# Wall-to-wall optimal transport

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# Wall to Wall Optimal Transport

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# The problem:

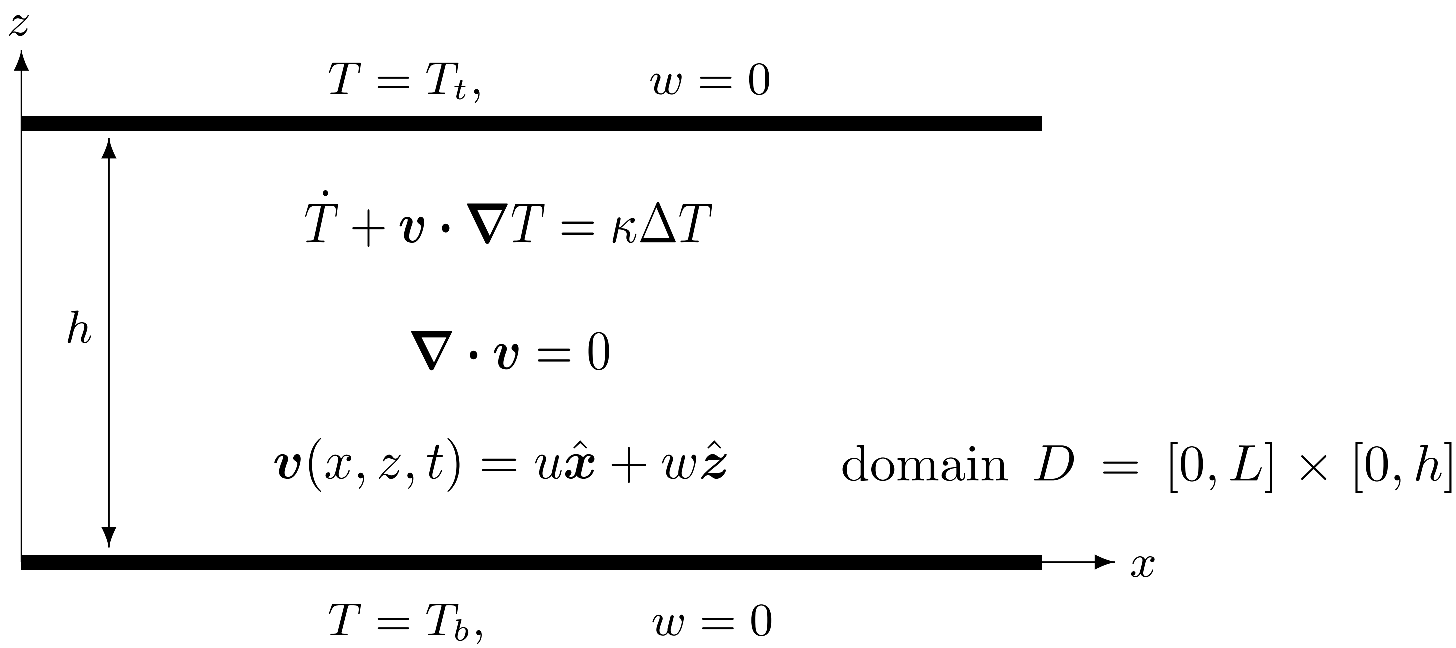


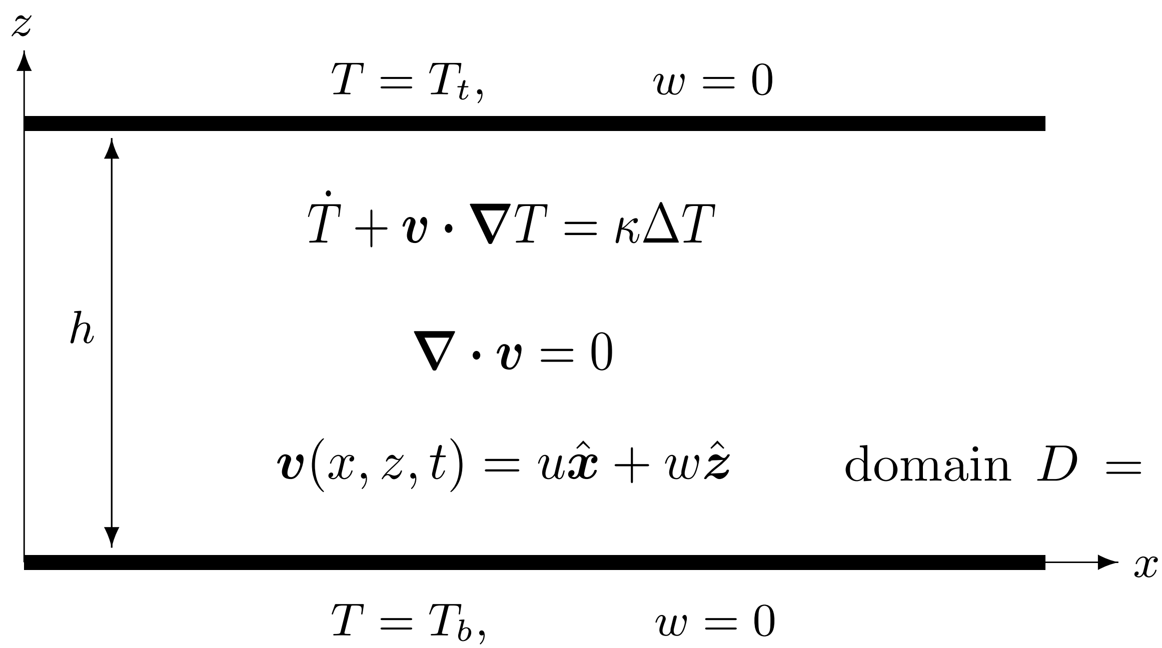

FIGURE 1. Schematic of the configuration. The top and bottom boundaries are impermeable and held at fixed temperatures.

All the system variables are periodic in the $x$ direction

## Question:

what flow moves the most heat from one wall to the other?

# The problem:




domain $D = [0, L] \times [0, h]$

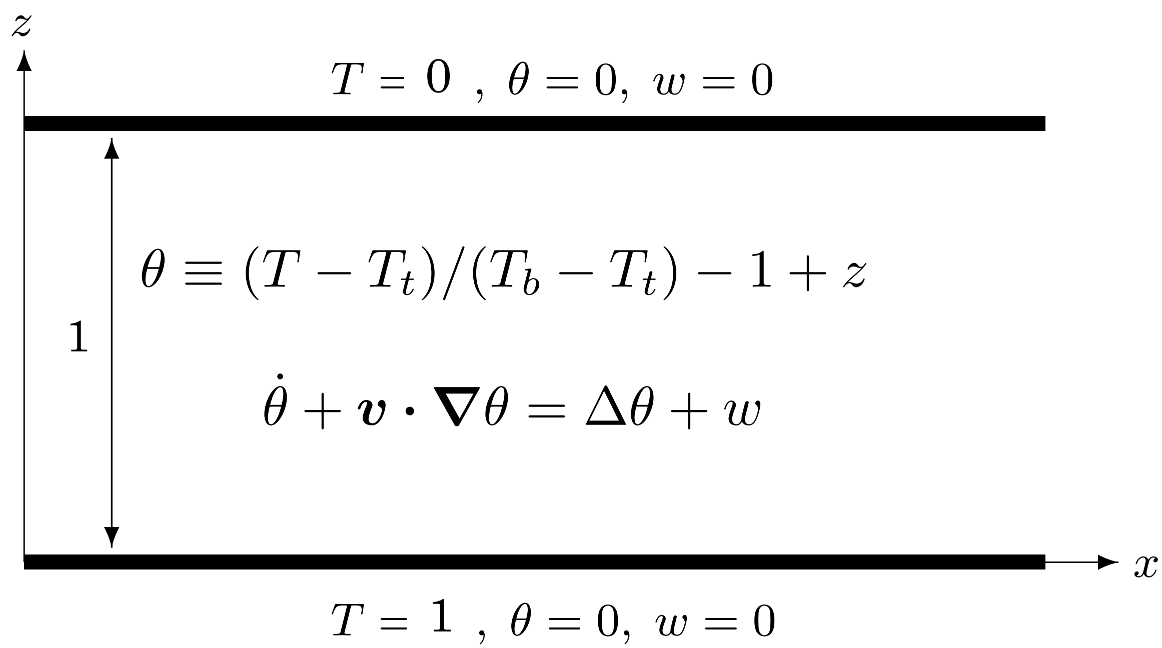
FIGURE 1. Schematic of the configuration. The top and bottom boundaries are impermeable and held at fixed temperatures.

All the system variables are periodic in the $x$ direction

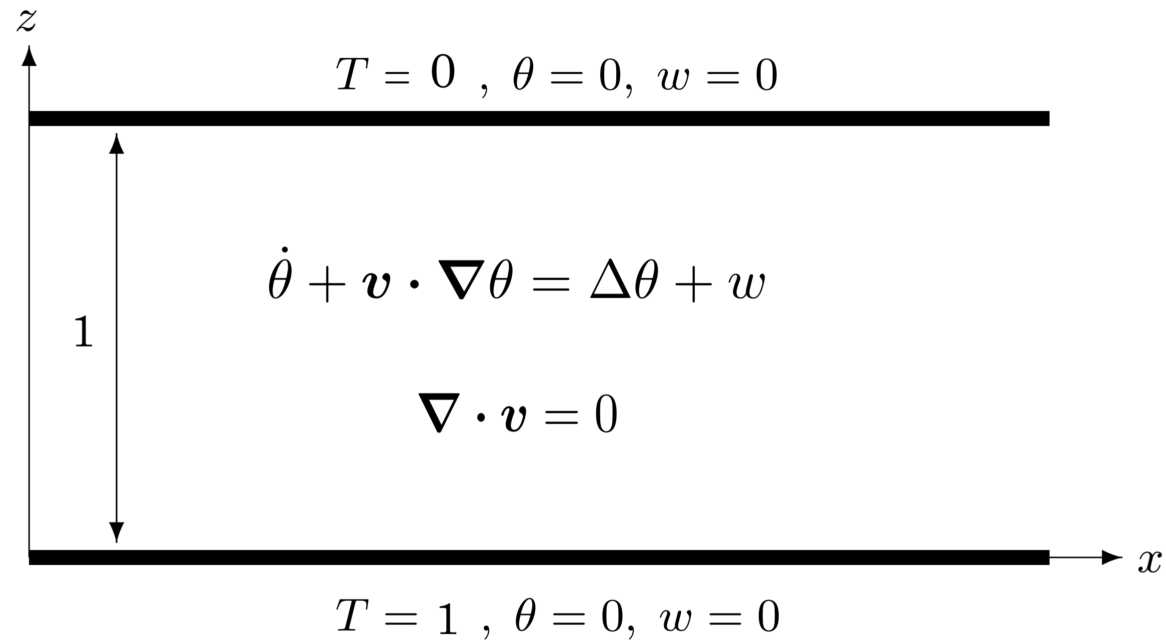
nondimensionalize the system using the spacing between the walls $h$ and time scale $h^2/\kappa$

and shift temperature scale and variable ...

