PT SYMMETRY in quantum mechanics and quantum field theory

Carl M. Bender
Washington University

Stat Mech Conference
Rutgers, 12 May 2019

How to find the positive root of

$x^5 + x = 1$ $\quad$ **(x = 0.75488)**

Insert ε: $x^5 + \varepsilon x = 1$ **(strong coupling)**

$$x(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n \qquad a_0 = 1$$

Match powers of ε

$$
\begin{aligned}
5a_1 + 1 &= 0, \\
5a_2 + 10a_1^2 + a_1 &= 0, \\
5a_3 + 20a_1a_2 + a_2 + 10a_1^3 &= 0, \\
5a_4 + 20a_1a_3 + a_3 + 10a_2^2 + 30a_1^2a_2 + 5a_1^4 &= 0.
\end{aligned}
$$

$$
a_1 = -\frac{1}{5}, \quad a_2 = -\frac{1}{25}, \quad a_3 = -\frac{1}{125},
$$

$$
a_4 = 0, \quad a_5 = \frac{21}{15625}, \quad a_6 = \frac{78}{78125}
$$

The perturbation series…

$$x(\epsilon) = 1 - \frac{1}{5}\epsilon - \frac{1}{25}\epsilon^2 - \frac{1}{125}\epsilon^3 + \frac{21}{15625}\epsilon^5 + \frac{78}{78125}\epsilon^6 + \cdots$$

Sum series at $\varepsilon = 1$:

Radius of convergence of series: $5/4^{4/5} = 1.64938\ldots$

Sixth-order result: $x(1) = 0.75434$

Exact answer: $x = 0.75488$

(0.07% error)

Field theorist's way to insert ε

Insert ε : $\varepsilon x^5 + x = 1$ **(weak coupling)**

Perturbation series:

$$x(\varepsilon) = 1 - \varepsilon + 5\varepsilon^2 - 35\varepsilon^3 + 285\varepsilon^4 - 2530\varepsilon^5 + 23751\varepsilon^6 - \ldots$$

Sum the series at ε = 1

Radius of convergence of this series: $4^4/5^5 = 0.08192$

Result: $x(1) = 21476$

But (3,3)-Padé gives 0.7% accuracy!

Another way to insert a small parameter

$x^{1+\delta} + x = 1$ **<u>δ measures the departure from linearity</u>**

Perturbation series: $x(\delta) = c_0 + c_1\delta + c_2\delta^2 + c_3\delta^3 + \cdots$

$$c_0 = \tfrac{1}{2}, \quad c_1 = \tfrac{1}{4}\ln 2, \quad c_2 = -\tfrac{1}{8}\ln 2\,,$$
$$c_3 = -\tfrac{1}{48}\ln^3 2 + \tfrac{1}{32}\ln^2 2 + \tfrac{1}{16}\ln 2\,,$$
$$c_4 = \tfrac{1}{32}\ln^3 2 - \tfrac{3}{64}\ln^2 2 - \tfrac{1}{32}\ln 2\,,$$
$$c_5 = \frac{1}{480}\ln^5 2 - \frac{7}{768}\ln^4 2 - \frac{3}{128}\ln^3 2 + \frac{3}{64}\ln^2 2 + \frac{1}{64}\ln 2\,,$$
$$c_6 = -\frac{1}{192}\ln^5 2 + \frac{35}{1536}\ln^4 2 + \frac{5}{768}\ln^3 2 - \frac{5}{128}\ln^2 2 - \frac{1}{128}\ln 2\,,$$

Radius of convergence of this series = 1

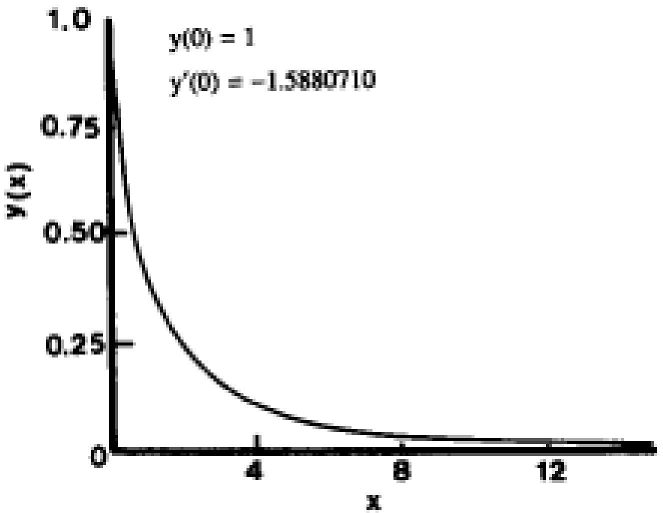
Must sum series at δ = 4

(3,3)-Padé gives 0.5% accuracy – better than weak coupling!

Thomas-Fermi equation (nuclear structure)

$$y''(x) = [y(x)]^{3/2}/\sqrt{x}$$

$$y(0) = 1, y(\infty) = 0$$



Delta expansion: $y''(x) = y(x)[y(x)/x]^{\delta}$ $\quad y(0) = 1, y(\infty) = 0$

$$y(x) = y_0(x) + \delta y_1(x) + \delta^2 y_2(x) + \cdots$$

Third-order calculation with (2,1)-Padé approximant gives 1.1% accuracy!

Blasius Equation (fluid mechanics)

$$y'''(x) + y''(x)y(x) = 0$$
$$y(0) = y'(0) = 0,\ y'(\infty) = 1$$

$$y''(0) = 0.46960...$$

$$y'''(x) + y''(x)[y(x)]^{\delta} = 0$$

$$y(0) = y'(0) = 0,\ y'(\infty) = 1$$

$$y(x) = y_0(x) + \delta y_1(x) + \delta^2 y_2(x) + \cdots$$

Second-order calculation with (1,1)-Padé approximant gives 8.7% accuracy!

Lane-Emden equation (stellar structure)

$$y''(x) + 2y'(x)/x + [y(x)]^n = 0$$

$$y''(x) + 2y'(x) + [y(x)]^{1+\delta} = 0$$

Korteweg-de Vries equation (nonlinear waves)

$$u_t + uu_x + u_{xxx} = 0$$

$$u_t + u^\delta u_x + u_{xxx} = 0$$

Quantum mechanics

Replace $H = p^2 + x^4$

with $$H = p^2 + x^2\,(x^2)^\delta$$

and expand in powers of δ. Note: we raise a ***positive*** quantity to the power δ to avoid complex numbers appearing as an artefact.

But, here's an interesting idea:

$$H = p^2 + x^2\,(\mathrm{i}x)^\varepsilon$$

This Hamiltonian is not Hermitian but it is ***PT*** **symmetric**.

