Field theory and the renormalizaiton group on non-metric spaces

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Motivation

Affinities, interactions → networks

→ Statistical physics of networks not embedded in metric spaces

The Structure and Dynamics of Networks, (Newman, Barabàsi and Watts 2006)

Large Scale Structure and dynamics of Complex Networks from Information Technology to Finance and Natural Science (Barrat, Barthelemy and Vespignani 2007)

Some recent samples:

"Functional implications of genome topology"(Cavalli and Misteli 2013)

"Global network topology of stock markets" (Shahzad et al. 2018)

Critical phenomena and RG on networks: challenges

- Hyperscaling relations do not work (c.f. Kauffmann and Griffiths 1981).
- "Exact Real Space RG" on hierarchical (diamond) lattice
  $\nu = 1/y_t = 1.3370$ $\quad d\,\nu = 2 - \alpha$ $\quad \rightarrow \quad \alpha = -\,0.6738$!!
- How to relate the spectral density $\rho(\omega) \sim \omega^{\beta}$, and the spectral dimension (cf. Bianconi *et al.* 2010)
  $\tilde{d} \equiv 2(1+\beta)$ to the critical exponents ?
- Lower and upper critical spectral dimensions?

Building a field theory: graph Laplacian, eigenvectors and eigenvalues

<u>The graph Laplacian</u>

$i, j = 1, \ldots N$ label the nodes of a network,

$A_{ij} = 1$ for i, j connected, zero otherwise.

$$L_{ij} = d_i \, \delta_{ij} - A_{ij}$$

Discrete Hamiltonian for a scalar field ψ living on the nodes of a network

$$H = \frac{1}{2} \sum_{ij}^{N} \psi(i)[r_0 \delta_{ij} + L_{ij}]\psi(j) - h \sum_i \psi(i) + v_0 \sum_i \psi^4(i)$$

A. Tuncer, AE, PRE (2015)

Expand the field ψ in eigenvectors $\mathbf{u}_\mu$ of the graph Laplacian

$$\psi(i) = N^{-1/2} \sum_\mu \hat{\psi}_\mu u_\mu(i)$$ $i=1,\ldots N$, $\mu = 1,\ldots N$

Fourier transformed non-interacting Hamiltonian,

$$H_0 = \frac{1}{2} \sum_{\mu=1}^{N} [r_0 + \omega_\mu] \hat{\psi}_\mu^2 - h \hat{\psi}_1 .$$

$$Z_0 = \int_{-\infty}^{\infty} \prod_\mu d\hat{\psi}_\mu e^{-H_0},$$

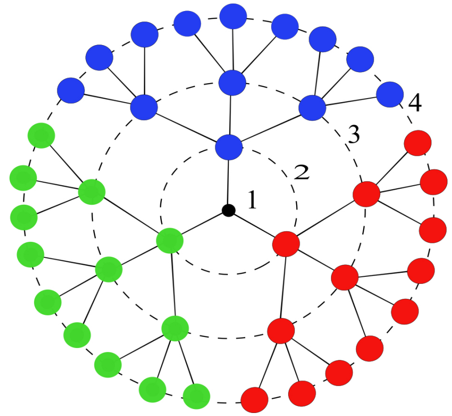
$k_B T = 1$

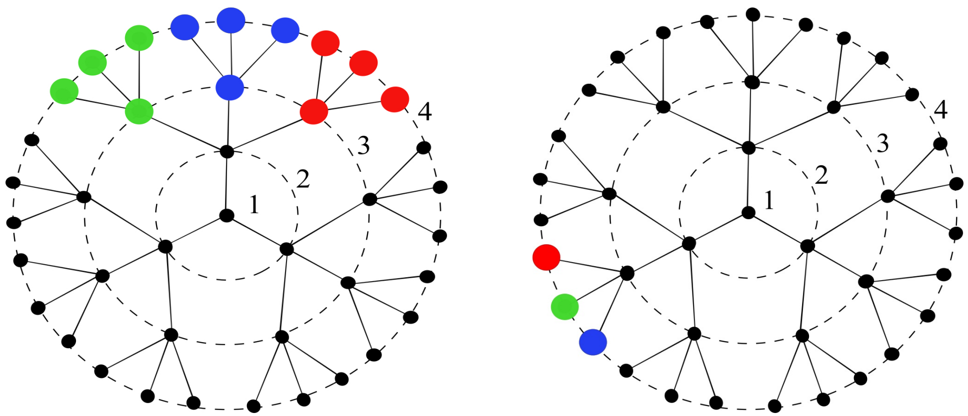
Eigenvectors of the graph Laplacian on the Cayley tree