# The problem:


$z$
$T = 0$ , $\theta = 0$, $w = 0$
$1$
$\theta \equiv (T - T_t)/(T_b - T_t) - 1 + z$
$\dot{\theta} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \theta = \Delta \theta + w$
$x$
$T = 1$ , $\theta = 0$, $w = 0$

# The problem:



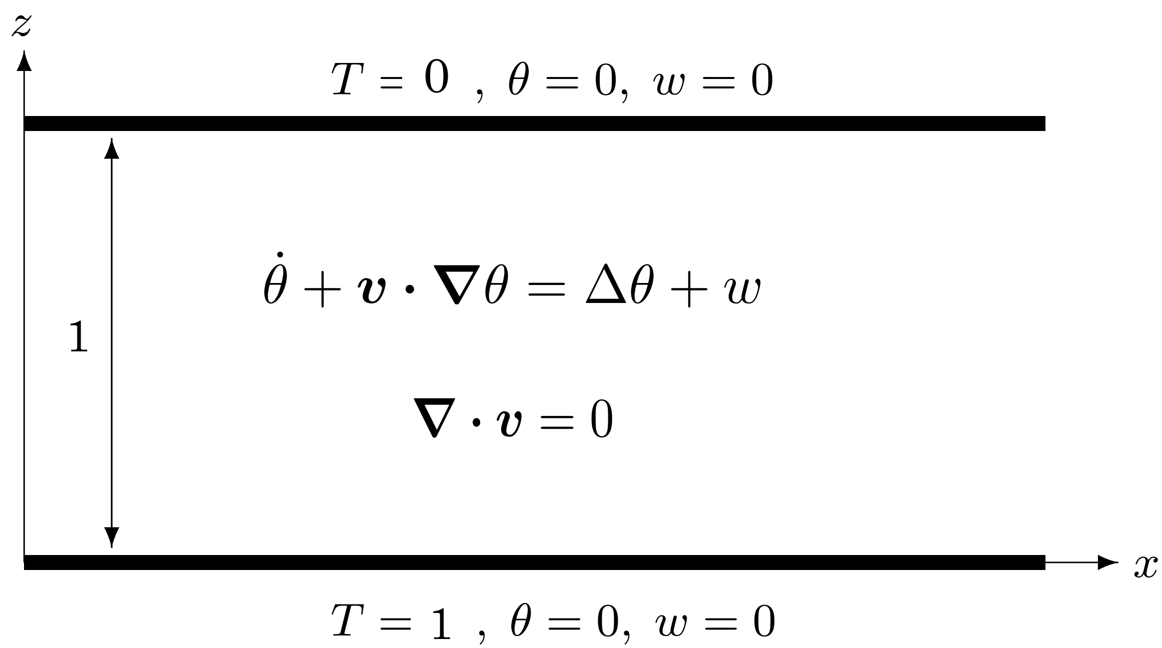

We consider velocity fields that either have fixed mean squared speed

$$U^2 = \frac{1}{hL} \int_D |\boldsymbol{v}|^2 \, \mathrm{d}x \, \mathrm{d}z$$

or fixed enstrophy density

$$\Omega^2 = \frac{1}{hL} \int_D |\boldsymbol{\omega}|^2 \, \mathrm{d}x \, \mathrm{d}z = \frac{1}{hL} \int_D |\boldsymbol{\nabla} \boldsymbol{v}|^2 \, \mathrm{d}x \, \mathrm{d}z$$

# The problem:




For the problem with fixed energy we define

$$Pe \equiv Uh/\kappa,$$

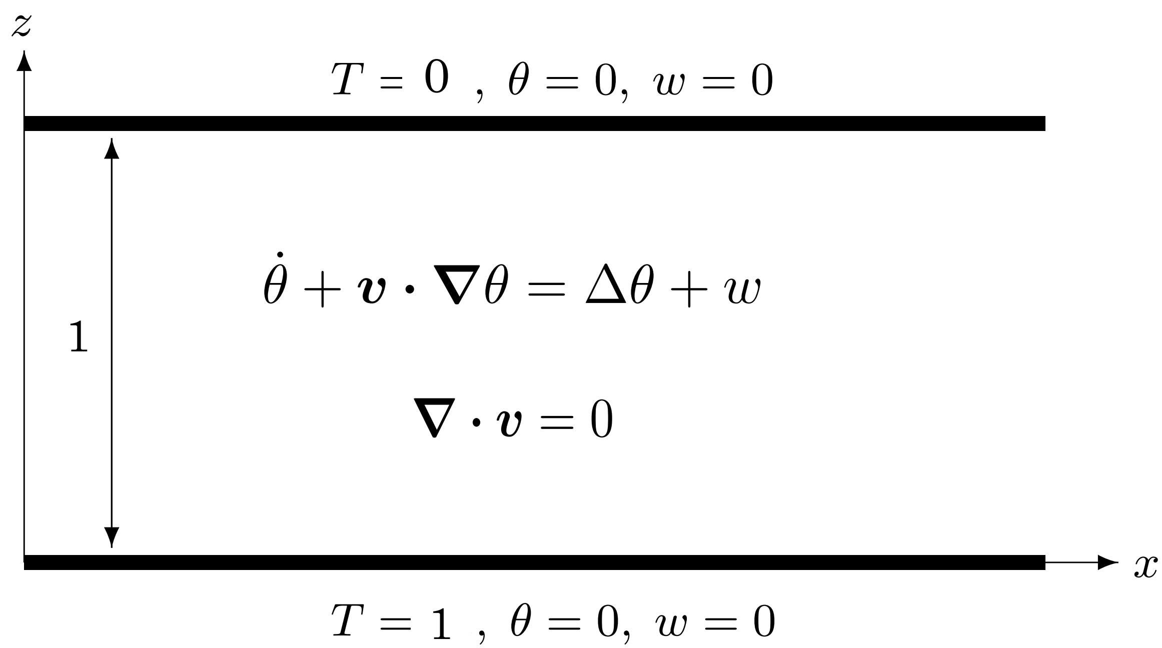
and for the problem with fixed enstrophy,

$$Pe \equiv \Omega h^2/\kappa.$$

We also define the aspect ratio $\mathit{\Gamma}$ as

$$\mathit{\Gamma} \equiv L/h.$$

# The problem:

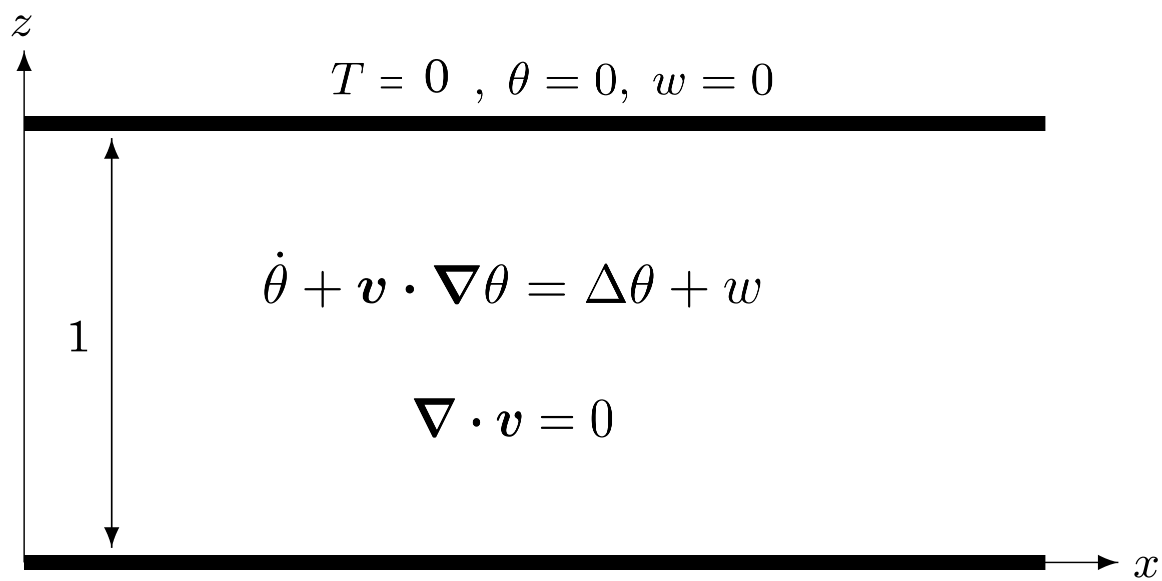



$$\langle \boldsymbol{a} \rangle \equiv \lim_{t\to\infty} \frac{1}{t} \int_0^t \left\{ \frac{1}{\Gamma} \int_D \boldsymbol{a}(x,z,s)\, \mathrm{d}x\, \mathrm{d}z \right\} \mathrm{d}s$$

$$Pe^2 = \left\langle |\boldsymbol{v}|^2 \right\rangle \quad \text{or} \quad Pe^2 = \left\langle |\boldsymbol{\nabla}\boldsymbol{v}|^2 \right\rangle$$

Vertical ***current***: $J_z = wT - \kappa \partial_z T$

# The problem:




$$T = 0\ ,\ \theta = 0,\ w = 0$$

$$\dot{\theta} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \theta = \Delta\theta + w$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$$

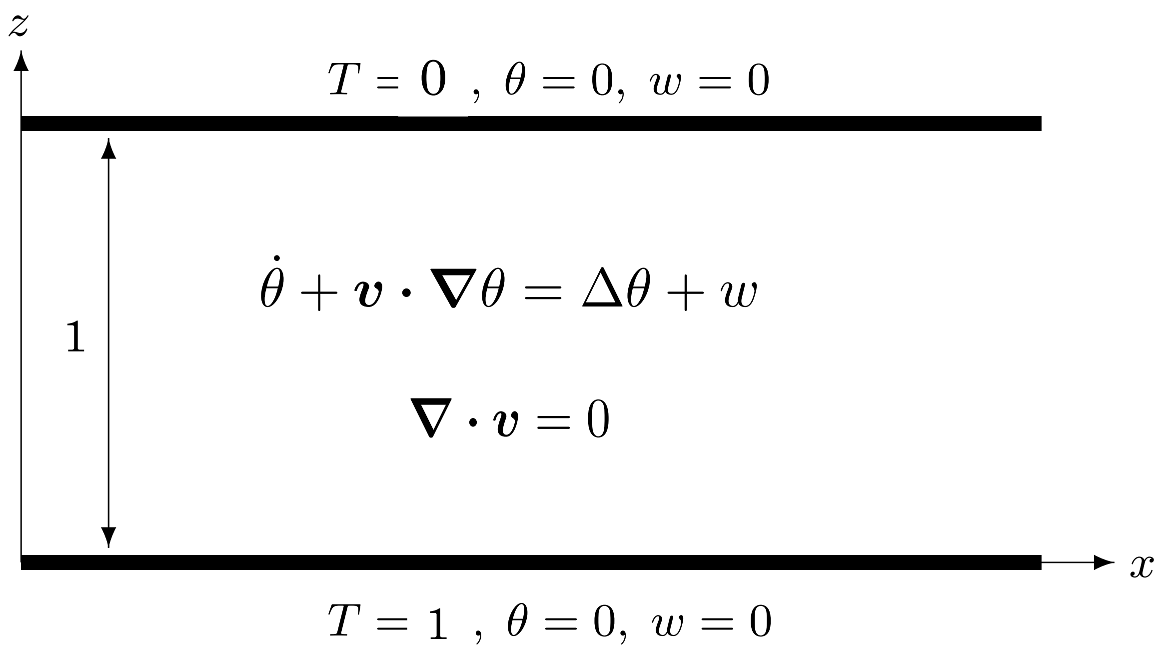
$$T = 1\ ,\ \theta = 0,\ w = 0$$

$$\langle \boldsymbol{a} \rangle \equiv \lim_{t \to \infty} \frac{1}{t} \int_0^t \left\{ \frac{1}{\Gamma} \int_D \boldsymbol{a}(x, z, s)\, \mathrm{d}x\, \mathrm{d}z \right\} \mathrm{d}s$$

$$Pe^2 = \left\langle |\boldsymbol{v}|^2 \right\rangle \quad \text{or} \quad Pe^2 = \left\langle |\boldsymbol{\nabla} \boldsymbol{v}|^2 \right\rangle$$

***Nusselt number*** $\boldsymbol{Nu} = \langle \textit{vertical current} \rangle = \left\langle \dfrac{J_z h}{\kappa(T_{bot} - T_{top})} \right\rangle$

# The problem:

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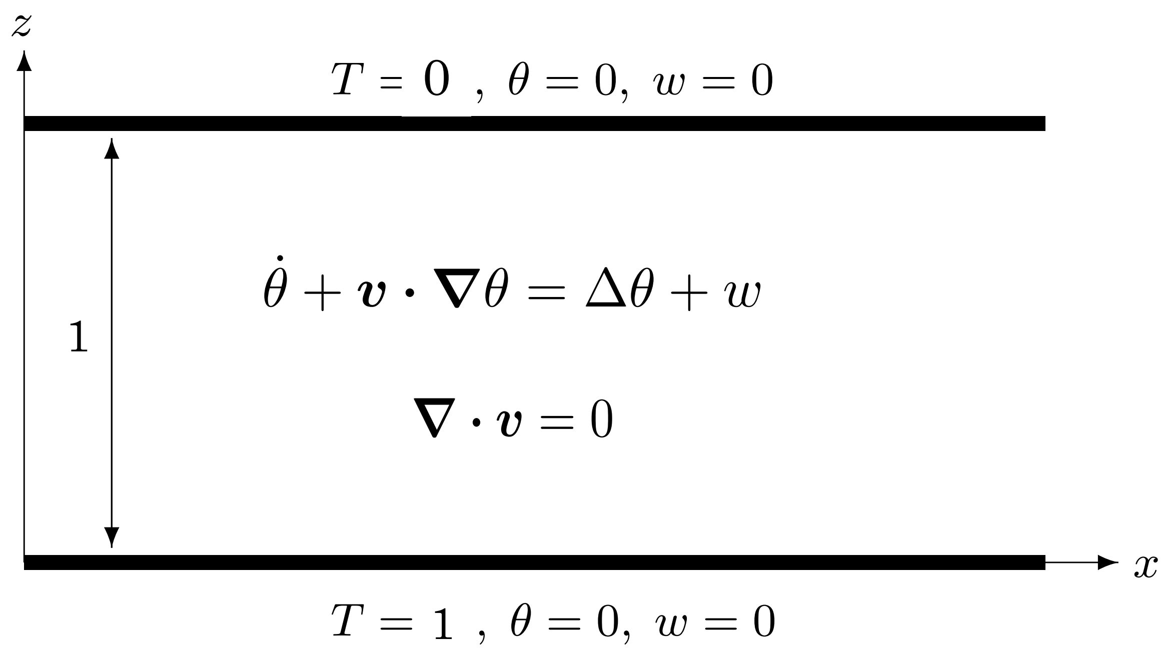
$$Pe^2 = \left\langle |\boldsymbol{v}|^2 \right\rangle \quad \text{or} \quad Pe^2 = \left\langle |\boldsymbol{\nabla v}|^2 \right\rangle$$