MESSAGE OF THIS TALK:

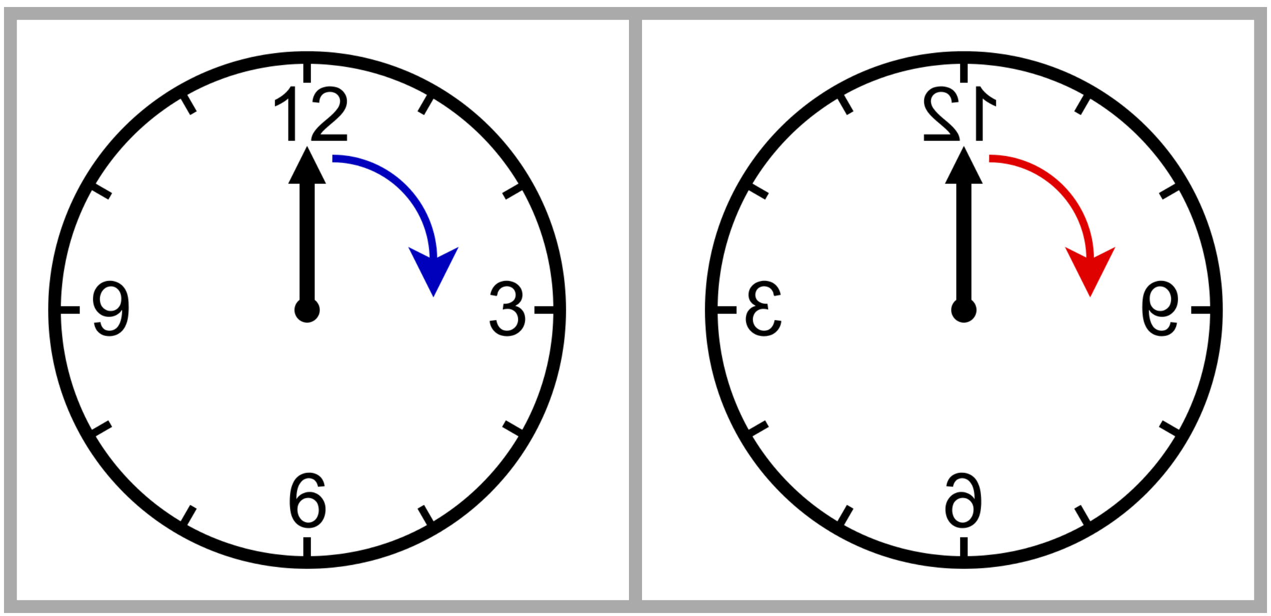
The *PT* expansion can be applied in quantum mechanics --- and *it can be extended to quantum field theory*

***PT*-symmetric quantum mechanics:**

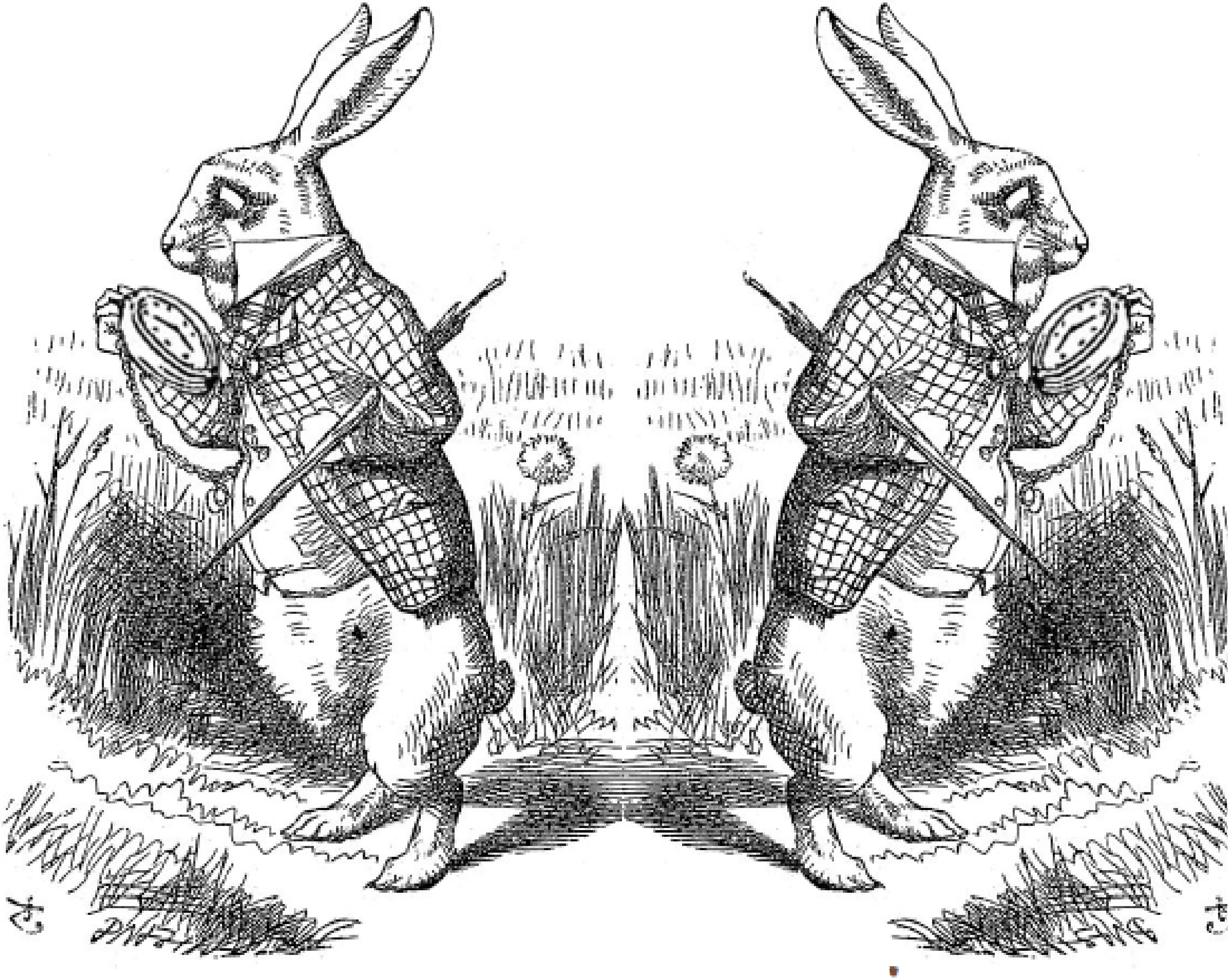
Extending quantum mechanics into the <u>complex</u> domain without violating any axioms of quantum mechanics

Generalizes but does not conflict with conventional quantum mechanics.

If you respect ***PT*** symmetry, ***<u>eigenvalues can remain real and unitarity can be preserved</u>*** even though Hamiltonian is not Hermitian.



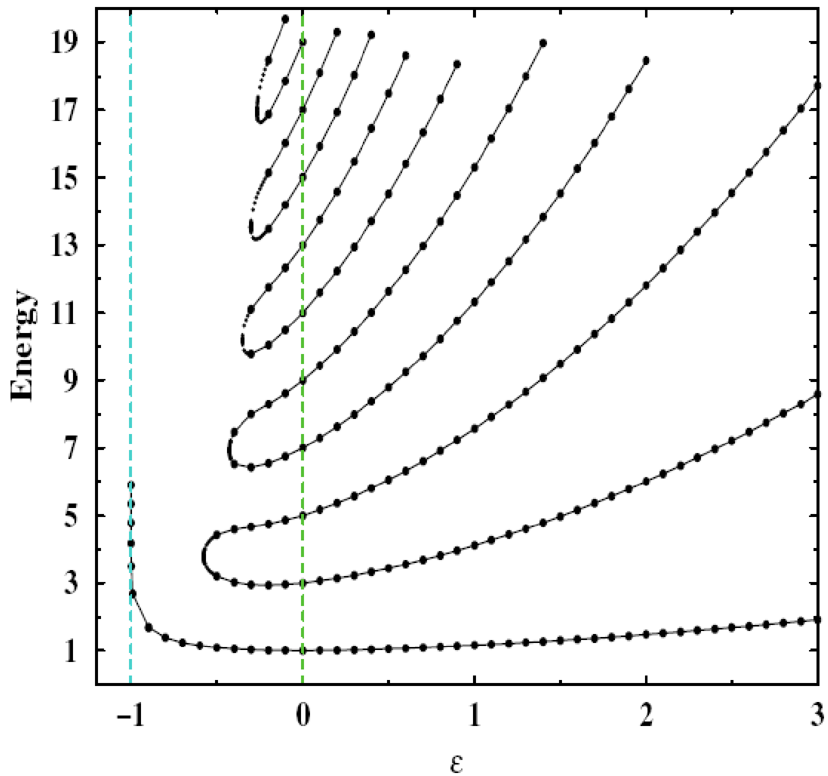
PT **reflection –** ***<u>simultaneous reflection of space and time</u>***



Space-time reflection

One-parameter family of *PT*-symmetric Hamiltonians obtained by *complex deformation* of the harmonic oscillator

$$H = p^2 + x^2(ix)^\varepsilon \qquad (\varepsilon \text{ real})$$



Look! *H* is not Hermitian but its eigenvalues are **all real**!

