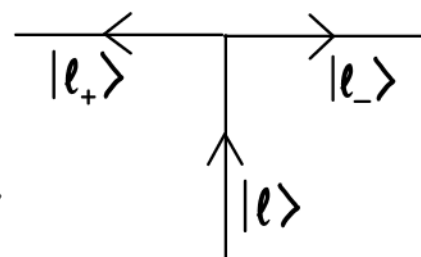
U_r diagonal in link basis:

$$U_r |l\rangle = |l\rangle e^{i\phi(l)}$$

$\phi(l)$ uncorrelated random phases,
uniformly distributed

U_s deterministic scattering at nodes:

$$U_s |l\rangle = |l_+\rangle a_+ + |l_-\rangle a_-$$



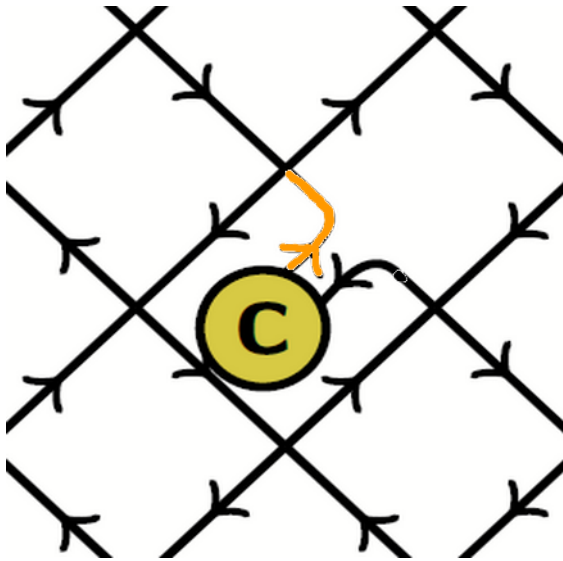
$$a_{\pm} = e^{i\pi/4} / \sqrt{2}$$

(at criticality)

Gaussian Free Field

Statistics of wave intensities.

Bondesan, Wieczorek & Z
PRL (2015), NPB-FS (2017) 45 pp



$\psi_c = U \psi_c$ stationary "scattering" state (quasi-energy zero)
for incoming-wave boundary conditions $\psi_c(c) = 1$.

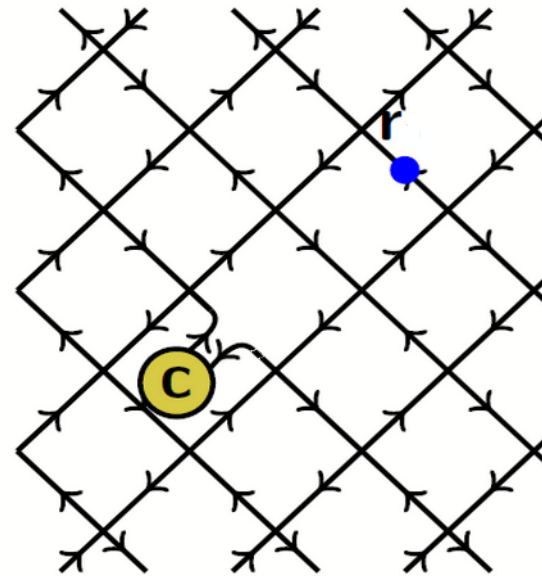
Observable: $|\psi_c(r)|^2$ for large distances $|r - c|$

Prediction from Abelian OPE,
crossing symmetry of 4-point fctn :