Wall to wall ***transport***: $Nu = 1 + \langle wT \rangle = 1 + \langle w\theta \rangle$

# The problem:



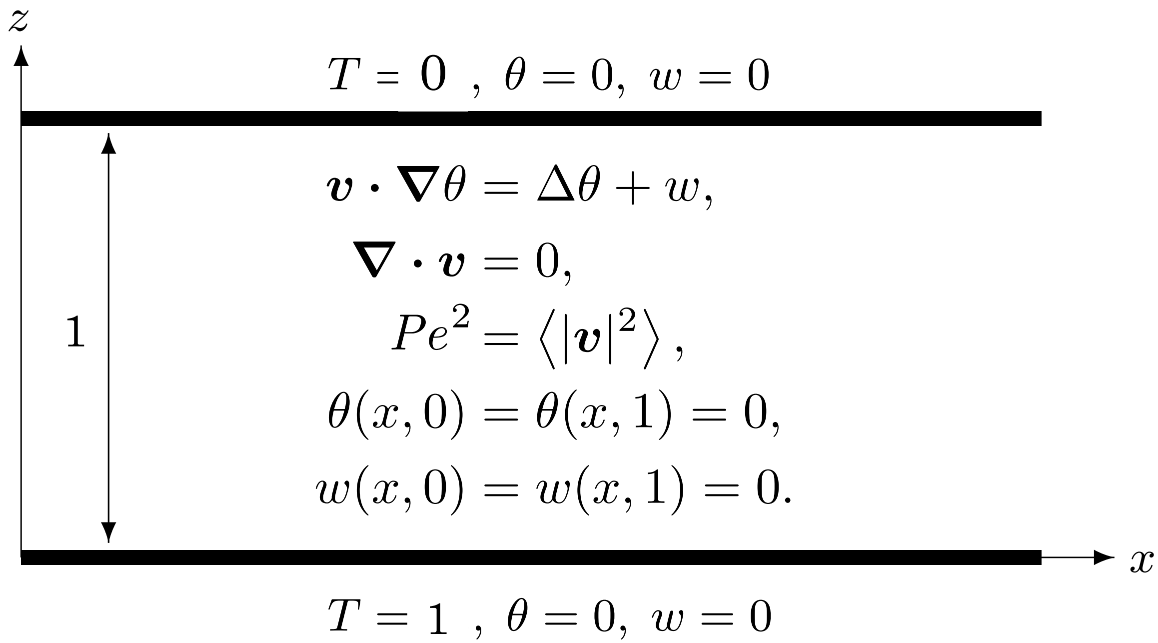

$$Nu = 1 + \langle wT \rangle = 1 + \langle w\theta \rangle$$

$$Nu_{\rm max}(Pe, \mathit{\Gamma}) \equiv \sup_{\boldsymbol{v}} \{ Nu(\boldsymbol{v}) \}$$

$$Nu_{\rm MAX}(Pe) \equiv \sup_{\mathit{\Gamma}} \{ Nu_{\rm max}(Pe, \mathit{\Gamma}) \}$$

Wall to wall ***optimal*** transport: $Nu_{\rm MAX}(Pe) = Nu_{\rm max}(Pe, \mathit{\Gamma}_{\rm opt}(Pe))$

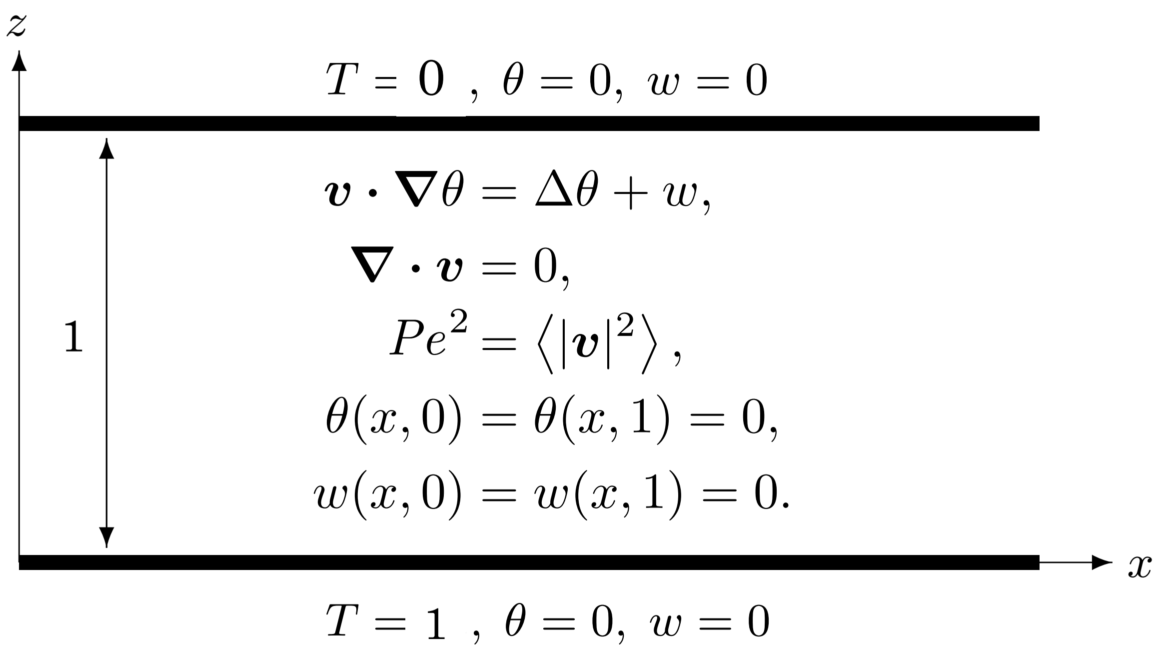
# Optimal transport with steady flow at fixed energy:




$$
\begin{aligned}
\boldsymbol{v} \cdot \boldsymbol{\nabla}\theta &= \Delta\theta + w, \\
\boldsymbol{\nabla} \cdot \boldsymbol{v} &= 0, \\
Pe^2 &= \left\langle |\boldsymbol{v}|^2 \right\rangle, \\
\theta(x,0) = \theta(x,1) &= 0, \\
w(x,0) = w(x,1) &= 0.
\end{aligned}
$$

**First**: an *a priori* estimate …

# Optimal transport with steady flow at fixed energy:




the maximum principle assures $0 \leqslant T \leqslant 1$ so $|T - 1/2| \leqslant 1/2$,

$$Nu = 1 + \langle wT \rangle = 1 + \langle w(T - 1/2) \rangle \leqslant 1 + \left\langle |w|^2 \right\rangle^{1/2} \left\langle |T - 1/2|^2 \right\rangle^{1/2}$$

$$\leqslant 1 + \frac{\left\langle |\boldsymbol{v}|^2 \right\rangle^{1/2}}{2} = 1 + \frac{Pe}{2}.$$

## Optimal transport with steady flow at fixed energy:

Calculus of variations ...

# Optimal transport with steady flow at fixed energy:

$$\mathcal{F} = \left\langle w\theta - \phi(x,z)\left(\boldsymbol{v}\cdot\boldsymbol{\nabla}\theta - \Delta\theta - w\right) + p(x,z)\left(\boldsymbol{\nabla}\cdot\boldsymbol{v}\right) - \frac{\mu}{2}\left(|\boldsymbol{v}|^2 - Pe^2\right)\right\rangle$$

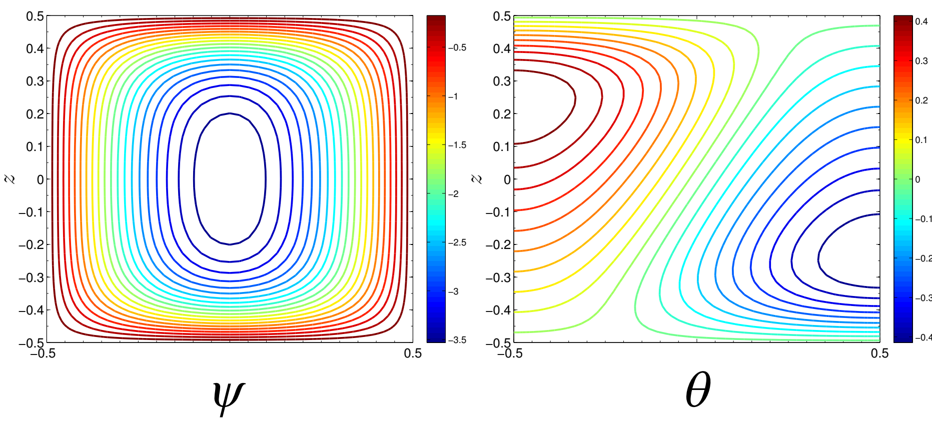
Euler-Lagrange equations are

$$\begin{aligned}
0 &= \frac{\delta\mathcal{F}}{\delta\boldsymbol{v}} = (\theta+\phi)\,\hat{\boldsymbol{z}} + \theta\,\boldsymbol{\nabla}\phi - \boldsymbol{\nabla}p - \mu\boldsymbol{v},\\
0 &= \frac{\delta\mathcal{F}}{\delta\theta} = \boldsymbol{v}\cdot\boldsymbol{\nabla}\phi + \Delta\phi + w,\\
0 &= \frac{\delta\mathcal{F}}{\delta\phi} = -\boldsymbol{v}\cdot\boldsymbol{\nabla}\theta + \Delta\theta + w,\\
0 &= \frac{\delta\mathcal{F}}{\delta p} = \boldsymbol{\nabla}\cdot\boldsymbol{v},\\
0 &= \frac{\partial\mathcal{F}}{\partial\mu} = Pe^2 - \left\langle|\boldsymbol{v}|^2\right\rangle.
\end{aligned}$$

$$\begin{aligned}
w(x,0) &= w(x,1) = 0,\\
\theta(x,0) &= \theta(x,1) = 0,\\
\phi(x,0) &= \phi(x,1) = 0.
\end{aligned}$$

# Optimal transport with steady flow at fixed energy:

As $\mu$ decreases and $Pe$ increases …




$$
\begin{aligned}
\mathrm{J}(\theta,\phi) + \mu\,\Delta\psi + (\theta+\phi)_x &= 0\\
-\mathrm{J}(\psi,\phi) + \Delta\phi - \psi_x &= 0\\
-\mathrm{J}(\psi,\theta) - \Delta\theta + \psi_x &= 0
\end{aligned}
$$

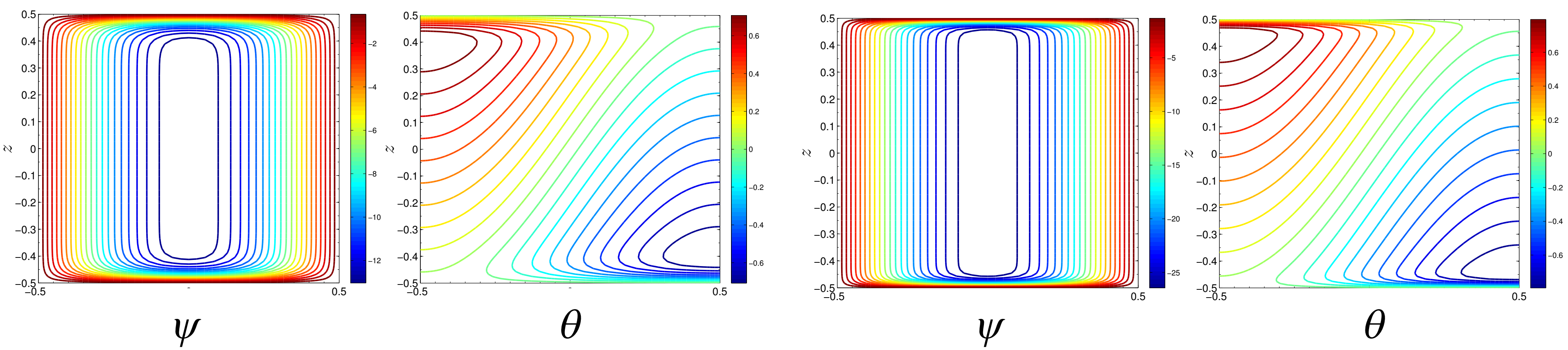



FIGURE 4. Evolution of the flow fields with $Pe$ for the case with $\Gamma = 1$. Panels on the left show $\psi$ and panels on the right show $\theta$. (a) and (b) $Pe = 10.0$, $Nu_{\max} = 2.4$; (c) and (d) $Pe = 59.4$, $Nu_{\max} = 9.7$; (e) and (f) $Pe = 161.3$, $Nu_{\max} = 20.7$. The resolution is $61^2$.