CMB and S. Boettcher
Phys. Rev. Lett. **80**, 5243 (1998)

P. Dorey, C. Dunning, and R. Tateo
J. Phys. A **40**, R205 (2007)

Special cases $\varepsilon = 1$: $H = p^2 + ix^3$ (Lee-Yang edge singularity, Reggeon field theory)

$\varepsilon = 2$: $H = p^2 - x^4$ (Upside-down potential!)

$\varepsilon = 4$: $H = p^2 + x^6$

PT-symmetric Hamiltonians are *complex deformations* of Hermitian Hamiltonians

You begin with a Hermitian Hamiltonian and introduce a *deformation parameter*…

Complex deformed squirrel

Complex deformed frog

Complex deformed parrot

Simple example: Complex deformed harmonic oscillator

$$H = p^2 + x^2 + \varepsilon i x$$

$$-\psi''(x) + x^2\psi(x) + \varepsilon i x\psi(x) = E\psi(x),$$

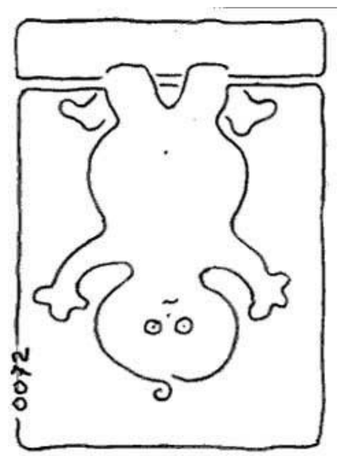
$$\psi(\pm\infty) = 0$$

$$E_n = 2n + 1 + \varepsilon^2/4 \qquad (n = 0, 1, 2, 3, \ldots)$$

How can a $-x^4$ potential (*upside-down*, *repulsive*) have a real positive spectrum??

Stability of upside-down potentials

$$V(x) = -x^4$$

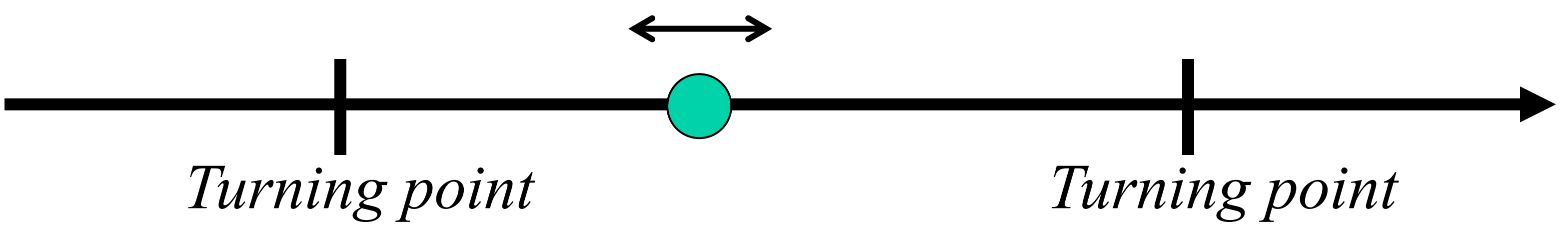


This upside-down potential looks *unstable* (*<u>on the real axis</u>*)
-- it is statically unstable but *dynamically* stable

Complex-variable theory explains why there are <u>bound states</u>

Heuristic freshman-physics explanation: Classical harmonic oscillator

Back and forth motion on the real-x axis:



$$E = p^2 + x^2$$

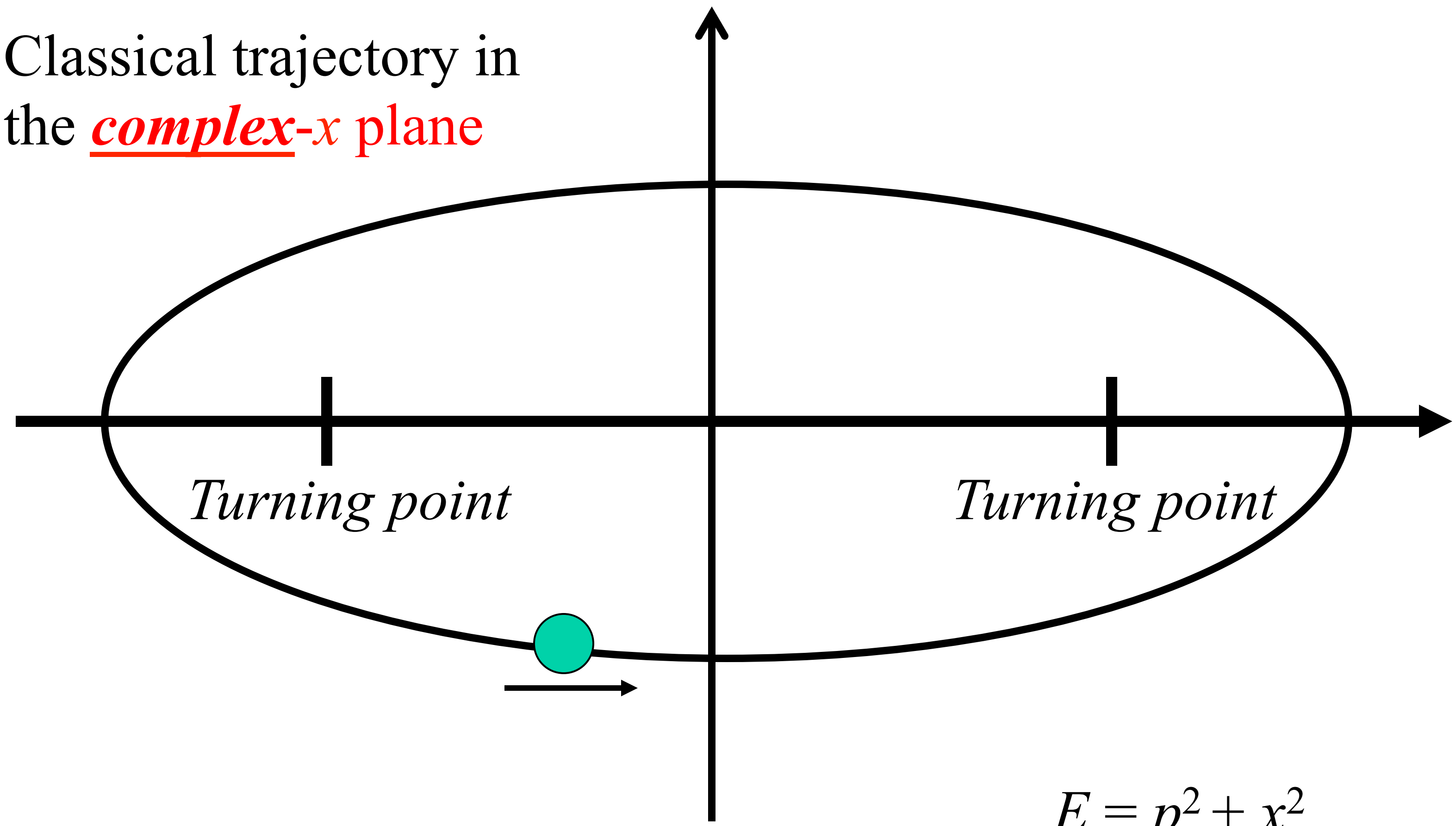
Classically allowed and ***classically forbidden*** regions…