for the primitive tree

$$\mathbf{u}^{(2)}_{\ell,(2)} = \begin{pmatrix} 0 \\ \mathbf{e}^{(\ell)} \end{pmatrix} \quad \cdot \quad \mathbf{e}^{(\ell)} \equiv \begin{pmatrix} e^{i\theta_1 \ell} \\ e^{i\theta_2 \ell} \\ \cdots \\ e^{i\theta_b \ell} \end{pmatrix}$$

$\theta_j = 2\pi j/b$, $j = 1..b$, $\ell = 1,..b\text{-}1$



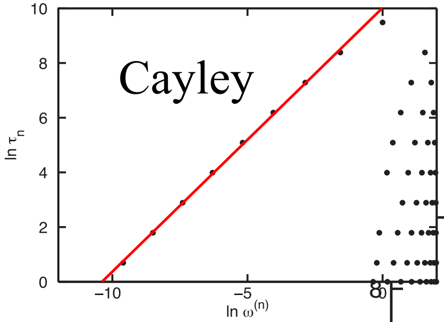


Shown for branching number b =3 and 4 generations. AE, A.Tuncer, arxiv 1806.01006

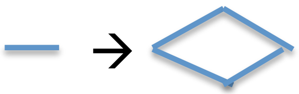
Iteratively determine
Eigenvalues //
Amplitudes of the eigenvectors,
on different shells
using network symmetries
and Cayley-Hamilton Theorem

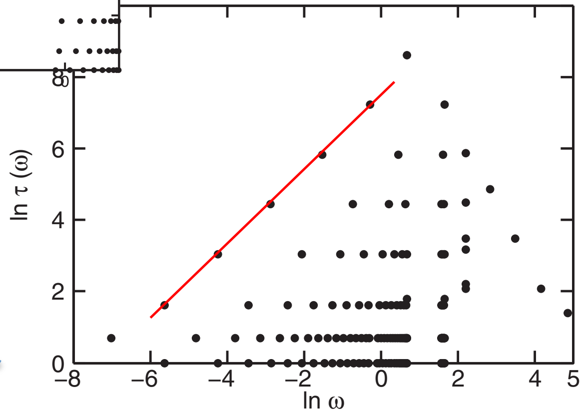
c.f. Chen, Onsager, Bonner and Nagle, "Hopping of ions in ice," J. Chem. Phys. 1974

Spectral densities, eigenvalues



diamond (hierarchical) lattice





scaling exponent β of the spectral density

$$\tau_n = b^{n-2}(b-1) \qquad \omega^{(n)} \simeq a_n b^{-(r-n+2)}$$

$$\tau_n \propto [\omega^{(n)}]^{\xi} \qquad \sum_n \tau_n = \frac{1}{\ln b}\int \omega^{\xi-1} d\omega$$

$\xi = 1 \rightarrow \quad \rho(\omega) \sim \omega^{\beta} \qquad \beta = 0$ (log)

n \ g	2	3	4	5	6	7	8	9	10
1	0	0	0	0	0	0	0	0	0
2	1	0.20871	0.05718	0.01751	0.0056281	0.0018477	0.00061209	0.00020353	6.78×10^{-05}
3		1	0.20871	0.05718	0.01751	0.0056281	0.0018477	0.00061209	0.00020353
4			1	0.20871	0.05718	0.01751	0.0056281	0.0018477	0.00061209
5				1	0.20871	0.05718	0.01751	0.0056281	0.0018477
6					1	0.20871	0.05718	0.01751	0.0056281
7						1	0.20871	0.05718	0.01751
8							1	0.20871	0.05718
9								1	0.20871
10									1

