

Integrable unitary dynamics and Knizhnik-Zamolodchikov equations

Emil Yuzbashyan



Coherent Many-Body Dynamics

(Far from Equilibrium Many-Body Physics)

Emerged around 2004 as one of the main fields in condensed matter physics

$$i\frac{\partial\Psi}{\partial t} = \hat{H}(t)\Psi$$


Interesting interacting many-body Hamiltonian of your choice

Reason: major experimental advances in 2000s

- ✓ Precise time-dependent control over system Hamiltonian
- ✓ Decoupling from environment

Examples:

- ✓ Pump-probe spectroscopy in solid state systems (e.g. superconducting films)
- ✓ Ultra-cold atomic gases
- ✓ Qubits and other quantum devices

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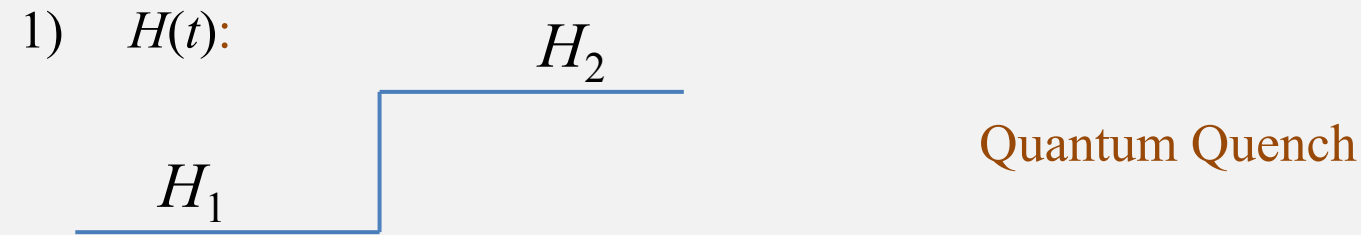

Interesting interacting many-body Hamiltonian of your choice

Want: $\Psi(t \rightarrow \infty) = ?$

Fundamental interest: New states of matter with properties unlike any equilibrium phase

For example, spontaneous emergence of oscillatory (in time) order, of spatial inhomogeneities, new topological properties, coexistence etc.

Three Main Cases

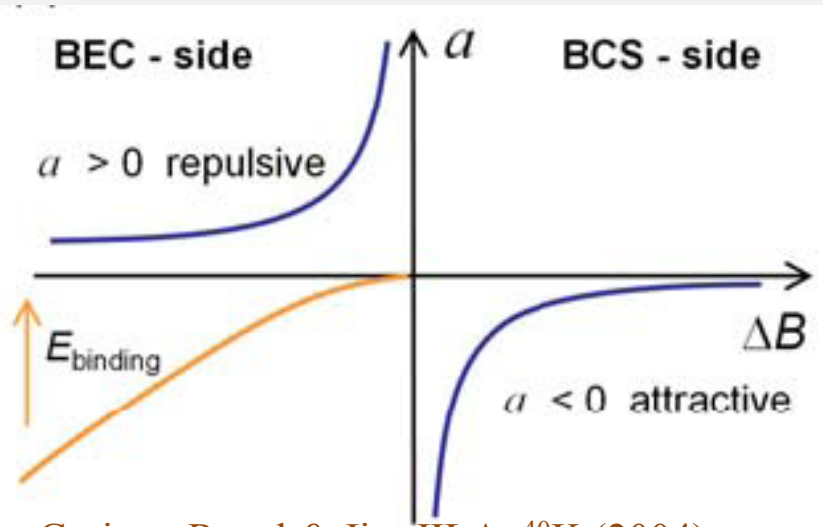


2) $H(t)$ – periodic Floquet physics

3) $H(t)$ – general smooth Landau-Zener physics

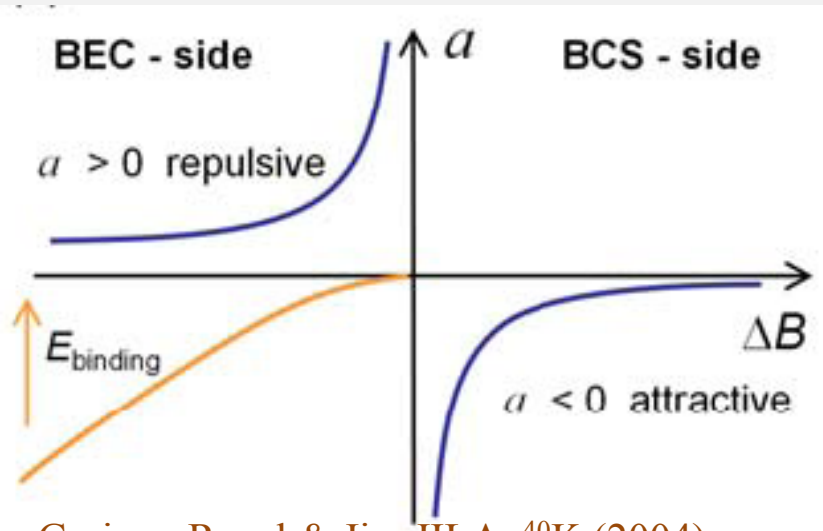
Breakdown of adiabaticity, Kibble-Zurek mechanism, adiabatic quantum computation, molecular production in cold gases