Classically allowed and *classically forbidden* regions

Classical harmonic oscillator in the complex plane

$$H = p^2 + x^2 \qquad (\varepsilon = 0)$$

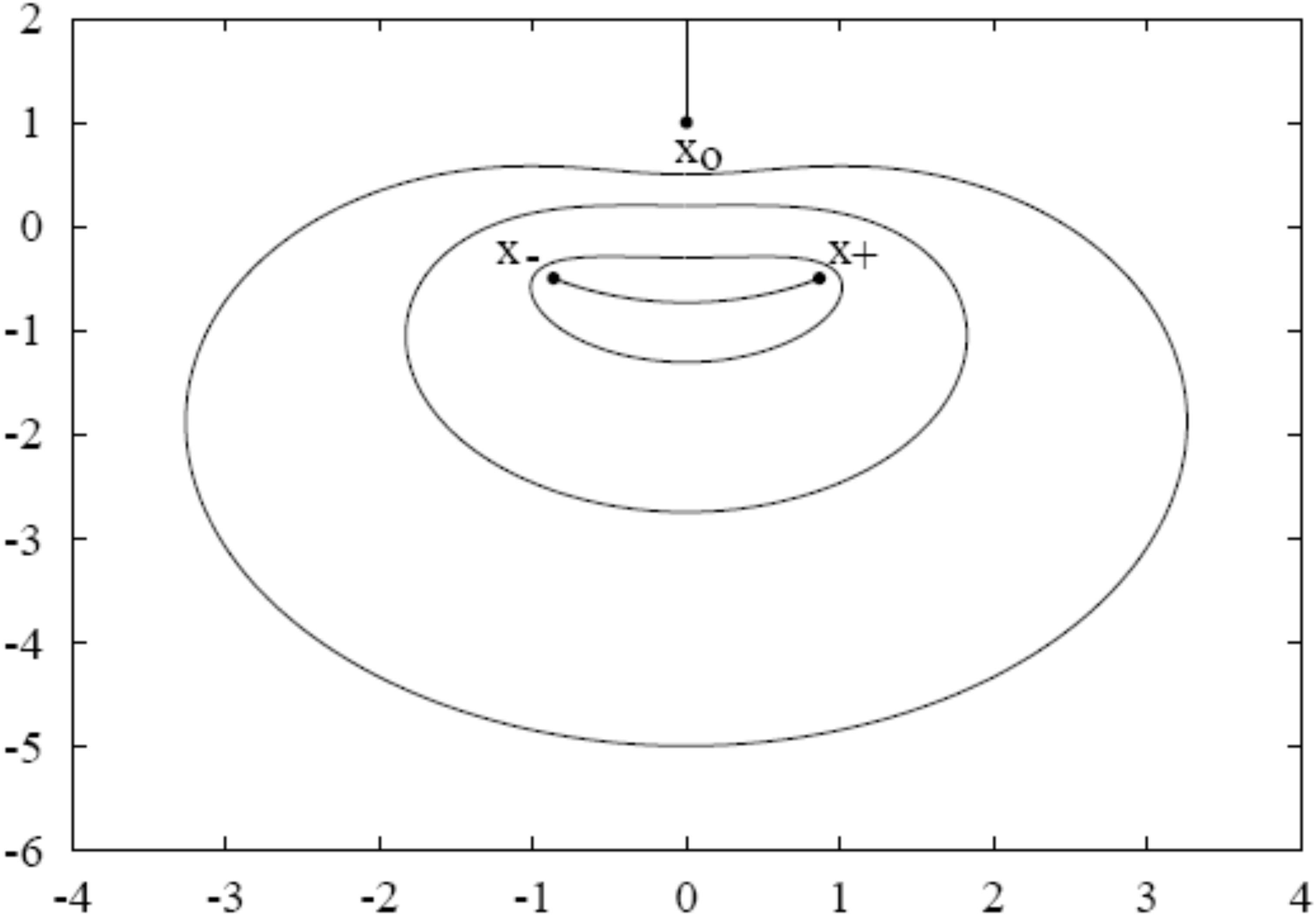
Classical trajectory in the ***complex***-x plane



$$E = p^2 + x^2$$

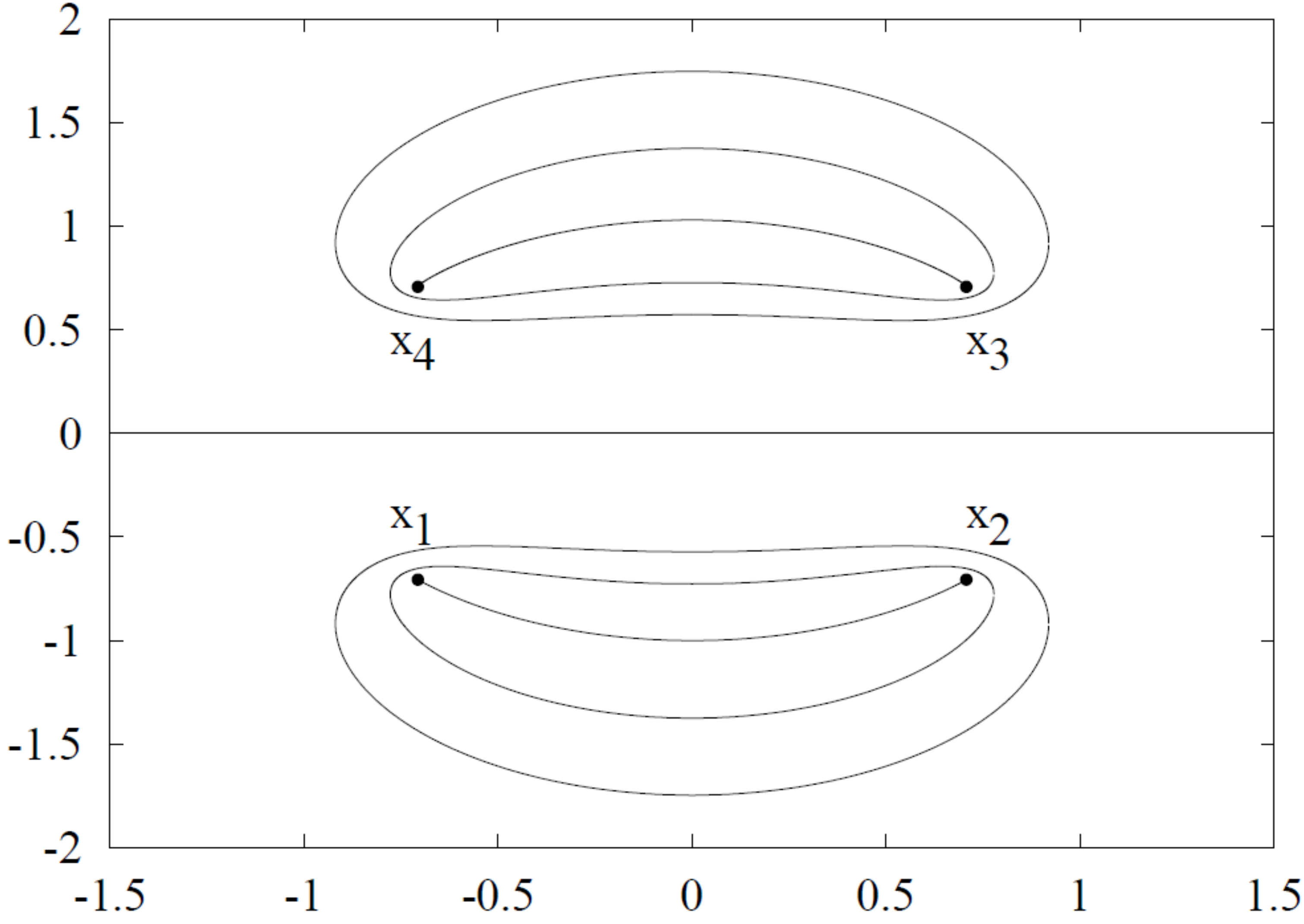
$H = p^2 + ix^3$ ($\varepsilon = 1$)