Cayley tree nested hierarchy of eigenvalues

g = number of generations,

I. Gaussian Hamiltonian, Truncation and rescaling "by volume"

$$H_0^< = \frac{1}{2}\sum_{\mu=1}^{\mu_B}[r_0 + \omega_\mu]\hat{\psi}_\mu^2 - h\hat{\psi}_1.$$

$$\boxed{\mu_B = N/B}$$

$$H_0' = \frac{1}{2}\sum_{\mu=1}^{N}[r_0 B^{-\phi_1} + B^{-\phi_1-\phi_2}\omega_\mu](\hat{\psi}_\mu')^2 - h'\hat{\psi}_1',$$

$$\boxed{r' = (\sigma_1^V)^{-1} z^2 r_0 = \sigma_2^V r_0 = B^{\phi_2} r_0.}$$

r_0 = reduced temperature

$$\boxed{z = \left(\sigma_1^V \sigma_2^V\right)^{1/2} = B^{(\phi_1+\phi_2)/2}.}$$

z fixes the coefficient of the Laplace term at unity

Wave function renormalization $\hat{\psi}' = z\hat{\psi}$

Defining the rescaling factors:

$$\sigma_1^V(B) \equiv \frac{\sum_{\mu=1}^{N} 1}{\sum_{\mu=1}^{\mu_B} 1} = B^{\phi_1}$$

$$\sigma_1^V(B)\sigma_2^V(B) \equiv \frac{\sum_{\mu=1}^{N} \omega_\mu}{\sum_{\mu=1}^{\mu_B} \omega_\mu} = B^{\phi_1+\phi_2},$$

ϕ_1=1 trivially ϕ_2= 1.06 ±0.04 numerical fit

The critical exponents: Not a metric space – no correlation length scaling interpretation for ϕ_2 !

Kadanoff rescaling (t is the reduced temperature as in original notation)

$$f(t,h) = B^{-1} f'\left(B^{Y_t^V} t, B^{Y_h^V} h\right).$$

$$f(t,0) \sim t^{(2-\alpha)} \qquad c_h \sim \partial^2 f(t,0)/\partial t^2 \sim t^{-\alpha}$$

Critical exponents from matching the renormalization factors

$$r' = B^{\phi_2} r_0 \rightarrow \quad Y_t^V = \phi_2 \quad \text{and} \quad h' = zh \rightarrow \quad Y_h^V = (\phi_1+\phi_2)/2$$

$t^{-\alpha} \sim B^{-1} B^{\phi_2(2-\alpha)} t^{-\alpha}$ and similarly, using $f(0, h) \sim h^{(1+1/\delta)}$

$$\alpha = 2 - \frac{1}{\phi_2}. \qquad\qquad \delta = \frac{1+\phi_2}{1-\phi_2}.$$

II. decimating eigenvalues beyond $\omega_{\max}/B$ and rescaling

$$\boxed{\mu_B = \sup\{\mu \in [1,N] : \omega_\mu < \Omega/B\}}$$

$$\sigma_1^{\Omega}(B) \equiv \frac{N}{\sum_{\mu=1}^{\mu_B} 1} = B^{p_1}, \qquad \sigma_1^{\Omega}(B)\sigma_2^{\Omega}(B) \equiv \frac{N\overline{\omega}}{\sum_{\mu=1}^{\mu_B} \omega_\mu} = B^{p_1+p_2}.$$