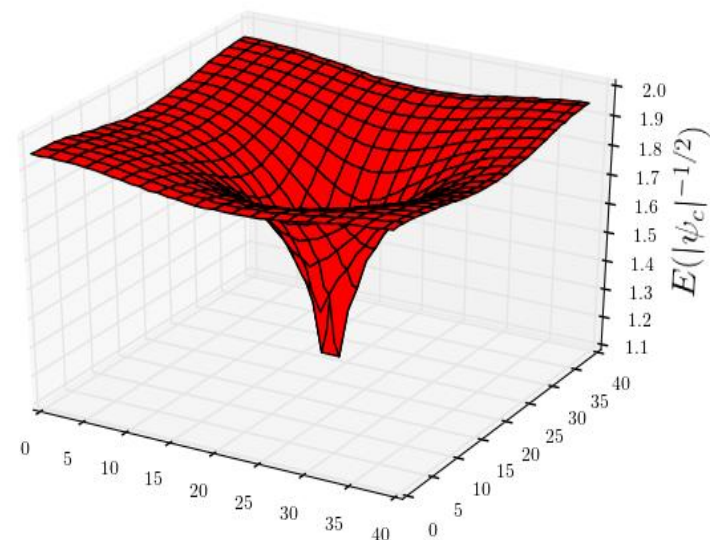
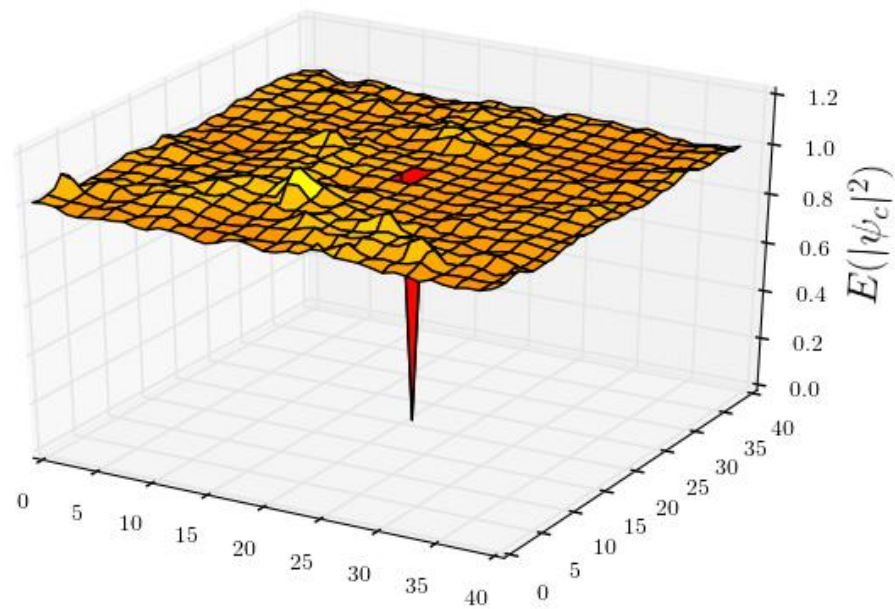
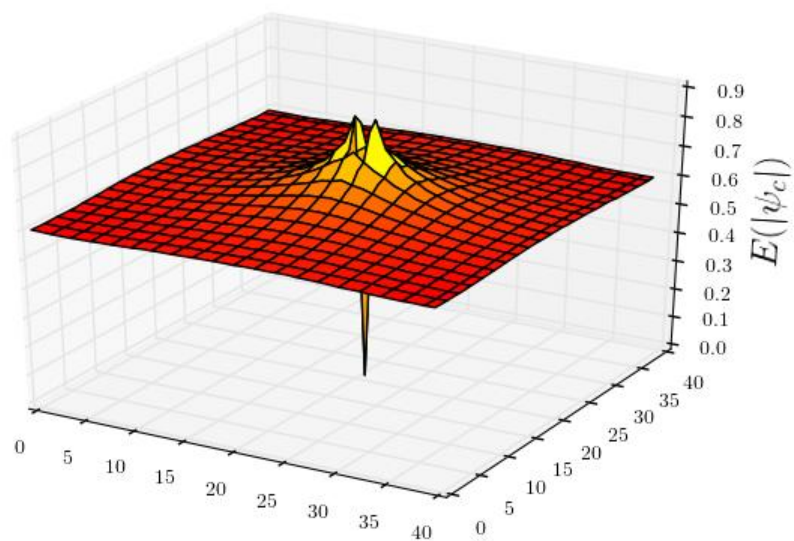
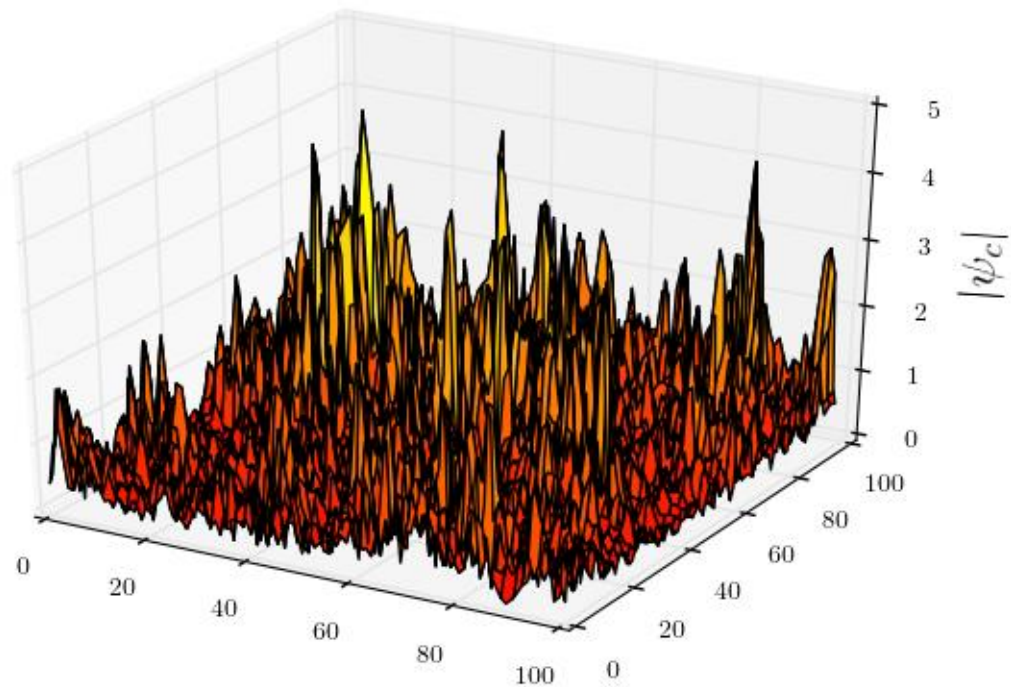
$$\mathbb{E}(|\psi_c(r)|^{2q}) \simeq |r - c|^{-2\Delta_q}, \quad \Delta_q = Xq(1-q).$$