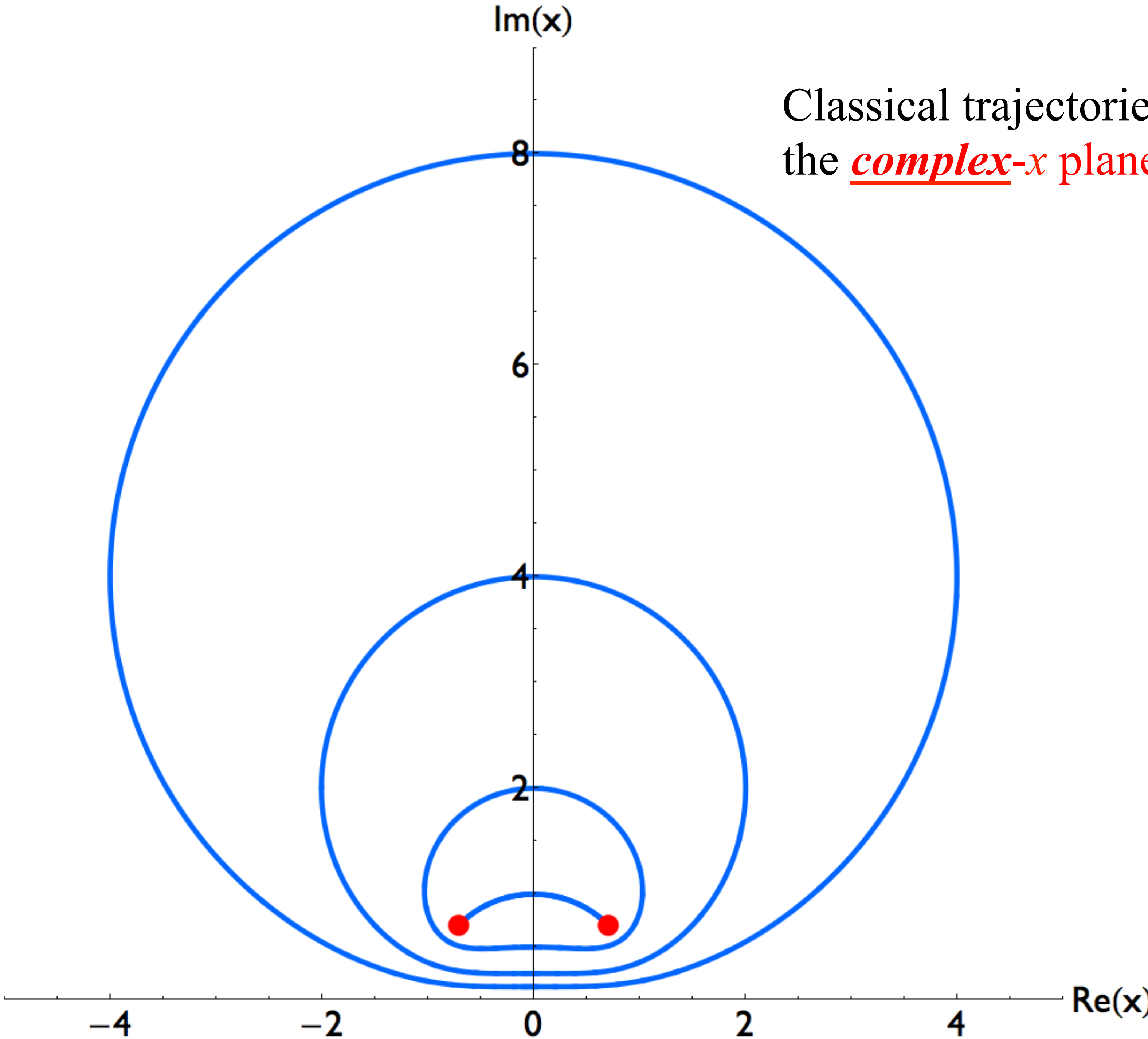
Classical trajectories in the ***complex**-x* plane