$$z = B^{(p_1+p_2)/2}.$$

again using Kadanoff scaling we get,

$$\alpha = 2 - p_1 \qquad \delta = (p_1+p_2)/(p_1-p_2)$$

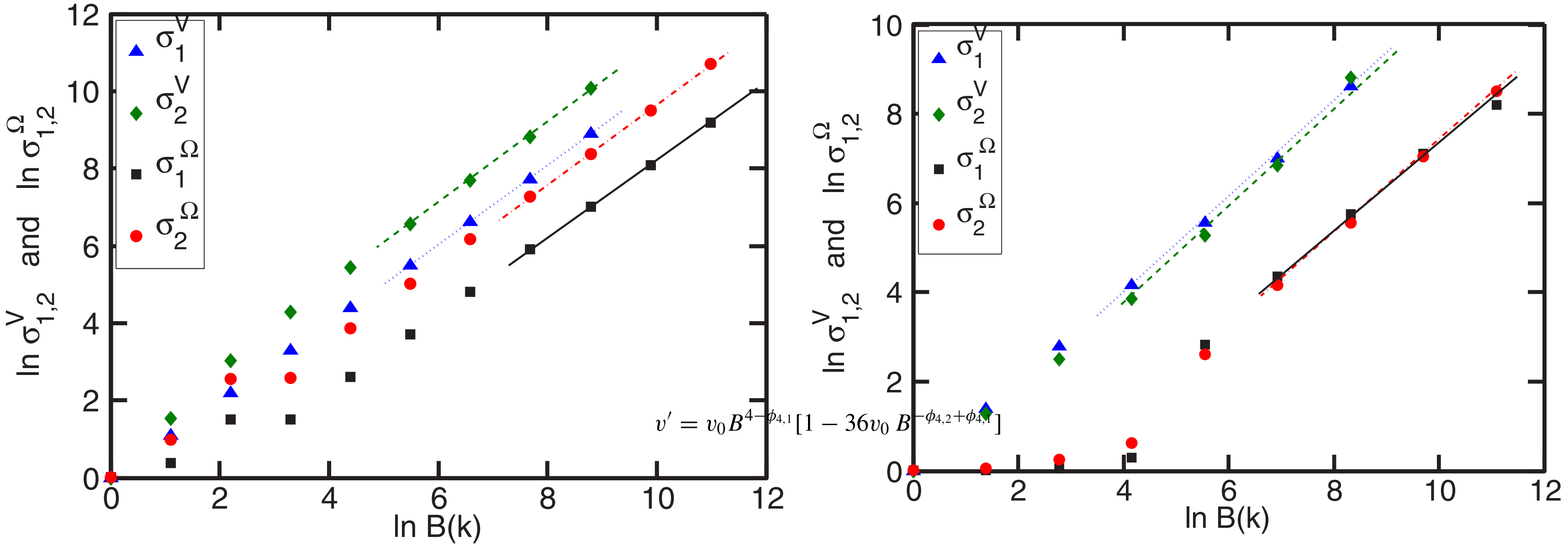
Going over to integrals rather than sums, using the spectral density,

$p_1 = \beta + 1$ and $p_2 = 1$

Relating the scaling exponents to the spectral density

$\rightarrow$ $\alpha = 1 - \beta \qquad \delta = (2+\beta)/\beta$

Gaussian model (Cayley and Diamond lattices)



Numerical determination of the exponents

$$p_1 \approx 1, \quad p_2 \approx 1.05 \pm 0.04 \quad (\beta = 0)$$

$$\alpha = 1 \qquad \delta = \infty$$

correcting for irrelevant dangerous variable:

$\alpha = 0 \qquad \delta = 3 \qquad$ (mean field values)

The ψ^4 theory

$$H_{\mathrm{int}} = v_0 \sum_{1,2,3,4} \hat{\psi}_1 \hat{\psi}_2 \hat{\psi}_3 \hat{\psi}_4 \, \Phi(1,2,3,4),$$

$$\Phi(1,2,3,4) = \sum_i^N u_1(i) u_2(i) u_3(i) u_4(i).$$

"1,2,3,4" shorthand for summing each of the four eigenvectors $\mathbf{u}_\mu$ over μ=1,…N, each independently