BCS-BEC Condensate of Ultracold Fermions (^{40}K , ^6Li)



Greiner, Regal & Jin, JILA, ^{40}K (2004)

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Detuning: $\omega_0 \approx 2\mu_B(B - B_0)$

Resonance width: $\gamma = \frac{g^2 \nu_F}{\varepsilon_F} \ll 1$

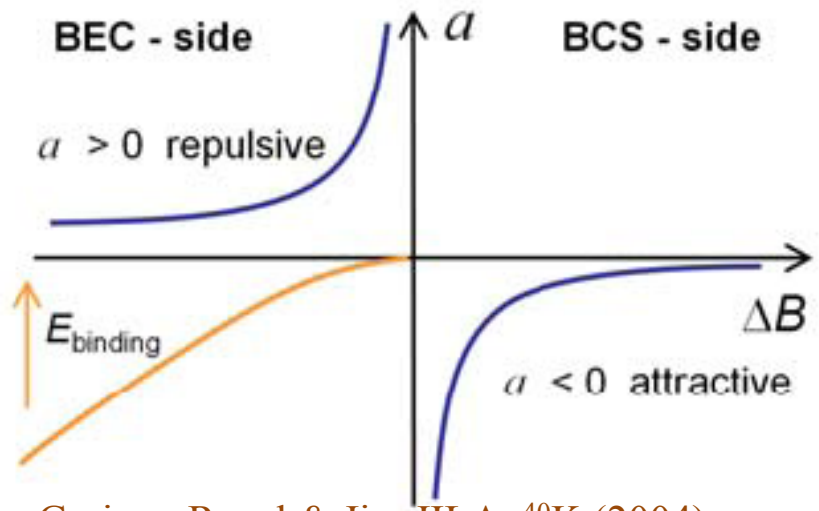
For a narrow resonance the BCS-BEC condensate is well described by the inhomogeneous Dicke model

$$\hat{H} = \sum_{\mathbf{k}, \sigma} \varepsilon_{\mathbf{k}} \hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{\mathbf{k}\sigma} + \omega_0 \hat{n}_b + g \sum_{\mathbf{k}} \left(\hat{b}^\dagger \hat{c}_{-\mathbf{k}\downarrow} \hat{c}_{\mathbf{k}\uparrow} + \hat{b} \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{-\mathbf{k}\downarrow}^\dagger \right)$$

atoms
molecules

$$\hat{n}_b = \hat{b}^\dagger \hat{b}$$

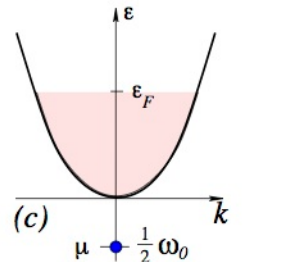
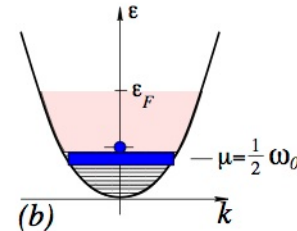
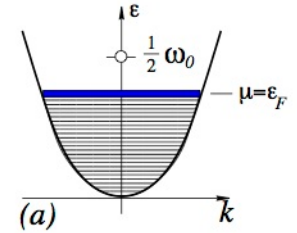
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$$\varepsilon_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m}$$

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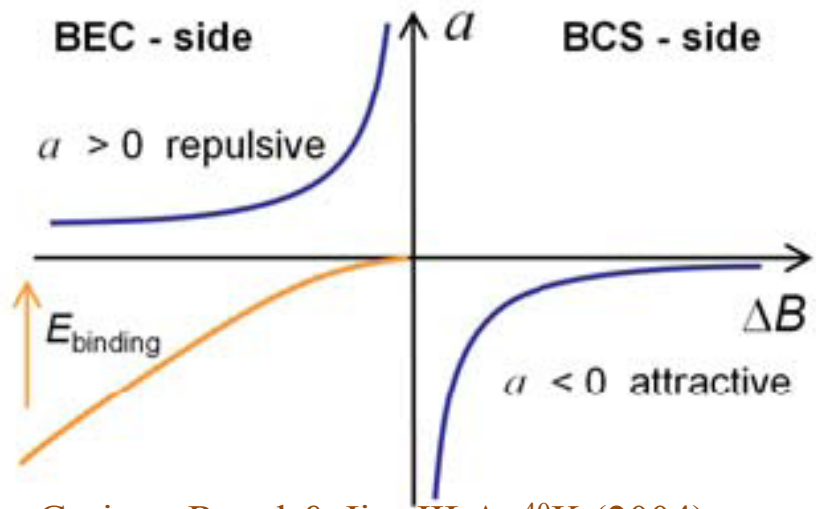
$$\hat{n}_b = \hat{b}^\dagger \hat{b}$$