$H = p^2 - x^4$ $(\varepsilon = 2)$

Classical trajectories in the ***complex***-*x* plane



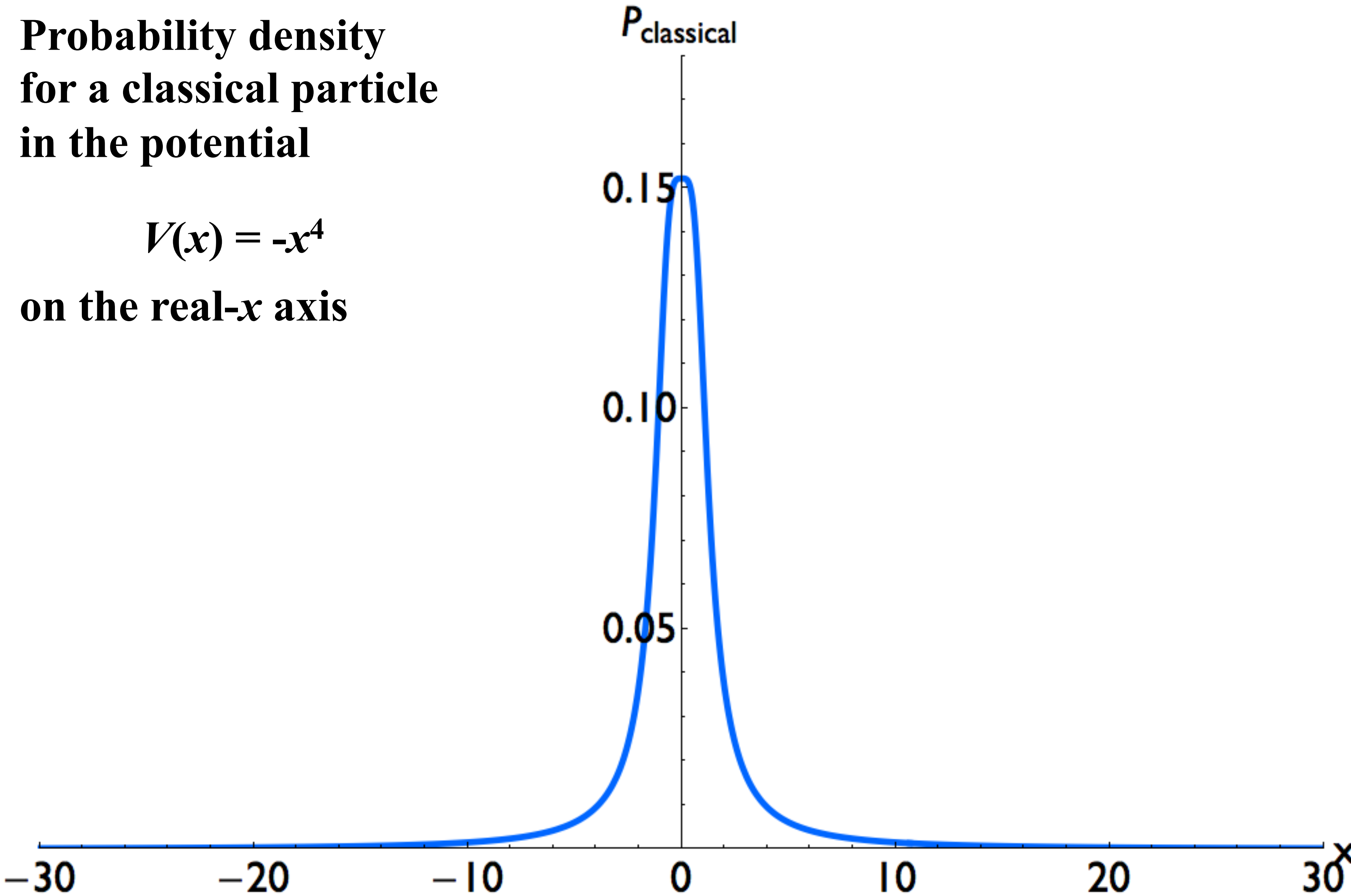


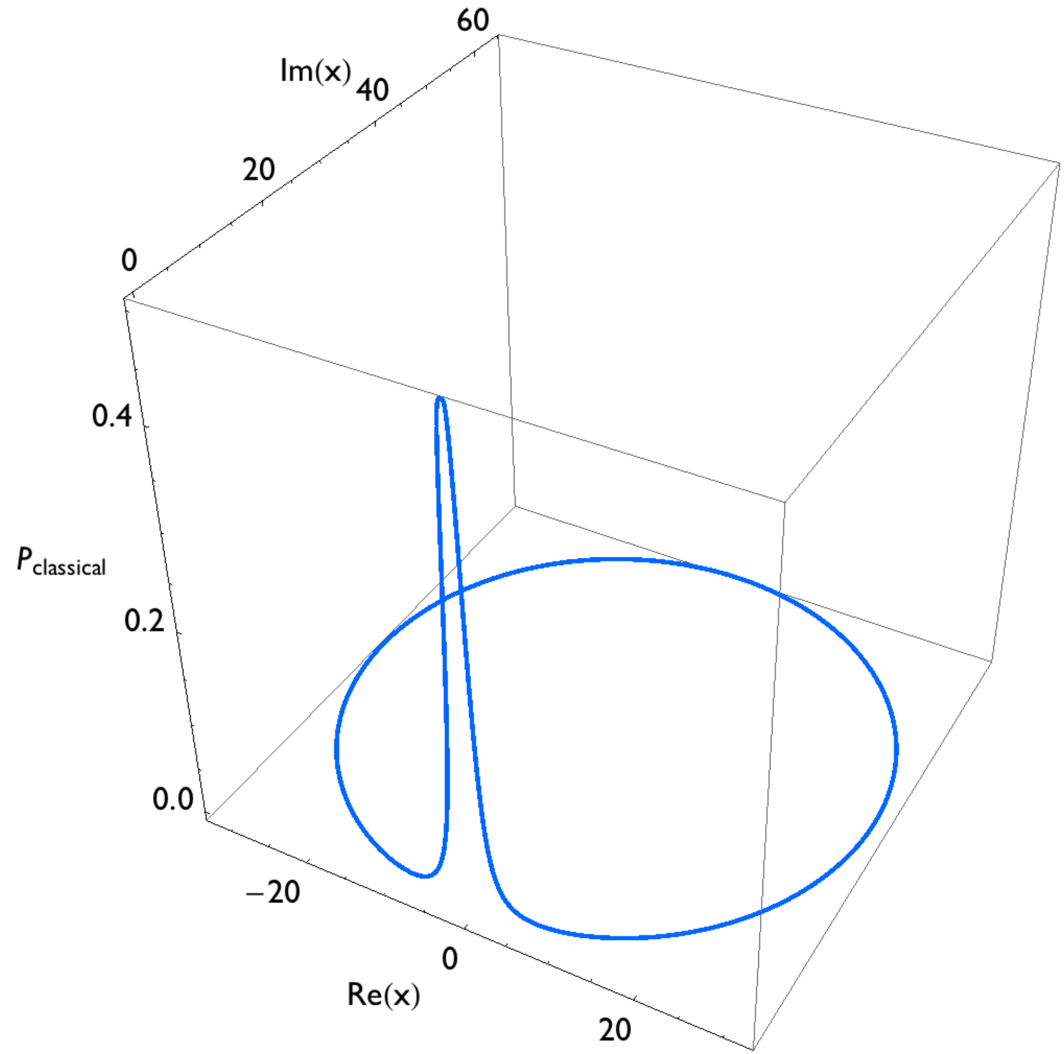
Classical trajectories in the ***complex***-x plane

Probability density for a classical particle in the potential

$$V(x) = -x^4$$

on the real-*x* axis





System with a static instability becomes *dynamically* stable in the complex domain!

Bohr-Sommerfeld Quantization of a complex atom

$$\oint dx\, p = \left(n + \tfrac{1}{2}\right)\pi$$

Bottom line: Instability at $x = 0$ is tamed

Complex analysis enables one to *tame instabilities*

Physical systems having static instabilities
can be dynamically *stable* in the complex domain

BASIC REASON:

If you extend real numbers to complex numbers, you lose *ordering* property of real numbers

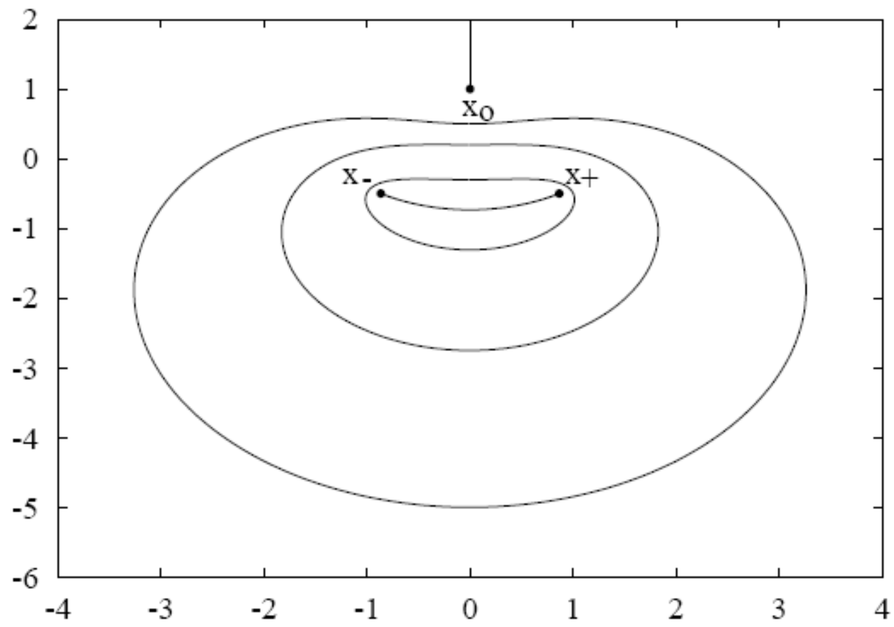
You lose the concept of > and <

Physical systems that *look* unstable may not be!
(examples: bicycles, tops, …)

Classical *PT* symmetry:

$H = p^2 + ix^3$ ($\varepsilon = 1$)

Classical trajectories in the ***complex***-x plane



Average location of the classical particle is a pure negative imaginary number. In quantum mechanics $\langle x \rangle$ is also negative and imaginary. In quantum field theory the one-point Green's function, say for a $-\phi^4$ field theory ($\varepsilon = 2$), should also be negative and imaginary.

Theoretical applications: renormalizing makes a Hamiltonian non-Hermitian, but it still is *PT* symmetric

- Lee model is unitary -- there are no ghosts (despite Pauli and Kallen!)
- Pais-Uhlenbeck model (no ghosts)
- Self-force on the electron (runaway modes)
- Double-scaling limit in QFT
- Stability of the Higgs vacuum
- Asymptotic behavior of the Painlevé transcendents
- Application to the Riemann hypothesis

…and many many many many more!