Defining the stream function $\psi$ by $u = \partial\psi/\partial z$ and $w = -\partial\psi/\partial x$

$$\text{where } \mathrm{J}(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial z} - \frac{\partial a}{\partial z}\frac{\partial b}{\partial x}$$

# Optimal transport with steady flow at fixed energy:

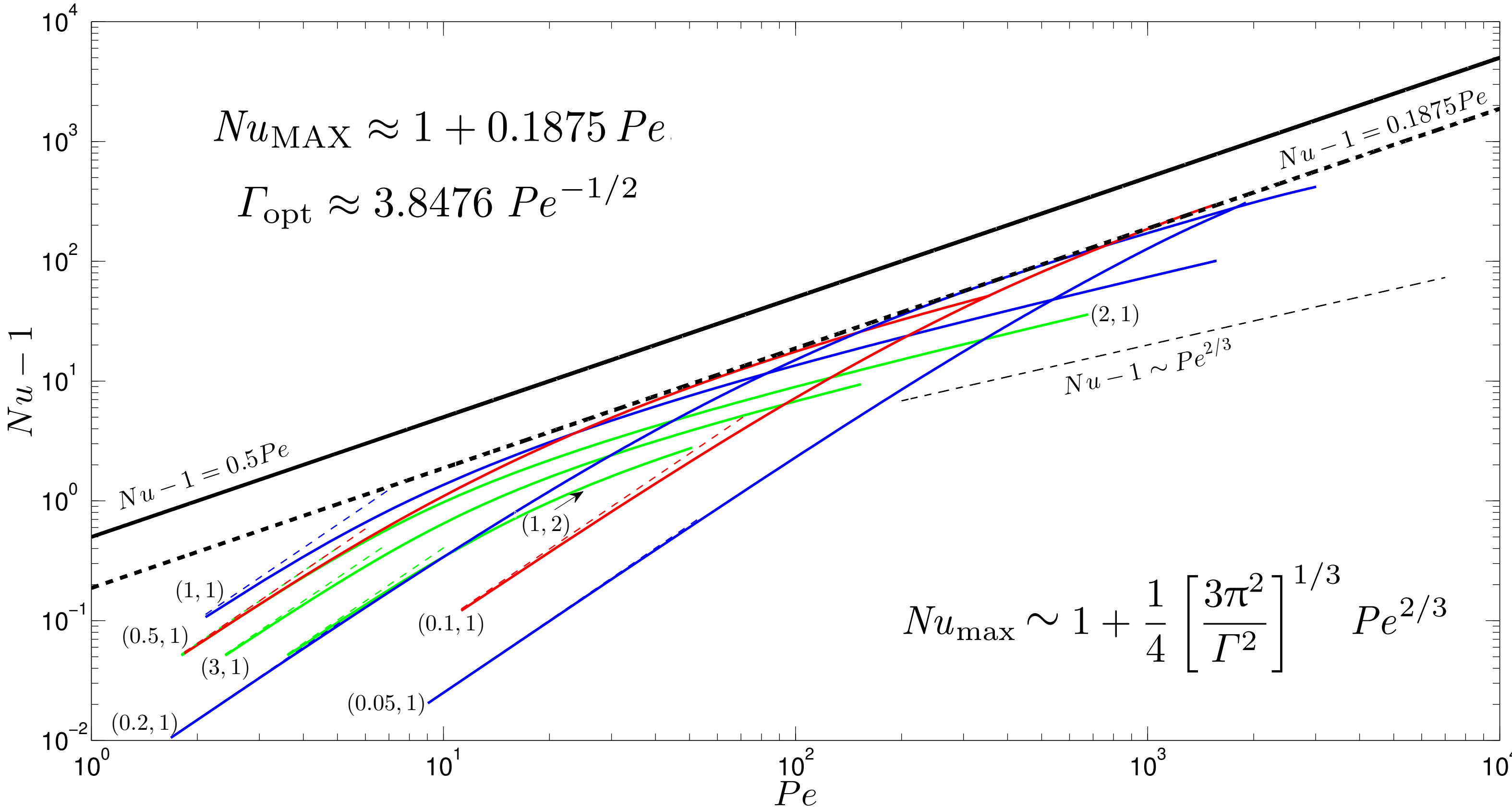
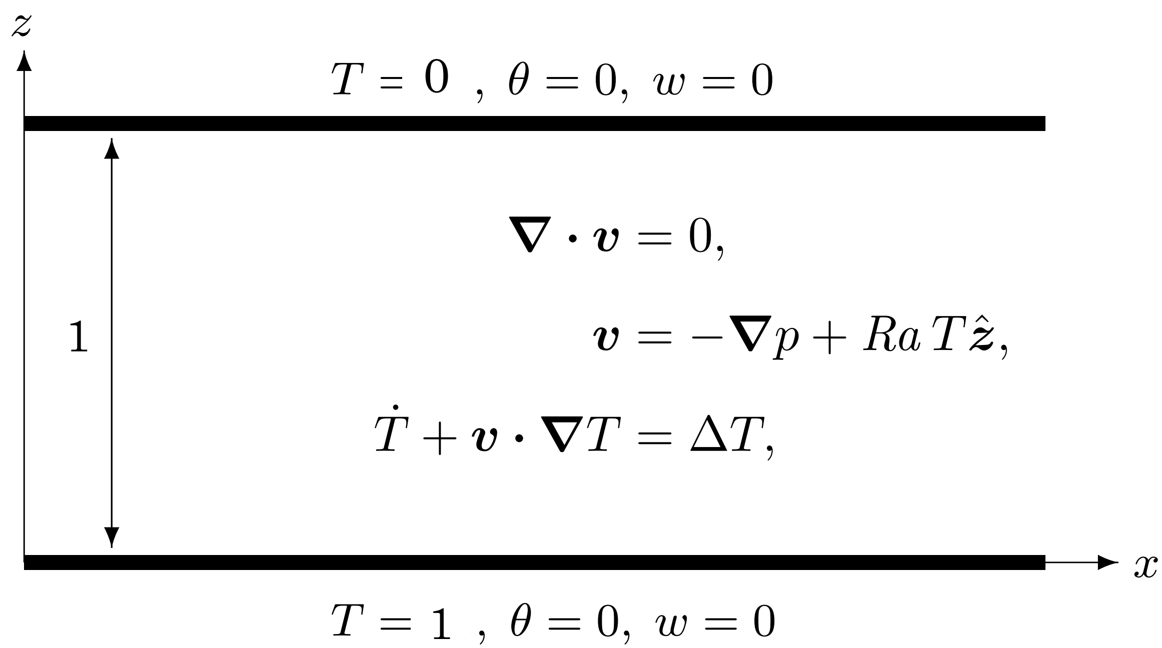


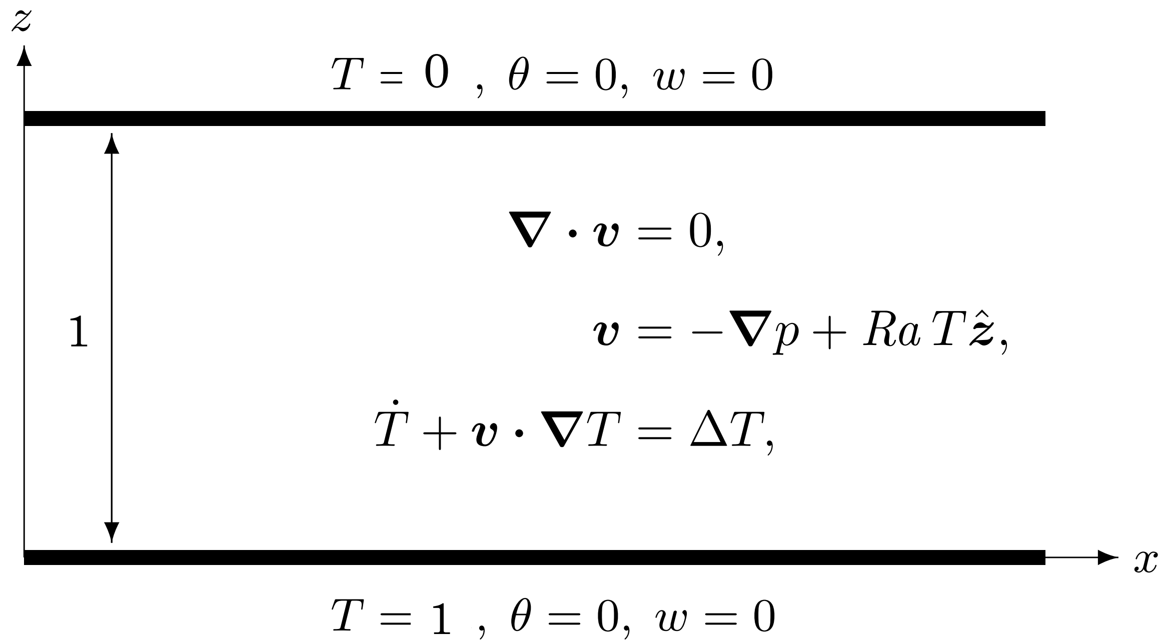

FIGURE 5. The numerically obtained $Nu_{\max}$ as a function of $Pe$ for various values of $\Gamma$ (non-black lines). The labels show $(\Gamma, m)$. For each case, the (short) dashed line of the same color, visible for most cases, shows the analytical $Nu_{\max}$ (2.31) in the small–$Pe$ limit. The thick black solid line shows the absolute upper bound (2.6), and the thick black dashed line shows the analytically obtained $Nu_{\mathrm{MAX}}$ (2.93). The thin black dashed line indicates the $Pe^{2/3}$ slope. Cases with $\Gamma > 1$ or $m > 1$ (green lines) do not produce $Nu_{\mathrm{MAX}}$. All results shown here have resolution $M = 61$. Using a higher resolution $M = 91$ results in negligible difference.