Ground state:

(a) $\omega_0 \rightarrow +\infty$ BCS of atoms – no molecules, weakly paired Fermi sea

(c) $\omega_0 \rightarrow -\infty$ BEC of molecules – no atoms, everything condensed into a single mode b

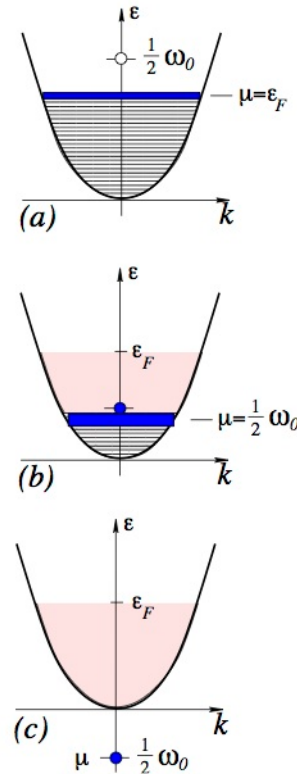
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atoms molecules

Linear sweep across the Feshbach resonance: $\omega_0 = -\nu t$ $i \frac{\partial \Psi}{\partial t} = \hat{H}(t) \Psi$

At $t \rightarrow -\infty$: $\langle \hat{n}_b \rangle = 0$, $\langle \hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{\mathbf{k}\sigma} \rangle = \theta(k - k_F)$

At $t \rightarrow +\infty$: $\langle \hat{n}_b \rangle = ?$, $\langle \hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{\mathbf{k}\sigma} \rangle = ?$

$$\hat{n}_b = \hat{b}^\dagger \hat{b}$$

Multi-level Landau-Zener problem

$$H(t) = A + Bt \qquad i\frac{\partial\Psi}{\partial t} = \hat{H}(t)\Psi$$

A, B – $N \times N$ time-independent Hermitian matrices

$$\Psi(t \rightarrow -\infty) = |\text{in}\rangle, \quad \Psi(t \rightarrow +\infty) = S|\text{in}\rangle$$

S – scattering matrix = ? Transition probabilities: $p_{i \rightarrow k} = |S_{ik}|^2$

B – diagonal (diabatic basis), $\text{Tr} B = 0$

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$N = 2$ Landau, Zener, Majorana, Stueckelberg (1932)

$$H(t) = \begin{pmatrix} 0 & g/2 \\ g/2 & 0 \end{pmatrix} + \begin{pmatrix} \lambda/2 & 0 \\ 0 & -\lambda/2 \end{pmatrix} t$$

$\Psi(t)$ – solution in terms of parabolic cylinder functions

Survival probability
(Landau-Zener formula): $p_{0 \rightarrow 0} = 1 - e^{-\frac{\pi g^2}{\lambda}} \rightarrow 1$ as $\lambda \rightarrow 0$
(adiabaticity)

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$N > 2$ No general solution, only certain special cases

Q: For which $H(t) = A + Bt$ is the multi-level Landau-Zener exactly solvable?
What is the solution?

Solvable: $p_{i \rightarrow k} = f_{\text{elem}}(A_{ij}, B_{ij})$

Exactly Solvable Multi-level Landau-Zener models

A. Trivial/reducible to 2×2 MLZ problems

$$H(t) = \begin{pmatrix} 0 & g/2 \\ g/2 & 0 \end{pmatrix} + \begin{pmatrix} \lambda/2 & 0 \\ 0 & -\lambda/2 \end{pmatrix} t = g \frac{\sigma_x}{2} + \lambda t \frac{\sigma_z}{2} \longrightarrow g S_x + \lambda t S_z$$

arbitrary rep of SU(2)

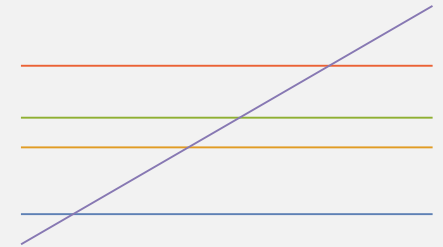
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B. Three irreducible exactly solvable MLZ problems since 1932

1. Demkov-Osherov model

Soviet Phys. JETP (1968)

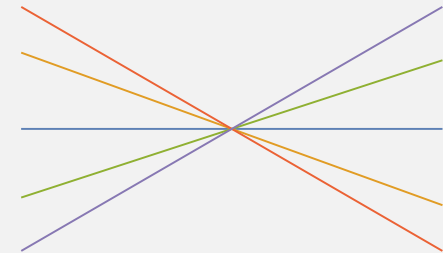
$$H(t) = \begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & a_N \end{pmatrix} + t \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$