Experimental Studies of *PT* symmetry:

- ***PT***-symmetric wave guides
- ***PT***-symmetric lasers
- ***PT***-symmetric electronic and mechanical systems
- Unidirectional transmission of light
- ***PT***-symmetric atomic diffusion
- ***PT***-symmetric superconducting wires
- ***PT***-symmetric optical graphene
- ***PT***-symmetric topological insulators

…and many many many many more!

Experimental studies of *PT*–symmetric systems

<u>First observation of *PT* transition using optical wave guides:</u>

“Observation of PT-symmetry breaking in complex optical potentials,” A. Guo, G. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. Siviloglou, and <u>D. Christodoulides</u>, *Physical Review Letters* **103**, 093902 (2009)

nature physics

LETTERS

PUBLISHED ONLINE: 24 JANUARY 2010 | DOI: 10.1038/NPHYS1515

Observation of parity–time symmetry in optics

Christian E. Rüter[1], Konstantinos G. Makris[2], Ramy El-Ganainy[2], Demetrios N. Christodoulides[2], Mordechai Segev[3] and Detlef Kip[1]*

One of the fundamental axioms of quantum mechanics is associated with the Hermiticity of physical observables[1]. In the case of the Hamiltonian operator, this requirement not only implies real eigenenergies but also guarantees probability conservation. Interestingly, a wide class of non-Hermitian Hamiltonians can still show entirely real spectra. Among these are Hamiltonians respecting parity–time (*PT*) symmetry[2–7]. Even though the Hermiticity of quantum observables was never in doubt, such concepts have motivated discussions on several fronts in physics, including quantum field theories[8], non-Hermitian Anderson models[9] and open quantum systems[10,11], to mention a few. Although the impact of *PT* symmetry in these fields is still debated, it has been recently realized that optics can provide a fertile ground where *PT*-related notions can be implemented and experimentally investigated[12–15]. In this letter we report the first observation of the behaviour of a *PT* optical coupled system that judiciously involves a complex index potential. We observe both spontaneous *PT* symmetry breaking and power oscillations violating left–right symmetry. Our results may pave the way towards a new class of *PT*-synthetic materials with intriguing and unexpected properties that rely on non-reciprocal light propagation and tailored transverse energy flow.

($\varepsilon > \varepsilon_{th}$), the spectrum ceases to be real and starts to involve imaginary eigenvalues. This signifies the onset of a spontaneous *PT* symmetry-breaking, that is, a 'phase transition' from the exact to broken-*PT* phase[7,20].

In optics, several physical processes are known to obey equations that are formally equivalent to that of Schrödinger in quantum mechanics. Spatial diffraction and temporal dispersion are perhaps the most prominent examples. In this work we focus our attention on the spatial domain, for example optical beam propagation in *PT*-symmetric complex potentials. In fact, such *PT* 'optical potentials' can be realized through a judicious inclusion of index guiding and gain/loss regions[7,12–14]. Given that the complex refractive-index distribution $n(x) = n_R(x) + in_I(x)$ plays the role of an optical potential, we can then design a *PT*-symmetric system by satisfying the conditions $n_R(x) = n_R(-x)$ and $n_I(x) = -n_I(-x)$.

In other words, the refractive-index profile must be an even function of position x whereas the gain/loss distribution should be odd. Under these conditions, the electric-field envelope E of the optical beam is governed by the paraxial equation of diffraction[13]:

$$i\frac{\partial E}{\partial z} + \frac{1}{2k}\frac{\partial^2 E}{\partial x^2} + k_0[n_R(x) + in_I(x)]E = 0$$

PT-symmetric photonic crystals – Stanford

PRL **116,** 203902 (2016) PHYSICAL REVIEW LETTERS week ending 20 MAY 2016

Exceptional Contours and Band Structure Design in Parity-Time Symmetric Photonic Crystals

Alexander Cerjan, Aaswath Raman, and Shanhui Fan*
Department of Electrical Engineering, and Ginzton Laboratory, Stanford University, Stanford, California 94305, USA
(Received 22 January 2016; published 20 May 2016)

We investigate the properties of two-dimensional parity-time symmetric periodic systems whose non-Hermitian periodicity is an integer multiple of the underlying Hermitian system's periodicity. This creates a natural set of degeneracies that can undergo thresholdless $\mathcal{PT}$ transitions. We derive a $\mathbf{k} \cdot \mathbf{p}$ perturbation theory suited to the continuous eigenvalues of such systems in terms of the modes of the underlying Hermitian system. In photonic crystals, such thresholdless $\mathcal{PT}$ transitions are shown to yield significant control over the band structure of the system, and can result in all-angle supercollimation, a $\mathcal{PT}$-superprism effect, and unidirectional behavior.

DOI: 10.1103/PhysRevLett.116.203902

PT-symmetric wireless power transfer – Stanford

LETTER

doi:10.1038/nature22404

Robust wireless power transfer using a nonlinear parity–time–symmetric circuit

Sid Assawaworrarit[1], Xiaofang Yu[1] & Shanhui Fan[1]

Considerable progress in wireless power transfer has been made in the realm of non-radiative transfer, which employs magnetic-field coupling in the near field[1–4]. A combination of circuit resonance and impedance transformation is often used to help to achieve efficient transfer of power over a predetermined distance of about the size of the resonators[3,4]. The development of non-radiative wireless power transfer has paved the way towards real-world applications such as wireless powering of implantable medical devices and wireless charging of stationary electric vehicles[1,2,5–8]. However, it remains a fundamental challenge to create a wireless power transfer system in which the transfer efficiency is robust against the variation of operating conditions. Here we propose theoretically and demonstrate experimentally that a parity–time-symmetric circuit incorporating a nonlinear gain saturation element provides robust wireless power transfer. Our results show that the transfer efficiency remains near unity over a distance variation of approximately one metre, without the need for any tuning. This is in contrast with conventional methods where high transfer efficiency can only be maintained by constantly tuning the frequency or the internal coupling parameters as the transfer distance or the relative orientation of the source and receiver units is varied. The use of a nonlinear parity–time-symmetric circuit should enable robust wireless power transfer to moving devices or vehicles[9,10].

PT-symmetric quantum field theory

[CMB, S. P. Klevansky, N. Hassanpour, and S. Sarkar, *Phys. Rev. D 98*, 125003 (2018)]

D-dimensional Euclidean-space quantum field theory with a pseudoscalar field

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2}\phi^2(i\phi)^\varepsilon \quad (\varepsilon \geq 0)$$

To avoid renormalization infinities assume $D < 2$.