# Rayleigh-Bénard convection in a porous medium:

***active*** transport problem



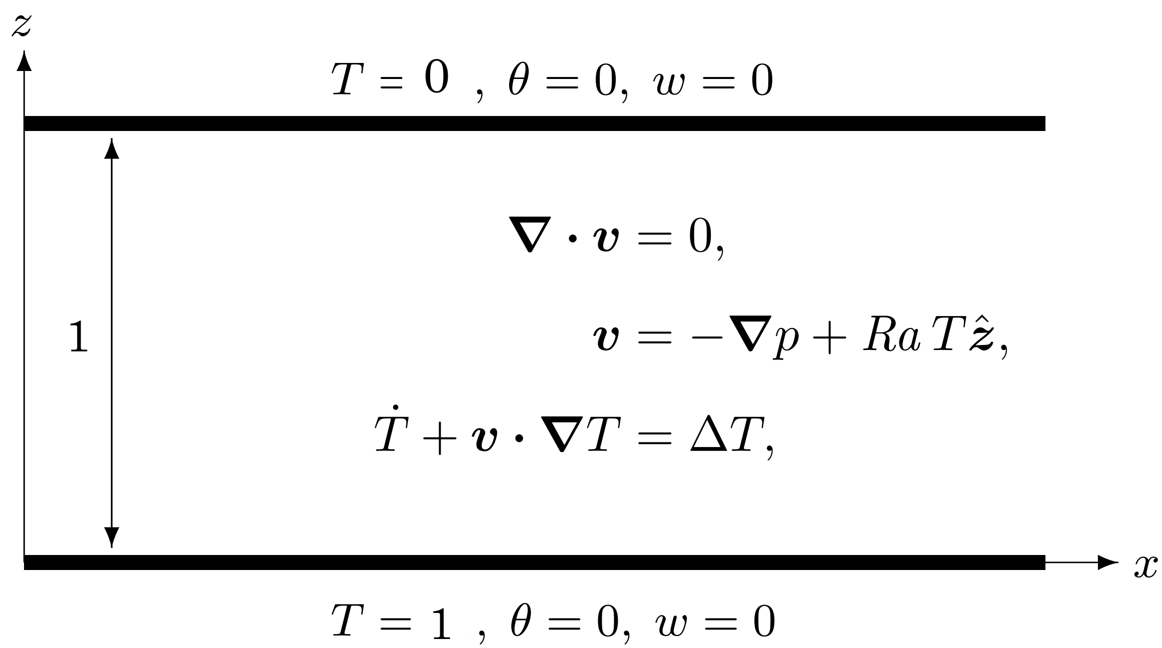
# Rayleigh-Bénard convection in a porous medium:




$J_z$ is a result of ***buoyancy***

Not to be confused with ***Jay-Z*** and ***Beyoncé***:

## Rayleigh-Bénard convection in a porous medium:




$$\left\langle |\boldsymbol{v}|^2 \right\rangle = Ra\ \langle wT \rangle$$

$$Pe^2 = Ra\,(Nu - 1)$$

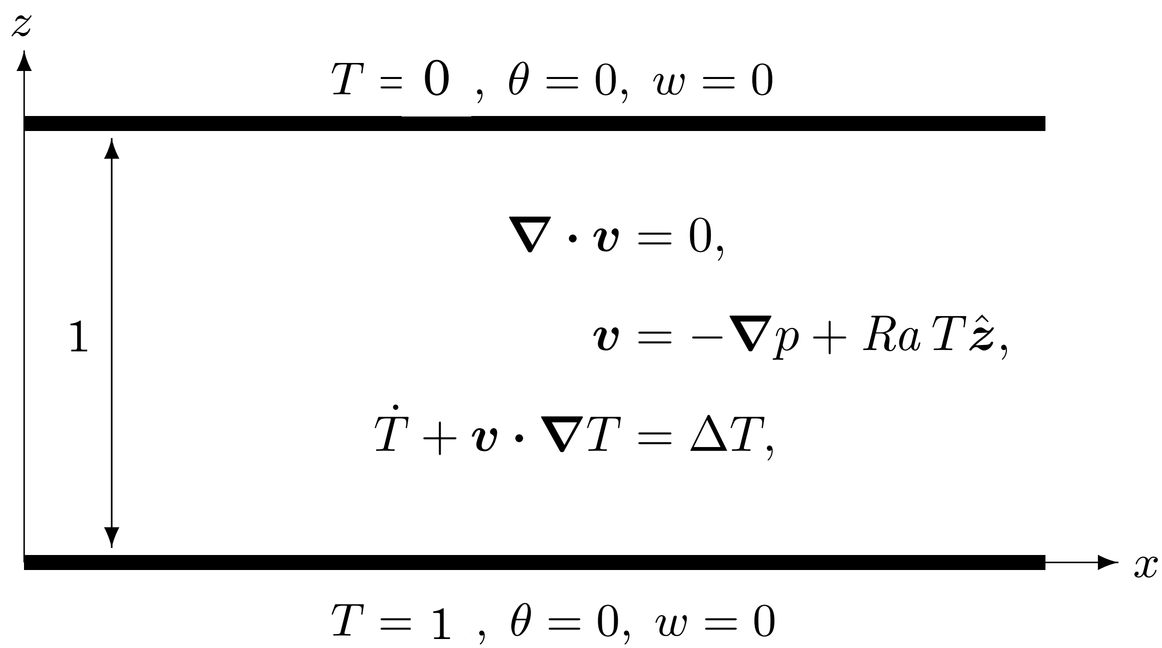
$J_z$ is a result of ***buoyancy***

$$Nu_{\max}(Ra, \mathit{\Gamma}) = 1 + \frac{\sqrt{3}\pi}{8\mathit{\Gamma}}\, Ra^{1/2},$$

$$Nu_{\mathrm{MAX}}(Ra) = 1 + 0.0352\, Ra,$$

$$\mathit{\Gamma}_{\mathrm{opt}} = 8.89\, Ra^{-1/2}.$$

# Rayleigh-Bénard convection in a porous medium:

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$$Nu_{\mathrm{MAX}}(Ra) = 1 + 0.0352\, Ra,$$

$$\mathit{\Gamma}_{\mathrm{opt}} = 8.89\, Ra^{-1/2}.$$

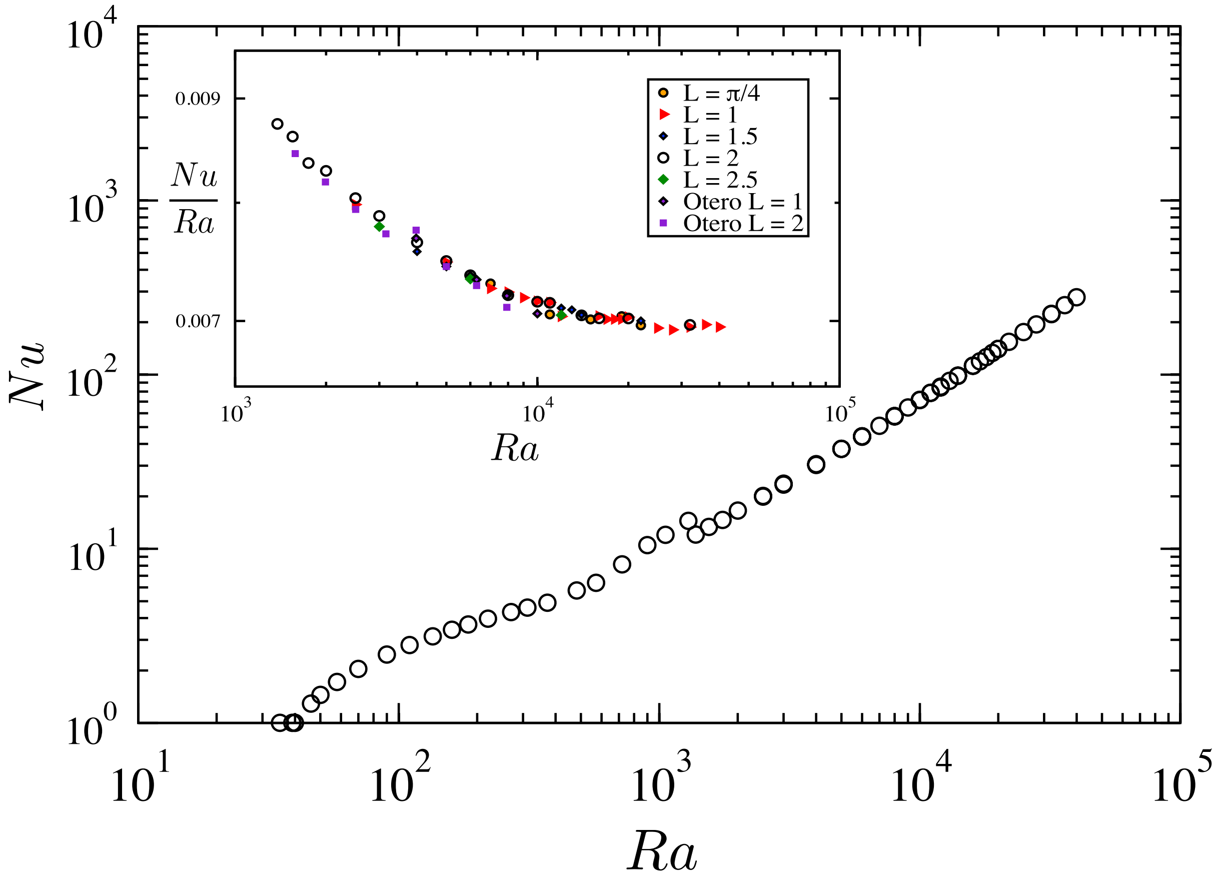
***More*** than ***steady*** transport via Darcy's Law allows!

# Rayleigh-Bénard convection in a porous medium:

PRL 108, 224503 (2012) PHYSICAL REVIEW LETTERS week ending 1 JUNE 2012

## Ultimate Regime of High Rayleigh Number Convection in a Porous Medium

Duncan R. Hewitt,[1] Jerome A. Neufeld,[1,2,3] and John R. Lister[1]
[1]Institute of Theoretical Geophysics, DAMTP, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, United Kingdom
[2]Department of Earth Sciences, University of Cambridge, Cambridge, United Kingdom
[3]BP Institute, University of Cambridge, Cambridge, United Kingdom




**Direct numerical simulations**

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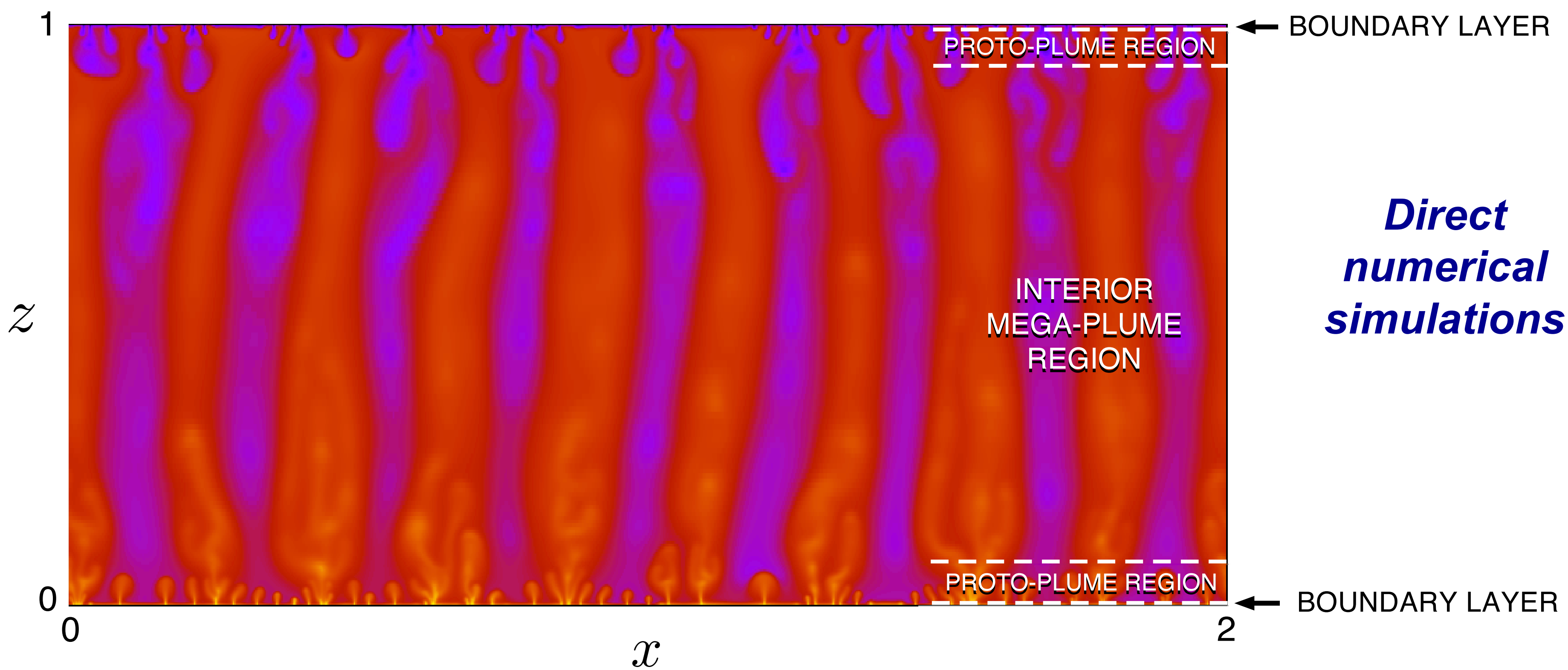