2. Bow-tie model

Ostrovsky & Nakamura, J. Phys. A (1997)

$$H(t) = \begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & 0 \end{pmatrix} + t \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}$$



3. Inhomogeneous Dicke model

Sinitsyn, Yuzbashyan, Chernyak, Patra & Sun, PRL (2018)

$$\hat{H}(t) = \sum_{\mathbf{k}, \sigma} \varepsilon_{\mathbf{k}} \hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{\mathbf{k}\sigma} - (\nu t) \hat{n}_b + g \sum_{\mathbf{k}} \left(\hat{b}^\dagger \hat{c}_{-\mathbf{k}\downarrow} \hat{c}_{\mathbf{k}\uparrow} + \hat{b} \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{-\mathbf{k}\downarrow}^\dagger \right)$$

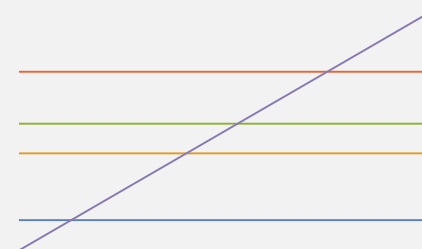
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$$S = \begin{pmatrix} p_2 \cdots p_N & q_2 p_3 \cdots p_N & q_3 p_4 \cdots p_N & q_4 p_5 \cdots p_N & \cdots & q_N \\ q_2 & p_2 & 0 & 0 & \cdots & 0 \\ p_2 q_3 & q_2 q_3 & p_3 & 0 & \cdots & 0 \\ p_2 p_3 q_4 & q_2 p_3 q_4 & q_3 q_4 & p_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p_2 \cdots p_{N-1} q_N & q_2 p_3 \cdots p_{N-1} q_N & q_3 p_4 \cdots p_{N-1} q_N & q_4 p_5 \cdots p_{N-1} q_N & \cdots & q_N \end{pmatrix}$$

$$p_k = e^{-\frac{\pi g_k^2}{\lambda}}, \quad q_k = \sqrt{1 - p_k^2}$$

$$S = S_{\text{LZ}}^{1N} \cdots S_{\text{LZ}}^{13} S_{\text{LZ}}^{12}$$

Exactly Solvable Multi-level Landau-Zener models

Q: What is special about these models? What sets them apart from any other Hamiltonian linear in time?

$$H(t) = A + Bt$$

A, B – $N \times N$ time-independent Hermitian matrices

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A: They are “Quantum integrable” matrices – have nontrivial commuting partners

$$\exists \quad \tilde{H}(t) = \tilde{A} + \tilde{B}t + \frac{\tilde{C}}{t} : \quad \left[\tilde{H}(t), H(t) \right] = 0 \quad \forall t$$

Patra & Yuzbashyan (2015)

$\tilde{A}, \tilde{B}, \tilde{C}$ – $N \times N$ Hermitian matrices

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For simplicity, let $\tilde{C} = 0$ (Demkov-Osherov, Dicke), i.e. $\tilde{H}(t) = \tilde{A} + \tilde{B}t$

Nontrivial: $\tilde{H}(t) \neq \alpha H(t) + (\beta + \gamma t)\mathbb{1}$

Typical $H(t) = A + Bt \not\equiv$ nontrivial $\tilde{H}(t) = \tilde{A} + \tilde{B}t$

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- ✓ Matrices that have nontrivial commuting partners are measure zero in the space of Hermitian matrices linear in t
- ✓ Have all hallmarks of Bethe Ansatz-solvable models (level crossings, Poisson level statistics, exact solution). We therefore proposed to call such matrices integrable.

[Owusu, Wagh & Yuzbashyan (2009), Owusu & Yuzbashyan (2011), Yuzbashyan & Shastry (2013)]

Exactly Solvable Multi-level Landau-Zener models

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Example: **inhomogeneous Dicke model**

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Anderson pseudospins: $s_{\mathbf{k}}^z \equiv \frac{1}{2} \left[c_{\mathbf{k}\uparrow}^{\dagger} c_{\mathbf{k}\uparrow} + c_{-\mathbf{k}\downarrow}^{\dagger} c_{-\mathbf{k}\downarrow} - 1 \right]$, $s_{\mathbf{k}}^{-} \equiv c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}$, $s_{\mathbf{k}}^{+} \equiv c_{\mathbf{k}\uparrow}^{\dagger} c_{-\mathbf{k}\downarrow}^{\dagger}$

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$$\hat{H}_{\mathbf{k}}(t) = (\varepsilon_{\mathbf{k}} - \nu t) s_{\mathbf{k}}^z + g (\hat{b}^{\dagger} s_{\mathbf{k}}^{-} + \hat{b} s_{\mathbf{k}}^{+}) + 2g^2 \sum_{p \neq k} \frac{\vec{s}_{\mathbf{k}} \cdot \vec{s}_{\mathbf{p}}}{\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{p}}}$$

$$[\hat{H}(t), \hat{H}_{\mathbf{k}}(t)] = [\hat{H}_{\mathbf{k}}(t), \hat{H}_{\mathbf{p}}(t)] = 0, \quad \forall t, \mathbf{k}, \mathbf{p}$$