Interpretation: $\log |\psi_c(r)|^2 \equiv \varphi(r)$

GFF with background charge $Q=1$, stiffness X .



Numerical simulations give $X \cong 0.26 (\dots 0.28) \leadsto$ Hypothesis $X = 1/k$ (level $k=4$)



SUSY Vertex Model

CC network model $\longrightarrow$ SUSY vertex model

A variant (due to N. Read) of the Wegner-Efetov supersymmetry method trades the task of taking disorder averages for a statistical mechanics problem of new (collective) variables that admit a **continuum limit** (at the critical point).

Uses **second quantization** on a Fock space for bosons and fermions.

Retarded sector: $U = e^X$, $\text{Re } X < 0$.

$$\begin{aligned} \text{bosons: } \text{Det}_{\mathbb{C}^N}^{-1}(1-U) &= \text{Tr}_{S(\mathbb{C}^N)} \rho_B(U) \\ \text{fermions: } \text{Det}(1-U) &= \text{STr} \rho_F(U) \end{aligned} \quad \text{where} \quad \begin{aligned} \rho_B(e^X) &= e^{b^\dagger X b} \\ \rho_F(e^X) &= e^{f^\dagger X f} \end{aligned}$$

Advanced sector: $U = e^X$, $\text{Re } X > 0$.

$$\begin{aligned} b^\dagger &\rightarrow -b, & b &\rightarrow b^\dagger, \\ f^\dagger &\rightarrow +f, & f &\rightarrow f^\dagger. \end{aligned}$$

Random phase average projects Fock space to subspace of $U(1)$ singlets

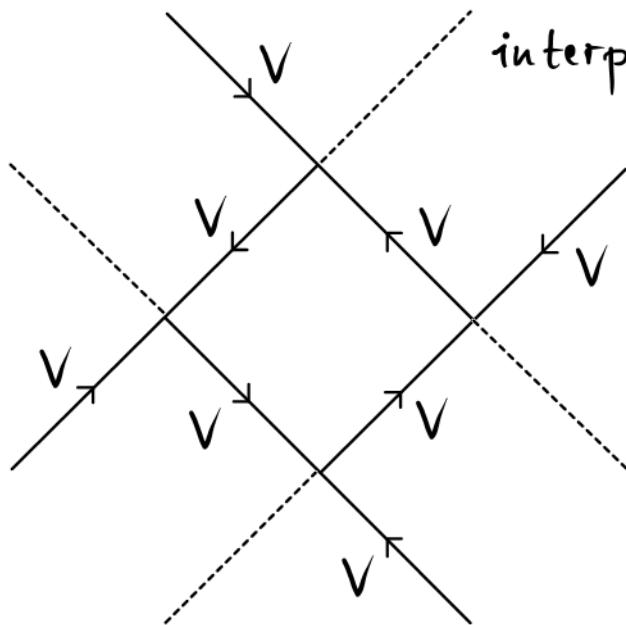
$\Lambda \quad \mathcal{H} = \bigotimes_{\text{links}} V(l)$ state space of SUSY vertex model