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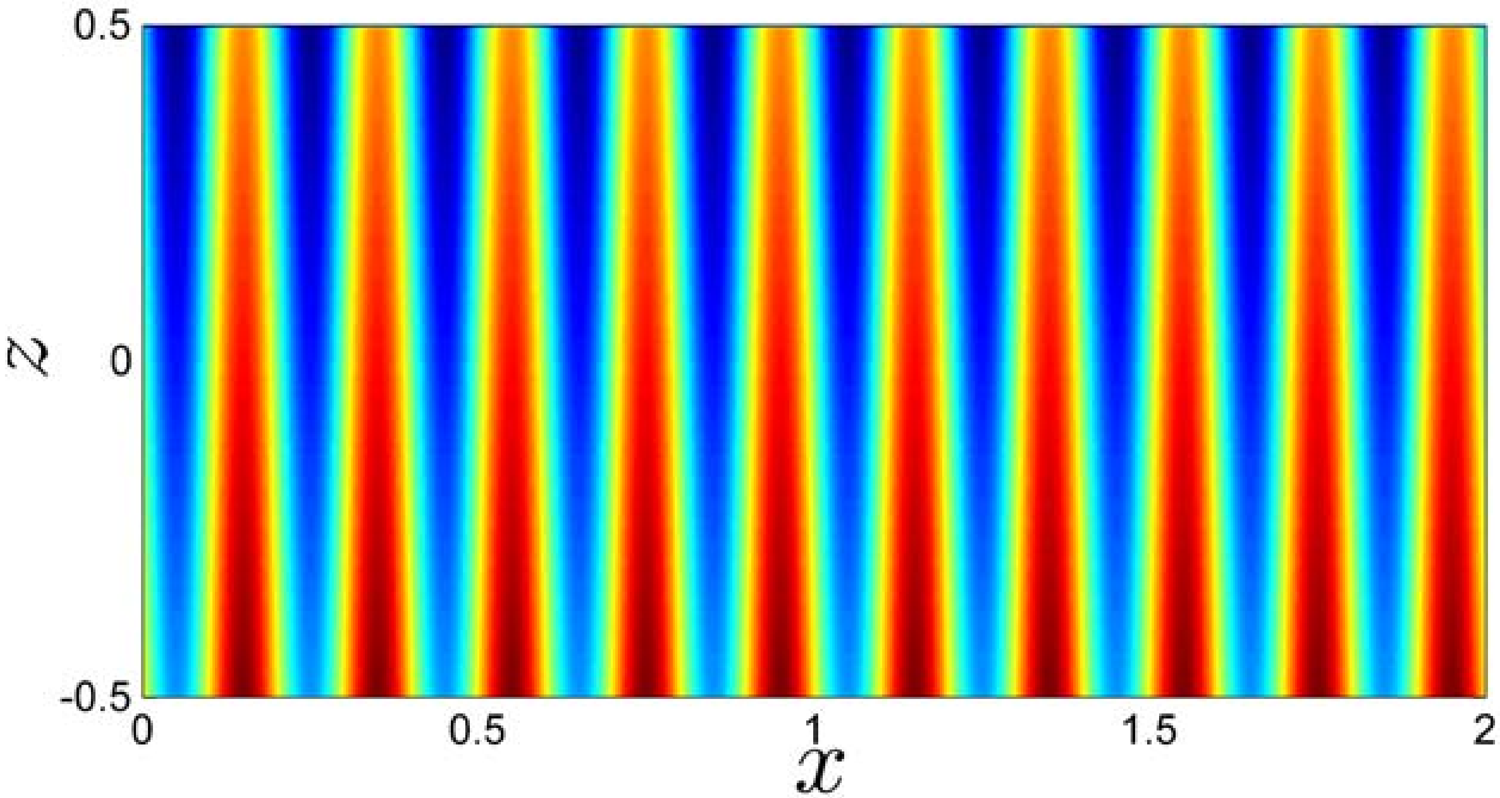


BOUNDARY LAYER

***Optimal (steady) transport***

BOUNDARY LAYER

# Rayleigh-Bénard convection in a porous medium:

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**Ultimate Regime of High Rayleigh Number Convection in a Porous Medium**

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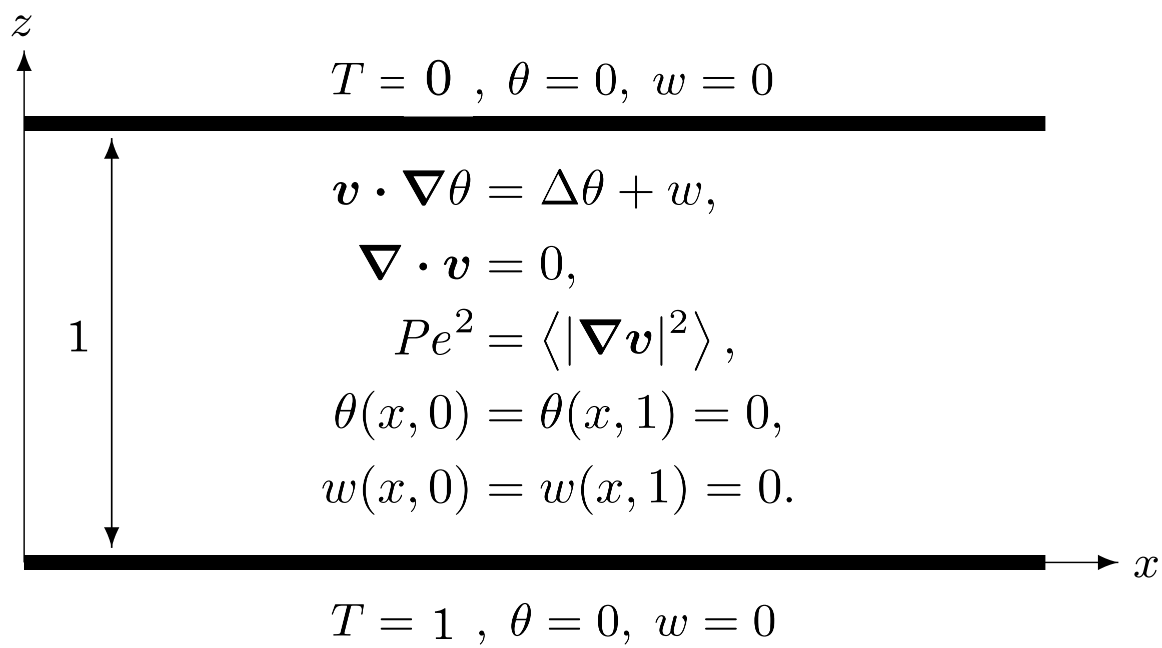
$\therefore$ Natural convection ***can*** realize $Nu \sim Nu_{MAX} \sim Ra$.

# Optimal transport with steady flow at fixed enstrophy:




$$
\begin{aligned}
\boldsymbol{v} \cdot \boldsymbol{\nabla}\theta &= \Delta\theta + w, \\
\boldsymbol{\nabla} \cdot \boldsymbol{v} &= 0, \\
Pe^2 &= \left\langle |\boldsymbol{\nabla} \boldsymbol{v}|^2 \right\rangle, \\
\theta(x,0) &= \theta(x,1) = 0, \\
w(x,0) &= w(x,1) = 0.
\end{aligned}
$$

Maximum principle & Cauchy-Schwarz yields $Nu_{\text{MAX}} \leq c\,Pe$

… but the '***background method***' yields $Nu_{\text{MAX}} \leq c\,Pe^{2/3}$

# An Optimal Control Approach to Bounding Transport Properties of Thermal Convection

by

Andre N. Souza

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Applied and Interdisciplinary Mathematics)
in the University of Michigan
2016

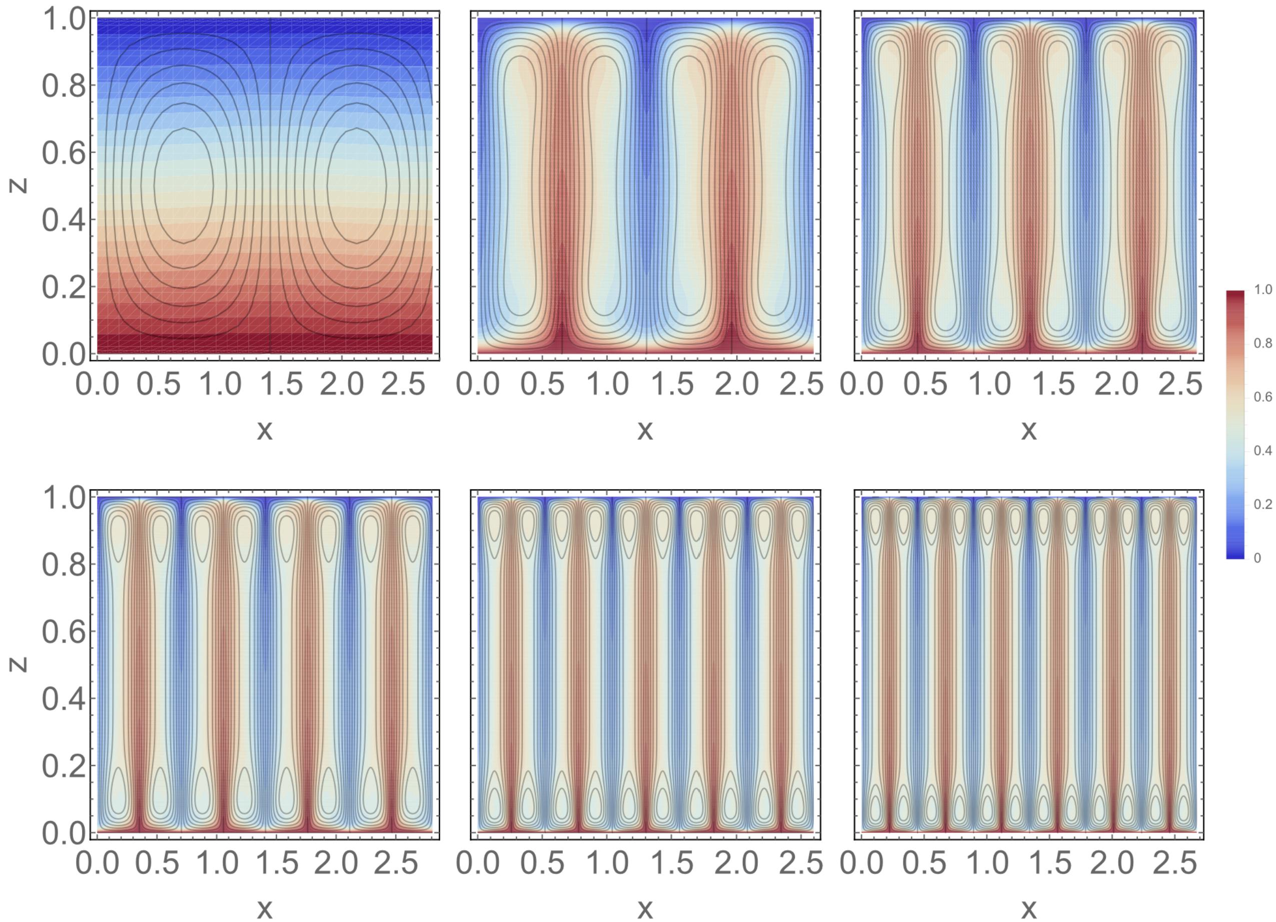



Figure 4.2: Optimal stress-free solutions for different enstrophy budgets in the full domain. The black contour lines are the streamlines and the colors represent the temperature field. From left to right, top to bottom the Péclet numbers are $4.0\times10^{-1}$, $5.0\times10^{2}$, $1.6\times10^{3}$, $3.2\times10^{3}$, $8.0\times10^{3}$, $1.2\times10^{4}$. The best known optimizers consist of a multiplicity of convection cells for a fixed domain size as the enstrophy budget increases.