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Patra & Yuzbashyan (2015)

Q: What is the role of these commuting partners? How do they help us solve for the dynamics of the system?

$$i \frac{\partial \Psi}{\partial t} = \hat{H}(t) \Psi$$

Want: $\Psi(t \rightarrow \infty) = ?$

They aren't conserved:

$$\frac{d\tilde{H}}{dt} = i[H, \tilde{H}] + \frac{\partial \tilde{H}}{\partial t} = \frac{\partial \tilde{H}}{\partial t} \neq 0$$

Idea: the non-stationary Schrodinger equation can be consistently embedded into a set of multi-time Schrodinger equations

Sinitsyn, Yuzbashyan, Chernyak, Patra & Sun, PRL (2018)

$$\left\{ \begin{array}{l} i \frac{\partial \Psi}{\partial t} = \hat{H} \Psi \\ i \nu \frac{\partial \Psi}{\partial x_k} = \hat{H}_k \Psi, \quad k = 1, \dots, n-1 \end{array} \right.$$

$$x_0 \equiv \nu t, \quad \hat{H}_0 \equiv \hat{H}, \quad \partial_k = \frac{\partial}{\partial x_k}, \quad \mathbf{x} = (x_0, \dots, x_{n-1})$$

$$i \nu \partial_k \Psi(\mathbf{x}) = \hat{H}_k \Psi(\mathbf{x})$$



Consistency: $\partial_j \hat{H}_k - \partial_k \hat{H}_j - i[\hat{H}_k, \hat{H}_j] = 0$

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Consistency: $\underbrace{\partial_j \hat{H}_k - \partial_k \hat{H}_j}_{\text{real}} - \underbrace{i[\hat{H}_k, \hat{H}_j]}_{\text{imaginary}} = 0$

$$\begin{cases} \partial_j \hat{H}_k = \partial_k \hat{H}_j \\ [\hat{H}_k, \hat{H}_j] = 0 \end{cases} \longleftarrow \text{Integrability}$$

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Example: inhomogeneous
Dicke model

$$\hat{H} = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} s_{\mathbf{k}}^z + \omega_0 \hat{n}_b + g \sum_{\mathbf{k}} \left(\hat{b}^\dagger s_{\mathbf{k}}^- + \hat{b} s_{\mathbf{k}}^+ \right)$$

$$\hat{H}_{\mathbf{k}} = (\varepsilon_{\mathbf{k}} + \omega_0) s_{\mathbf{k}}^z + g(\hat{b}^\dagger s_{\mathbf{k}}^- + \hat{b} s_{\mathbf{k}}^+) + 2g^2 \sum_{p \neq k} \frac{\vec{s}_{\mathbf{k}} \cdot \vec{s}_{\mathbf{p}}}{\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{p}}}$$

$$x_0 = -\omega_0 = \nu t, \quad x_k = \varepsilon_{\mathbf{k}}$$

$$\frac{\partial H_{\mathbf{k}}}{\partial \varepsilon_{\mathbf{p}}} = 2g^2 \frac{\vec{s}_{\mathbf{k}} \cdot \vec{s}_{\mathbf{p}}}{(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{p}})^2} = \frac{\partial H_{\mathbf{p}}}{\partial \varepsilon_{\mathbf{k}}}$$

$$\frac{\partial H}{\partial \varepsilon_{\mathbf{k}}} = s_{\mathbf{k}}^z = \frac{\partial H_{\mathbf{k}}}{\partial \omega_0}$$

$$\begin{cases} \partial_j \hat{H}_k = \partial_k \hat{H}_j \\ [\hat{H}_k, \hat{H}_j] = 0 \end{cases} \longleftarrow \text{Integrability}$$

Multi-level Landau-Zener problem

$$\begin{cases} i \frac{\partial \Psi}{\partial t} = \hat{H} \Psi \\ i \nu \frac{\partial \Psi}{\partial x_k} = \hat{H}_k \Psi \end{cases}$$

Formal solution: $\Psi(\boldsymbol{x}) = T \exp \left(-i \int_{\mathcal{P}} \hat{H}_k dx_k \right) \Psi(\boldsymbol{x}_0)$

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Path-independent

$\partial_j \hat{H}_k - \partial_k \hat{H}_j - i[\hat{H}_k, \hat{H}_j] = 0 \implies$ the curvature of non-abelian gauge field $\mathcal{A}_k = -i\hat{H}_k$ is zero