where $V = \bigoplus_{n_b^+ + n_f^+ = n_b^- + n_f^-} \left(S^{n_b^+}(\mathbb{C}) \otimes \Lambda^{n_f^+}(\mathbb{C}) \otimes S^{n_b^-}(\mathbb{C}^*) \otimes \Lambda^{n_f^-}(\mathbb{C}^*) \right)$

is irreducible highest-weight representation for $\mathcal{U} = U(1,1|2)$.

r replicas: $\mathcal{U} = U(r,r|2r)$

unit cell



interpretation $\hookleftarrow$ Feynman paths

$$\mathbb{E}(|\psi_c(r)|^{2q}) = \langle \pi_{\text{vac}}(c) (B^\dagger B)^q(r) \rangle_{\text{v.h.}}$$

$$\langle A \rangle_{\text{v.h.}} = \text{Str } A \rho(u_s)$$

Global symmetry $\mathcal{U} = U(r,r|2r)$

Continuum Limit from
Dirac Approximation

Pure network model ($U \equiv U_s$) at long wave lengths

$$1 - U_s \approx iD \quad (\text{Dirac operator} \leadsto \text{discrete holomorphic derivative})$$

Treat the network model with random phase disorder as a perturbation...

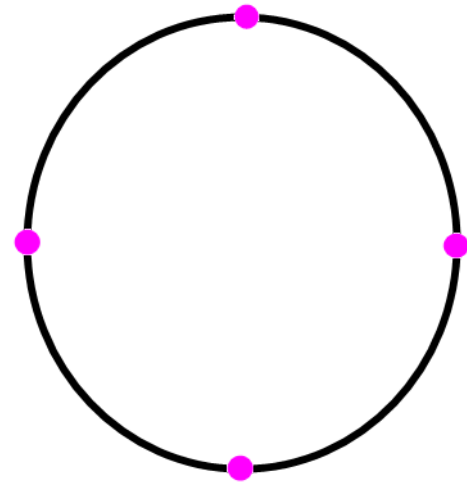
$\leadsto$ (SUSY) Dirac theory with random gauge potential