# An Optimal Control Approach to Bounding Transport Properties of Thermal Convection

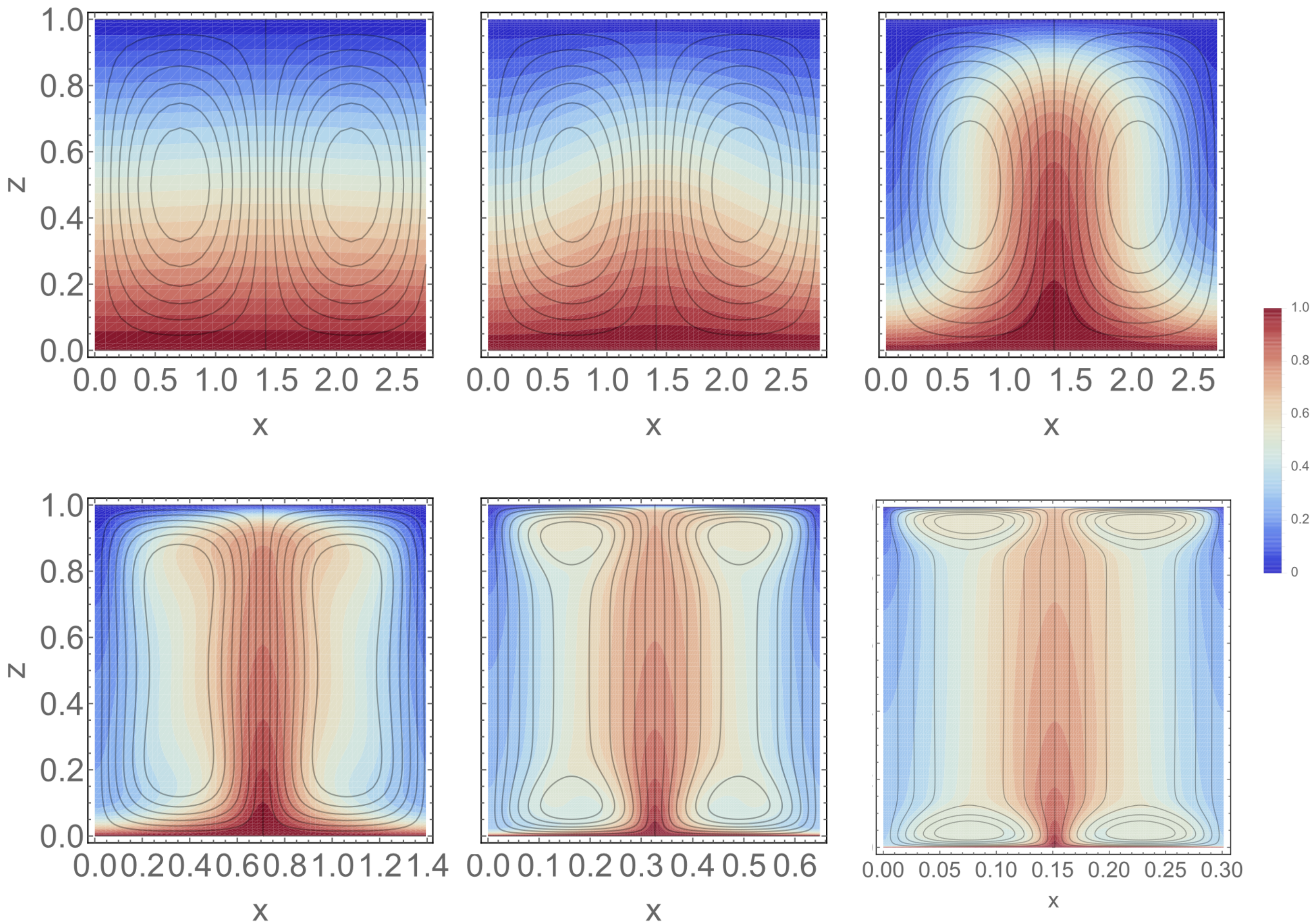



Figure 4.4: Optimal stress-free solutions for different enstrophy budgets in a single cell. The black contour lines are the streamlines and the colors represent the temperature field. From left to right, top to bottom the Péclet numbers are $4.0\times10^{-1}, 4.0\times10^{0}, 4.0\times10^{1}, 4.0\times10^{2}, 4.0\times10^{3}, 4.0\times10^{4}$. The domain size in the horizontal $x$ direction shrinks as the Péclet number increases.

# An Optimal Control Approach to Bounding Transport Properties of Thermal Convection

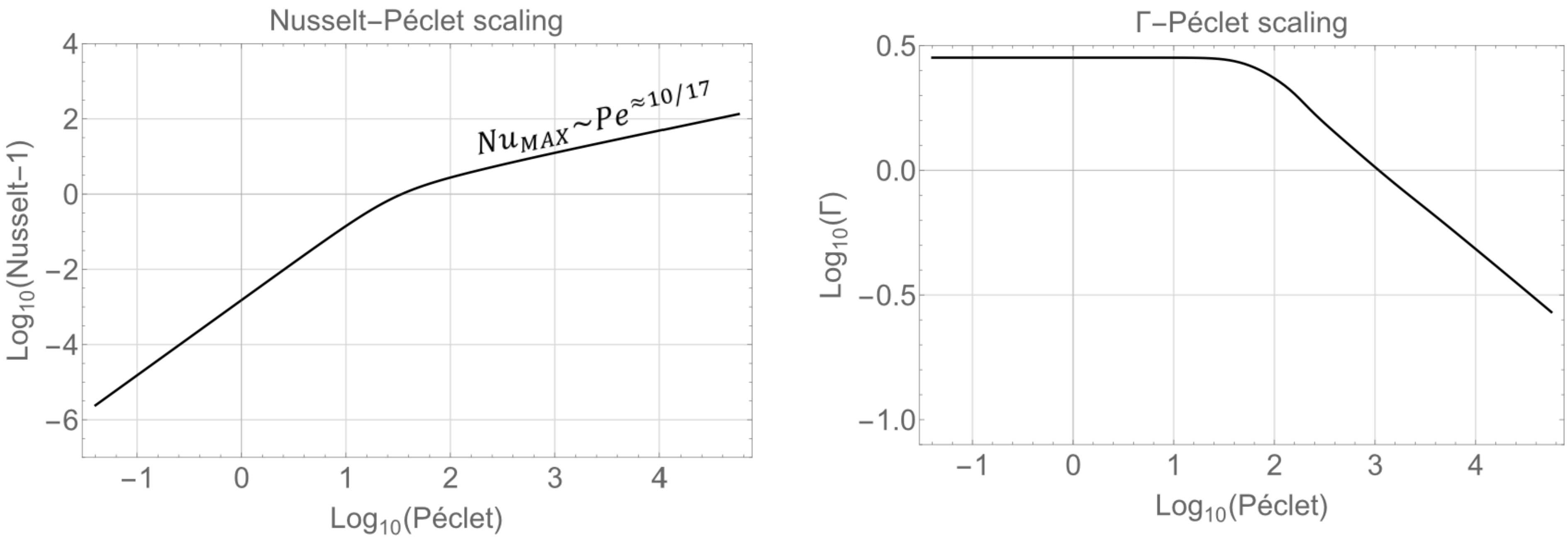



Figure 4.6: Computed optimal Nusselt number (Nu) and aspect ratio ($\Gamma$) as a function of the enstrophy budget (Pe), for stress-free boundary conditions.

# Optimal Wall-to-Wall Transport by Incompressible Flows

Charlie

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**Optimal Wall-to-Wall Transport by Incompressible Flows**

$$Nu(\mathbf{u}) - 1 = \min_{\eta:\eta|_{\partial\Omega}=0} \left\langle |\nabla\eta|^2 \right\rangle + \left\langle |\nabla\Delta^{-1}(-w + \mathbf{u}\cdot\nabla\eta)|^2 \right\rangle$$

$$= \max_{\xi:\xi|_{\partial\Omega}=0} 2\left\langle w\xi \right\rangle - \left\langle |\nabla\Delta^{-1}\mathbf{u}\cdot\nabla\xi|^2 \right\rangle - \left\langle |\nabla\xi|^2 \right\rangle$$

$$\therefore\ Nu_{MAX} = \max_{\mathbf{u},\xi} \uparrow$$

**Optimal Wall-to-Wall Transport by Incompressible Flows**

**Theorem:** there exist incompressible steady 2D flows $\mathbf{u}_{Pe}$ with

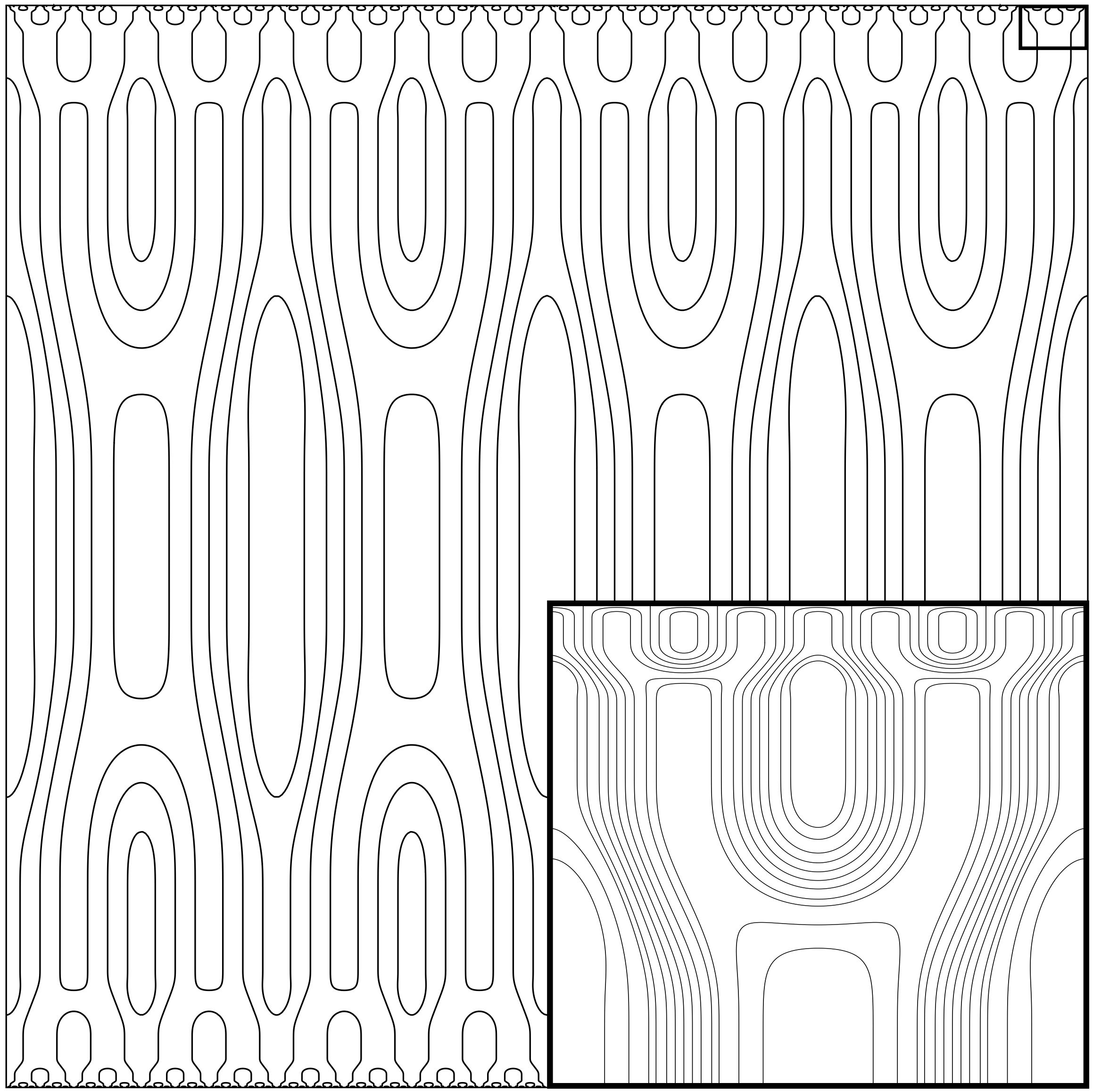
$$\left\langle \left| \nabla \mathrm{u}_{Pe}^2 \right| \right\rangle \leq Pe^2$$