Multi-level Landau-Zener problem

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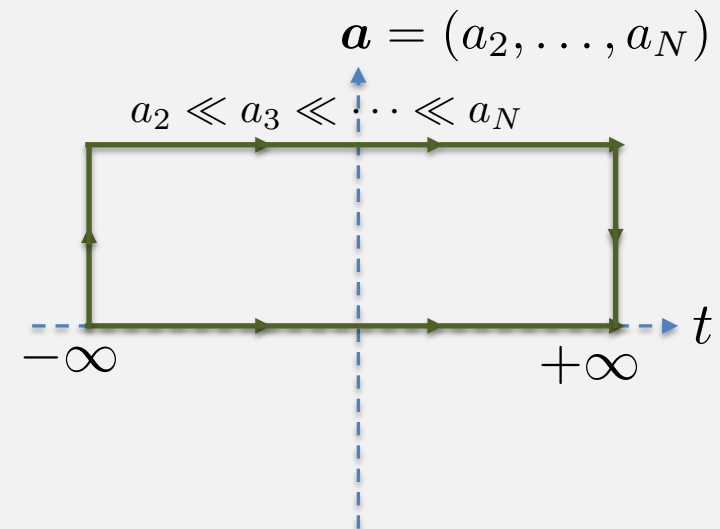
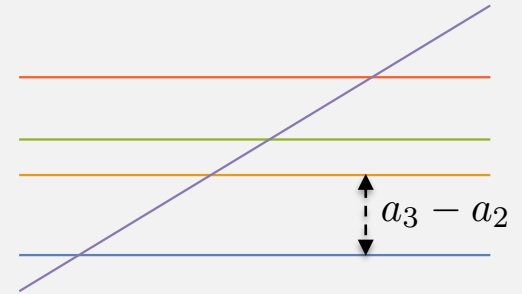
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Path-independent

Demkov-Osherov model:

$$H(t) = \begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & a_N \end{pmatrix} + t \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$x_k = a_k$$



Multi-level Landau-Zener problem

$$\begin{cases} i \frac{\partial \Psi}{\partial t} = \hat{H} \Psi \\ i \nu \frac{\partial \Psi}{\partial x_k} = \hat{H}_k \Psi \end{cases}$$

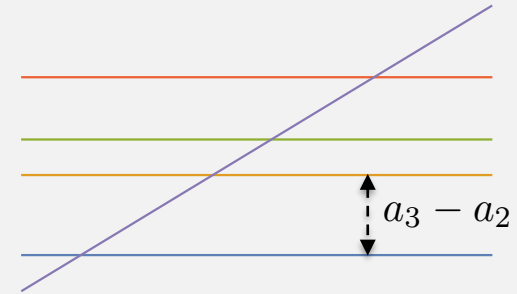
Formal solution: $\Psi(\mathbf{x}) = T \exp \left(-i \int_{\mathcal{P}} \hat{H}_k dx_k \right) \Psi(\mathbf{x}_0)$

Path-independent

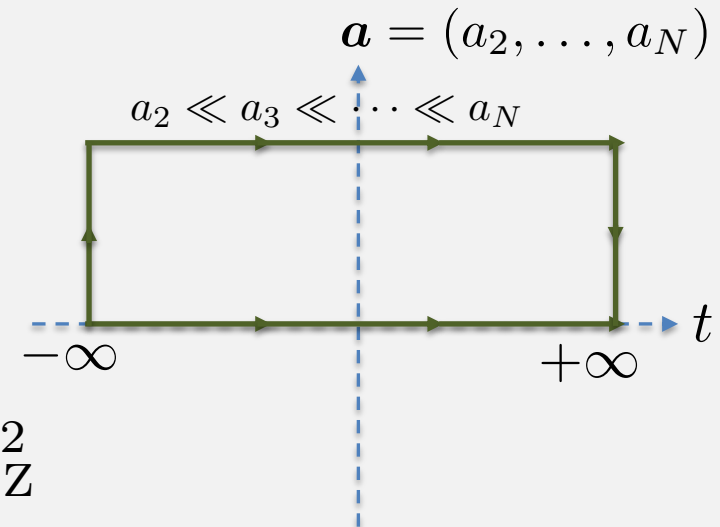
Demkov-Osherov model:

$$H(t) = \begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & a_N \end{pmatrix} + t \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$x_k = a_k$$



Energy levels well separated (i.e. evolution is adiabatic and no transitions occur) everywhere along the contour except near crossings where 2×2 scattering events take place



$$\Rightarrow S = S_{\text{LZ}}^{1N} \cdots S_{\text{LZ}}^{13} S_{\text{LZ}}^{12}$$

Knizhnik-Zamolodchikov equations

$$i\nu \frac{\partial \Psi}{\partial \varepsilon_j} = \hat{H}_j \Psi$$

$$\hat{H}_j = 2Bs_j^z - \sum_{k \neq j} \frac{\vec{s}_j \cdot \vec{s}_k}{\varepsilon_j - \varepsilon_k} \text{ -- Gaudin magnets}$$

More generally, any semisimple Lie algebra and/or hyperbolic, trigonometric Gaudins

$$[\hat{H}_j, \hat{H}_k] = 0, \quad [S_{\text{tot}}^z, \hat{H}_k] = 0, \quad S_{\text{tot}}^z = \sum_k s_k^z$$