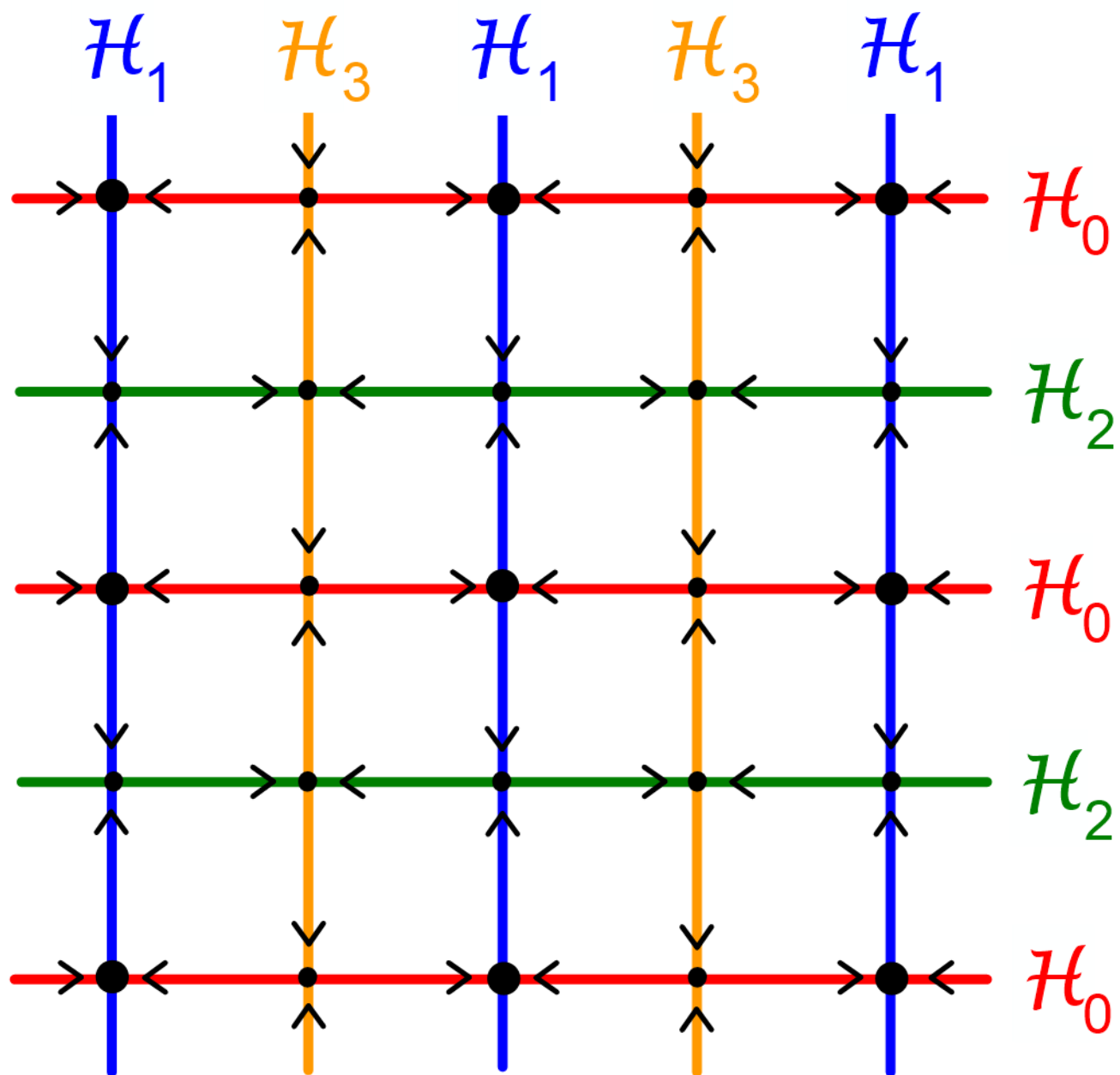
and random scalar potential

$\mathbb{Z}_4$ spectral symmetry

$\leadsto$ 4 Dirac points (at 4th roots of unity)

$$e^{im\pi/2} \cdot 1 - U_s = iD_m \quad (m = 0, 1, 2, 3)$$





Preparatory step:

Assume A with values in $\mathfrak{su}(k) \subset \mathfrak{u}(k)$ ($k=4$).

Bosonize.

CFT describing continuum limit of disordered critical point is

(Bernard & LeClair 2002; Tsvetlik et al. 2001):

" $GL(2r|2r)$ " WZW model of level $k=4$.

Target space = supermanifold over Riemannian symmetric space of type A/A

$$\begin{array}{ccc} \left(GL(2r)/U(2r) \right) \times U(2r) & & \\ \text{noncompact} & \times & \text{compact} \\ bb & & ff \end{array}$$

What happens when the $U(1)$ gauge field and the random scalar potential are turned on?

Gauged WZW action functional:

notation: $d = \partial + \bar{\partial}$, $\partial = dz \wedge \frac{\partial}{\partial z}$

$$S_k^{\text{WZW}}[M; A = A^{10} + A^{01}] = \frac{ik}{4\pi} \int_{\Sigma} \left(\text{STr } M^{-1} \partial M \wedge M^{-1} \bar{\partial} M + \frac{1}{3} d^{-1} \text{STr } (M^{-1} dM)^{\wedge 3} \right) \\ + \frac{ik}{2\pi} \int_{\Sigma} \text{STr} \left(\partial M \cdot M^{-1} \wedge A^{01} - A^{10} \wedge M^{-1} \bar{\partial} M - A^{10} \wedge M^{-1} A^{01} M + A^{10} \wedge A^{01} \right).$$

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Remark: predicts multifractal scaling exponents $\Delta_q = \frac{1}{k} q(1-q)$.

Thank you,

And Happy Birthday to Jürg, Tom & Herbert !