Objective: Calculate vacuum energy density, renormalized mass, and *all* Green's functions

$$G_1, \;\; G_2(x\text{-}y), \;\; \ldots$$

as perturbation series in powers of ε

If we expand in ε we get logarithmic terms in the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2}\phi^2 + \tfrac{1}{2}\varepsilon\phi^2 \log(i\phi) + \mathrm{O}(\varepsilon^2)$$

Unperturbed Lagrangian is usual free (linear) field theory

$$\mathcal{L}_0 = \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2}\phi^2$$

How do we interpret logarithm term log($i\phi$) ??

$$\log(i\phi) = \tfrac{1}{2}i\pi + \log(\phi) \quad (\phi > 0)$$

$$\log(i\phi) = -\tfrac{1}{2}i\pi + \log(-\phi) \quad (\phi < 0)$$

$$\boxed{\log(i\phi) = \tfrac{1}{2}i\pi\,\frac{|\phi|}{\phi} + \log(|\phi|) = \tfrac{1}{2}i\pi\,\frac{|\phi|}{\phi} + \tfrac{1}{2}\log(\phi^2)}$$

Imaginary term is odd in ϕ
and real term is even in ϕ

--- <u>this enforces ***PT*** symmetry</u>

Free propagator in momentum space:

$$\tilde{\Delta}(p) = \frac{1}{p^2 + 1}$$

coordinate space:

$$\Delta(x_1 - x_2) = (2\pi)^{-\frac{D}{2}} |x_1 - x_2|^{1-\frac{D}{2}} \mathrm{K}_{1-\frac{D}{2}}(|x_1 - x_2|)$$

$$\Delta(0) = (4\pi)^{-D/2} \Gamma(1 - D/2)$$

Partition function:

$$Z(\varepsilon) = \int \mathcal{D}\phi \exp\left(-\int d^D x \mathcal{L}\right)$$

$$Z(0) = \int \mathcal{D}\phi \exp\left(-\int d^D x \mathcal{L}_0\right)$$

Expand $Z(\varepsilon)$ to first order in ε

$$Z(\varepsilon) = \int \mathcal{D}\phi e^{-\int d^D x \mathcal{L}_0}\left[1 - \frac{\varepsilon}{4}\int d^D y \left\{i\pi\phi(y)|\phi(y)| + \phi^2(y)\log\left[\phi^2(y)\right]\right\}\right]$$

Note: Imaginary part is <u>*odd*</u> and real part is <u>*even*</u>

Ground-state energy density

$$\Delta E = \frac{\varepsilon}{4Z(0)V} \int \mathcal{D}\phi e^{-\int d^D x \mathcal{L}_0} \int d^D y \phi^2(y) \log\left[\phi^2(y)\right]$$

Replica trick (Parisi): $\log A = \lim_{N\to 0} \frac{1}{N}\left(A^N - 1\right)$

We do something simpler: $A^2 \log(A^2) = \lim_{N\to 1} \frac{\mathrm{d}}{\mathrm{d}N} A^{2N}$

***Note*: Replica trick not rigorous. Involves a continuous limit. Validity not proved but when compared with known results (in low dimension) it has always worked.**

Final result:

$$\boxed{\begin{aligned}&\Delta E = \tfrac{1}{4}\varepsilon\Delta(0)\left\{\log[2\Delta(0)] + \psi\left(\tfrac{3}{2}\right)\right\}\\ &\psi\left(\tfrac{3}{2}\right) = 2 - \gamma - 2\log 2\\ &\Delta(0) = (4\pi)^{-D/2}\Gamma(1 - D/2)\end{aligned}}$$

Special case $D = 0$ for which $\Delta(0) = 1$: $\Delta E = \frac{\varepsilon}{4}\left[\psi\left(\tfrac{3}{2}\right) + \log 2\right]$

Special case $D = 1$: In quantum mechanics ΔE to leading order in ε is the expectation value of the interaction Hamiltonian

$$H_I = \tfrac{1}{2}\varepsilon x^2 \log(ix)$$

in the unperturbed ground-state wave function $\psi_0(x) = \exp\left(-\tfrac{1}{2}x^2\right)$

$$\Delta E = \frac{\varepsilon}{2}\int_{-\infty}^{\infty} dx\, e^{-x^2} x^2 \log(ix) \Big/ \int_{-\infty}^{\infty} dx\, e^{-x^2} = \tfrac{1}{8}\varepsilon\psi\left(\tfrac{3}{2}\right)$$

At $D = 1$ $\quad \Delta(0) = \frac{1}{2}$

One-point Green's function

$$G_1(a) = \frac{1}{Z(0)} \int \mathcal{D}\phi\, \phi(a) e^{-\int d^Dx \mathcal{L}_0} \left[-\frac{\varepsilon}{4} \int d^D y\, i\pi\phi(y)|\phi(y)| \right]$$

(keeping terms odd in the field)

$$\phi|\phi| = \frac{2}{\pi}\phi^2 \int_0^\infty \frac{dt}{t} \sin(t\phi)$$

$$G_1(a) = -\frac{i\varepsilon}{2} \int_{t=0}^\infty dt \sum_{n=0}^\infty \frac{(-1)^n t^{2n}}{(2n+1)!} \int d^D y$$
$$\times \frac{1}{Z(0)} \int \mathcal{D}\phi\, e^{-\int d^Dx\, \mathcal{L}_0} \phi(a)[\phi(y)]^{2n+3}$$

The second line is $2n+3$ points at y and one point at a:

$$\frac{1}{Z(0)}\int \mathcal{D}\phi\, e^{-L_0}\phi(0)[\phi(y)]^{2n+3} = (2n+3)!!\Delta(y-a)\Delta^{n+1}(0)$$

$$\int d^D y \Delta(y-a) = 1$$

$$\begin{aligned} G_1 &= -\frac{i\varepsilon}{2}\int_{t=0}^{\infty} dt \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}\Delta^{n+1}(0)(2n+3)!!}{(2n+1)!} \\ &= -\frac{i\varepsilon}{2}\int_{t=0}^{\infty} dt\, \Delta(0)\left[3-\Delta(0)t^2\right]e^{-\frac{1}{2}\Delta(0)t^2} \end{aligned}$$

$$\boxed{G_1 = -i\varepsilon\sqrt{\Delta(0)\pi/2}}$$

This is ***negative*** and ***imaginary***

This result is **exact to order ε for all *D***

Check:

When $D = 0$ we get

$$G_1 = -i\varepsilon\sqrt{\pi/2}.$$

When $D = 1$ we get

$$G_1 = -i\varepsilon\sqrt{\pi/4}.$$

This generalizes to higher order in ε and for all Green's functions…

n-point Green's function for n = 1, 3, 4, 5, …

$$G_n(x_1, x_2, \ldots x_n) = -\tfrac{1}{2}\varepsilon(-i)^n\Gamma\left(\tfrac{1}{2}n - 1\right)$$
$$\times\left[\tfrac{1}{2}\Delta(0)\right]^{1-n/2}\int d^D u \prod_{k=1}^{n}\Delta\left(x_k - u\right)$$

2-point Green's function

$$G_2(a-b) = \Delta(a-b) - \varepsilon K \int d^D y \, \Delta(a-y)\Delta(y-b)$$

$$\tilde{G}_2(p) = \frac{1}{p^2+1} - \varepsilon \frac{K}{(p^2+1)^2} + \mathrm{O}\left(\varepsilon^2\right)$$

$$\begin{aligned} K &= \tfrac{1}{2} + \tfrac{1}{2}\Gamma'\left(\tfrac{3}{2}\right)/\Gamma\left(\tfrac{3}{2}\right) + \tfrac{1}{2}\log[2\Delta(0)] \\ &= \tfrac{3}{2} - \tfrac{1}{2}\gamma + \tfrac{1}{2}\log\left[\tfrac{1}{2}\Delta(0)\right] \end{aligned}$$

Read off renormalized mass:

$$\tilde{G}_2(p) = \frac{1}{p^2 + 1 + \varepsilon K + \mathrm{O}\left(\varepsilon^2\right)}$$

$$M_{\mathrm{R}}^2 = 1 + K\varepsilon + \mathrm{O}\left(\varepsilon^2\right)$$

$$\begin{aligned} K &= \tfrac{1}{2} + \tfrac{1}{2}\Gamma'\left(\tfrac{3}{2}\right)/\Gamma\left(\tfrac{3}{2}\right) + \tfrac{1}{2}\log[2\Delta(0)] \\ &= \tfrac{3}{2} - \tfrac{1}{2}\gamma + \tfrac{1}{2}\log\left[\tfrac{1}{2}\Delta(0)\right] \end{aligned}$$

Comment on renormalization…

for coming to my talk!