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$$\sum_k 2\varepsilon_k \hat{H}_k \propto \hat{H}_{\text{BCS}} = \sum_k 2\varepsilon_k s_k^z - \frac{1}{2B} \sum_{j,k} s_j^+ s_k^- \quad \text{BCS model of superconductivity}$$

$$\Downarrow$$
$$[\hat{H}_{\text{BCS}}, \hat{H}_k] = 0$$

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$$\Downarrow$$
$$[\hat{H}_{\text{BCS}}, \hat{H}_k] = 0$$

$$\frac{\partial \hat{H}_k}{\partial B} = 2s_k^z = \frac{\partial \hat{H}_{\text{BCS}}}{\partial \varepsilon_k} \Rightarrow i\nu \frac{\partial \Psi}{\partial B} = \hat{H}_{\text{BCS}} \Psi$$

Knizhnik-Zamolodchikov-BCS equations

$$\begin{cases} i\nu \frac{\partial \Psi}{\partial \varepsilon_j} = \hat{H}_j \Psi \\ i\nu \frac{\partial \Psi}{\partial B} = \hat{H}_{\text{BCS}} \Psi \end{cases}$$

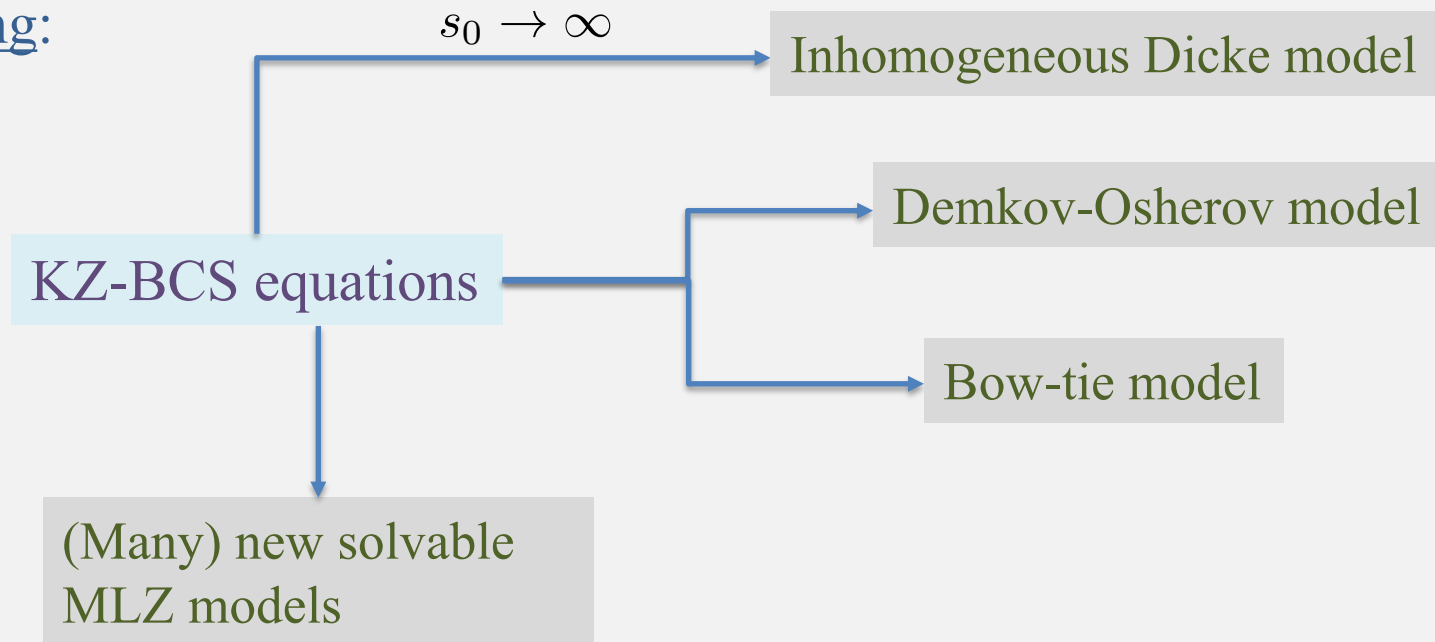
KZ-BCS equations

$$\hat{H}_j = 2Bs_j^z - \sum_{k \neq j} \frac{\vec{s}_j \cdot \vec{s}_k}{\varepsilon_j - \varepsilon_k} - \text{Gaudin magnets}$$

$$\hat{H}_{\text{BCS}} = \sum_k 2\varepsilon_k s_k^z - \frac{1}{2B} \sum_{j,k} s_j^+ s_k^-$$

BCS model of superconductivity

Mapping:



KZ-BCS equations

Demkov-Osherov model

$$\hat{H}_j = 2Bs_j^z - \sum_{k \neq j} \frac{\vec{s}_j \cdot \vec{s}_k}{\varepsilon_j - \varepsilon_k}$$

$$H(t) = \begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & a_N \end{pmatrix} + t \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$[S_{\text{tot}}^z, \hat{H}_j] = 0 \Rightarrow H_j = \begin{pmatrix} \boxed{1 \times 1} & & & \\ & \boxed{S_{\text{tot}}^z = \min + 1} & & \\ & & \boxed{N \times N} & \\ & & & \ddots \end{pmatrix}$$

Block-diagonal

$$s_1 = 1, \quad \varepsilon_1 = 0, \quad s_k = \frac{g_k^2}{a_k^2}, \quad \varepsilon_k = -\frac{1}{a_k}, \quad 2B = t - \sum_{k=2}^N \frac{g_k^2}{a_k}$$

$N \times N$ block of $\hat{H}_1 \longrightarrow$ Demkov-Osherov $H(t)$

$N \times N$ blocks of $\hat{H}_j \longrightarrow$ commuting partners of $H(t)$

KZ-BCS equations

Demkov-Osherov model

$$\hat{H}_j = 2Bs_j^z - \sum_{k \neq j} \frac{\vec{s}_j \cdot \vec{s}_k}{\varepsilon_j - \varepsilon_k}$$

$$H(t) = \begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & a_N \end{pmatrix} + t \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Knizhnik-Zamolodchikov equations are replaced with:

$$\begin{cases} i \frac{\partial \Psi}{\partial t} = H(t) \Psi & \text{non-stationary Schrodinger eq. for Demkov-Osherov} \\ i \frac{\partial \Psi}{\partial a_k} = H_k(t) \Psi & a_k \text{ evolution eqs.} \end{cases}$$

$$s_1 = 1, \quad \varepsilon_1 = 0, \quad s_k = \frac{g_k^2}{a_k^2}, \quad \varepsilon_k = -\frac{1}{a_k}, \quad 2B = t - \sum_{k=2}^N \frac{g_k^2}{a_k}$$

$N \times N$ block of $\hat{H}_1 \longrightarrow$ Demkov-Osherov $H(t)$

$N \times N$ blocks of $\hat{H}_j \longrightarrow$ commuting partners of $H(t)$

New integrable time-dependent Hamiltonians

$$\left\{ \begin{array}{l} i\nu \frac{\partial \Psi}{\partial \varepsilon_j} = \hat{H}_j \Psi \end{array} \right.$$

$$\left\{ \begin{array}{l} i\nu \frac{\partial \Psi}{\partial B} = \hat{H}_{\text{BCS}} \Psi \end{array} \right.$$

BCS model of superconductivity:

$$\hat{H}_{\text{BCS}} = \sum_k 2\varepsilon_k s_k^z - \frac{1}{2B} \sum_{j,k} s_j^+ s_k^-$$

Let $B = B(t)$

BCS Hamiltonian with coupling proportional to $1/t$

$$B(t) = \nu t$$

$$\hat{H}_{\text{BCS}}(t) = \sum_j 2\varepsilon_j \hat{s}_j^z - \frac{1}{\nu t} \sum_{j,k} \hat{s}_j^+ \hat{s}_k^-$$

Floquet BCS Hamiltonian

$$B(t) = \cos \nu t$$

$$\hat{H}_{\text{BCS}}(t) = \sin(\nu t) \sum_j 2\varepsilon_j \hat{s}_j^z - \tan(\nu t) \sum_{j,k} \hat{s}_j^+ \hat{s}_k^-$$

General solution of the non-stationary Schrodinger equation

KZ theory: integral representation of the general solution of Knizhnik-Zamolodchikov equations using off-shell Bethe Ansatz

Babujian J. Phys. A (1993)



Extending this technique and using the mapping to KZ-BCS equations, we derive the general solution of the non-stationary Schrodinger equation for all our models (inhomogeneous Dicke, Demkov-Osherov, bow-tie & time-dependent BCS)

Yuzbashyan, Ann. Phys. (2018)



Moreover, using techniques developed in KZ theory for relating asymptotic solutions, can get the scattering matrices (connection matrices in KZ language) for these models

Transition probabilities in multi-level Landau-Zener problem:

$$p_{i \rightarrow k} = q\text{-deformed } 6j \text{ symbols}$$

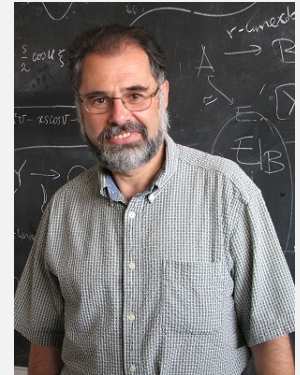
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Wayne State Univ.

Publications:

- ✓ Patra & Yuzbashyan, J. Phys. A (2015)
- ✓ Sinitsyn, Yuzbashyan, Chernyak, Patra & Sun, Phys. Rev. Lett (2018)
- ✓ Yuzbashyan, Ann. Phys. (2018)