and

$$Nu(\mathrm{u}_{Pe}) - 1 > \frac{Pe^{2/3}}{(\log Pe)^{4/3}}$$

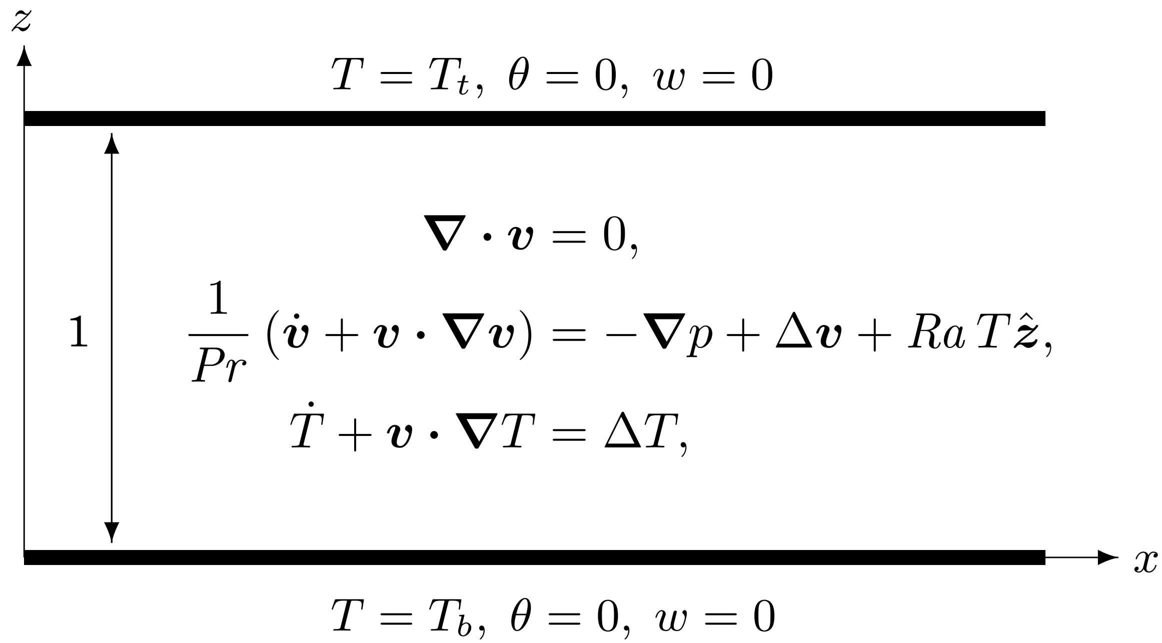
$$\left( \text{Recall: } Nu_{\text{MAX}} \leq c\, Pe^{2/3} \right)$$

# Optimal Wall-to-Wall Transport by Incompressible Flows



# Rayleigh-Bénard convection:

*active* transport problem



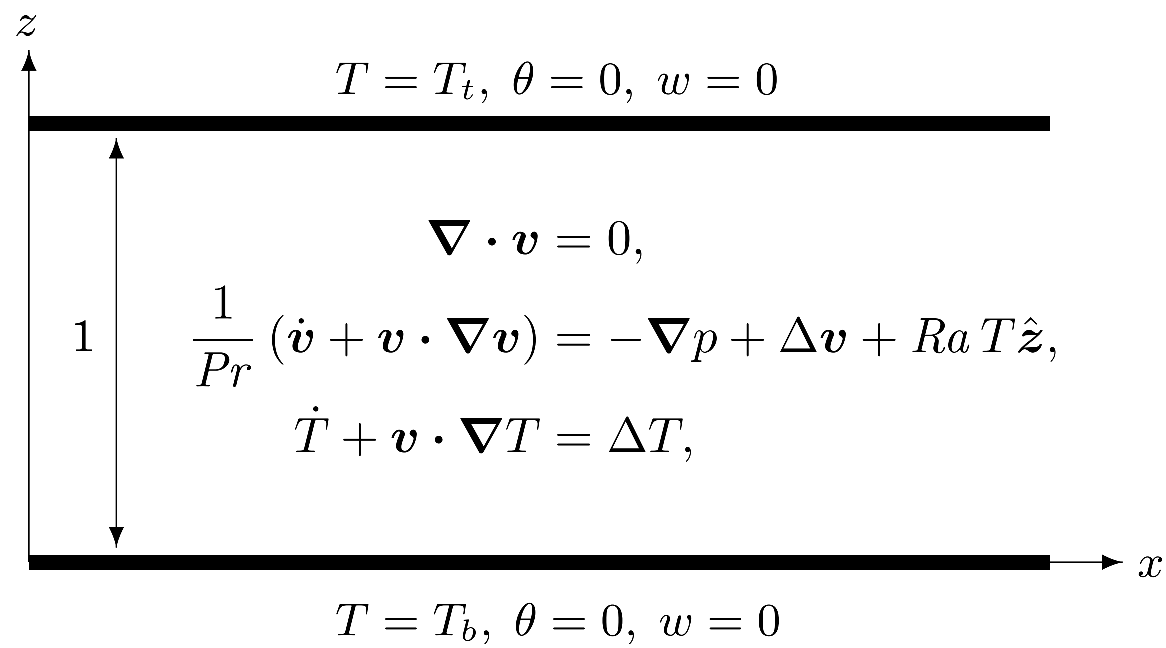

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0,$$

$$\frac{1}{Pr}\left(\dot{\boldsymbol{v}} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}\right) = -\boldsymbol{\nabla} p + \Delta \boldsymbol{v} + Ra\, T \hat{\boldsymbol{z}},$$

$$\dot{T} + \boldsymbol{v} \cdot \boldsymbol{\nabla} T = \Delta T,$$

**Lord Rayleigh 1916**

# Rayleigh-Bénard convection:



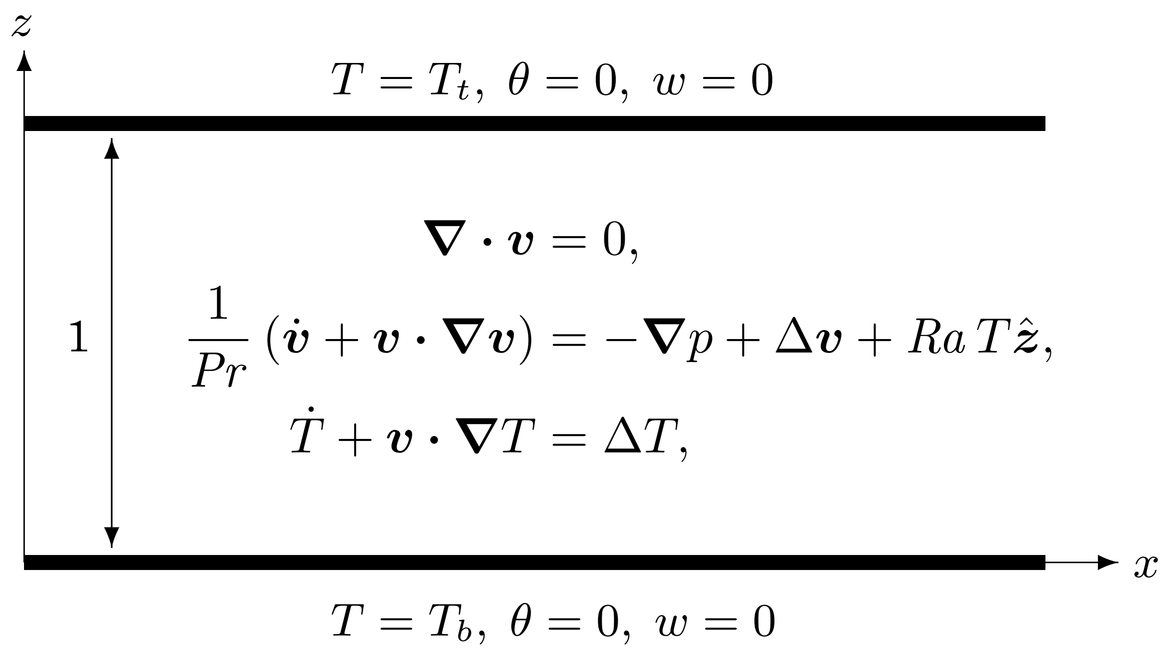

$$\begin{aligned} \nabla \cdot \boldsymbol{v} &= 0, \\ \frac{1}{Pr}\left(\dot{\boldsymbol{v}} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\right) &= -\nabla p + \Delta\boldsymbol{v} + Ra\,T\hat{\boldsymbol{z}}, \\ \dot{T} + \boldsymbol{v}\cdot\nabla T &= \Delta T, \end{aligned}$$

$$\left\langle |\nabla \boldsymbol{v}|^2 \right\rangle = Ra\ \langle wT \rangle$$

$$Pe^2 = Ra\,(Nu - 1)$$

$$c\,Pe^{2/3}/(\log Pe)^{4/3} \le\ Nu_{\text{MAX}} \le\ c'\,Pe^{2/3}$$

# Rayleigh-Bénard convection:




$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0,$$

$$\frac{1}{Pr}\left(\dot{\boldsymbol{v}} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}\right) = -\boldsymbol{\nabla} p + \Delta \boldsymbol{v} + Ra\, T \hat{\boldsymbol{z}},$$

$$\dot{T} + \boldsymbol{v} \cdot \boldsymbol{\nabla} T = \Delta T,$$

$$\left\langle |\boldsymbol{\nabla} \boldsymbol{v}|^2 \right\rangle = Ra\ \langle wT \rangle$$

$$Pe^2 = Ra\,(Nu - 1)$$

i.e., $c_1 \dfrac{\boldsymbol{Ra}^{1/2}}{(\log \boldsymbol{Ra})^2} \leq Nu_{MAX} \leq c_2\, \boldsymbol{Ra}^{1/2}$

# Rayleigh-Bénard convection:

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## Ultimate State of Two-Dimensional Rayleigh-Bénard Convection between Free-Slip Fixed-Temperature Boundaries

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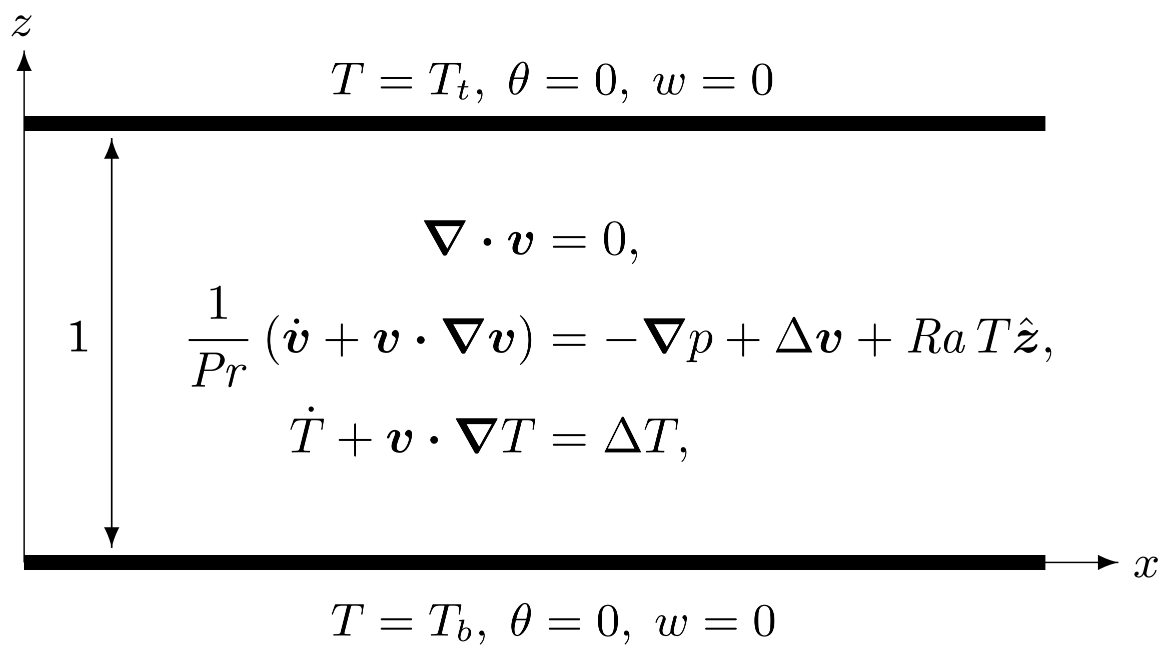
**Theorem:** $$\mathrm{Nu} \leq \frac{5^{7/12} \times 3^{3/4}}{2^{13/3}} \mathrm{Ra}^{5/12} - \frac{1}{4} \lesssim 0.2891 \mathrm{Ra}^{5/12}$$

# Rayleigh-Bénard convection:




Lord Rayleigh 1916

Can natural convection realize $Nu \sim Nu_{MAX} \sim Ra^{1/2}$ (mod logs)?

$$\frac{5}{12} < \frac{6}{12} = \frac{1}{2}$$

$\therefore$ **NOT GENERALLY!**

# THANKS FOR YOUR ATTENTION