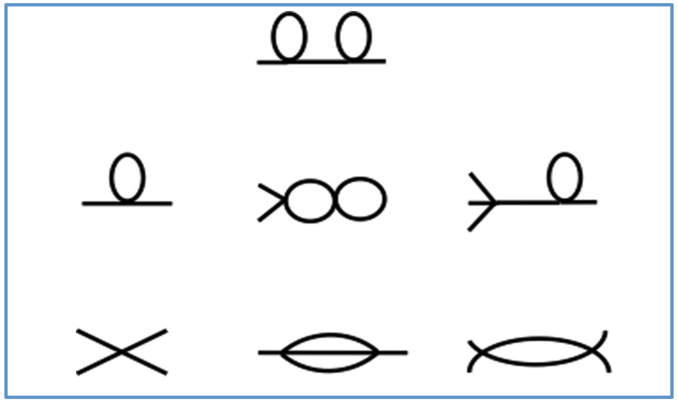
No simple sum rule for the eigenvectors, numerically calculated sums over products of eigenvectors

Perturbation expansion up to second order in the coupling constant

$$Z(r_0, v_0) = Z_0^{>} \int_{-\infty}^{\infty} \prod_{\mu<\mu_B} d\hat{\psi}_\mu e^{-H_0^{<}} \langle e^{-H_{\text{int}}} \rangle_0^{>}$$

$$\langle e^{-x} \rangle \simeq e^{-\langle x \rangle} e^{\frac{1}{2}[\langle x^2 \rangle - \langle x \rangle^2]}$$

Feynman diagrams for quadratic and quartic interactions

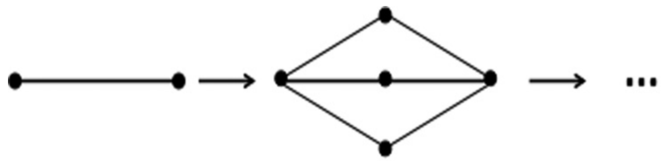


Gaussian fixed point stable with respect to 1st and 2nd order perturbations $v^* = 0$ for Cayley tree and hierarchical lattice

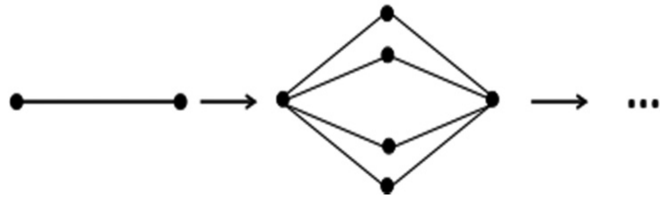
for $\beta = 0\,,\ \tilde{d} = 2$

(true with Wilson momentum shell RG as well !)

Itzykson and Luck (Proceedings Brasov Conference, 1985)
Generalized Hierarchical lattice



Number of parallel paths p



On hierarchical lattices (modified diamond lattices) with $\tilde{d} > 2$ Gaussian fixed point becomes unstable to second order ψ^4 terms; for $\tilde{d} \geq 4$ Gaussian fixed point stable

p	β	$\tilde{d}$	υ^*	α	δ
2	0	2	0	0	3
3	0.40	2.80	0.0040	1.14	6
4	0.66	3.32	0.0054	0.58	4.03
5	0.83	3.46	0.0048	0.75	3.41
7	1.06	4.12	0	0	3

←Mean field, corrected for dangerous irrelevant coupling

summary

- Can do "field theoretic" renormalization group on networks not embedded in metric spaces, by changing the upper cutoff of the eigenvalues of the Laplace op. or the number of nodes (volume) and rescaling
- Eigenvectors can be explicitly constructed for some non-periodic lattices (see Rojo et al., LAA for non-regular lattices)
- Critical exponents of ψ^4 model depend upon the spectral dimension and not upon the embedding or the fractal dimension. Non-Gaussian critical exponents for $2 < \tilde{d} < 4$.

If spectral density does not scale, have to resort to different methods (e.g., Goltsev, Dorogovtsev Mendes PRE 2003)