Large N Tensor Models

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How I Began Working on Large N

- In winter of 1984 I asked Curt Callan to be my Ph.D. adviser.

- He said: “Well, let’s see how things go. Why don’t you think about what infinite nuclear matter would look like in large N QCD.”

Skyrmion Crystal

- After reading REAL nuclear physics papers and doing some estimates, we decided that at large N it would form a crystal!
- Mass and nucleon-nucleon potential ~N, while kinetic energy if of order 1.
- In the Skyrme model a baryon is described by a topological soliton in the non-linear sigma model with unit winding number for the map S^3 -> S^3.

$$B = \frac{1}{24\pi^2}\varepsilon^{ijk}\int d^3x\, \mathrm{Tr}\left(\partial_i U U^\dagger \partial_j U U^\dagger \partial_k U U^\dagger\right)$$

$$\mathcal{L} = \tfrac{1}{16} f_\pi^2\, \mathrm{Tr}\, \partial_\mu U \partial^\mu U^\dagger + \frac{1}{32e^2}\mathrm{Tr}\left[\partial_\mu U U^\dagger, \partial_\nu U U^\dagger\right]^2$$

Over a year later

Nuclear Physics B262 (1985) 133–143

NUCLEAR MATTER IN THE SKYRME MODEL

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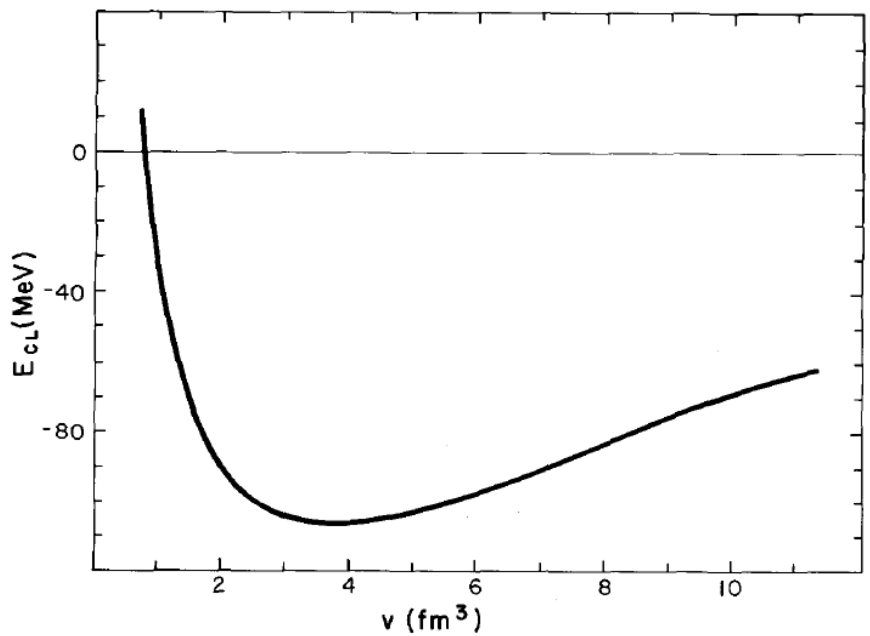
The applicability of the Skyrme model to the analysis of nuclear matter is discussed. Constraints on an ansatz that describes a cubic skyrmion crystal are presented. Properties of this ansatz are studied numerically. Results are used to discuss nuclear matter in the large-N limit and at $N = 3$.

- The energy per nucleon is minimized at densities higher than nuclear density, where individual skyrmions “lose their identity.”



- Such a state may be relevant to the core of a neutron star.

Baby Skyrmion Crystals

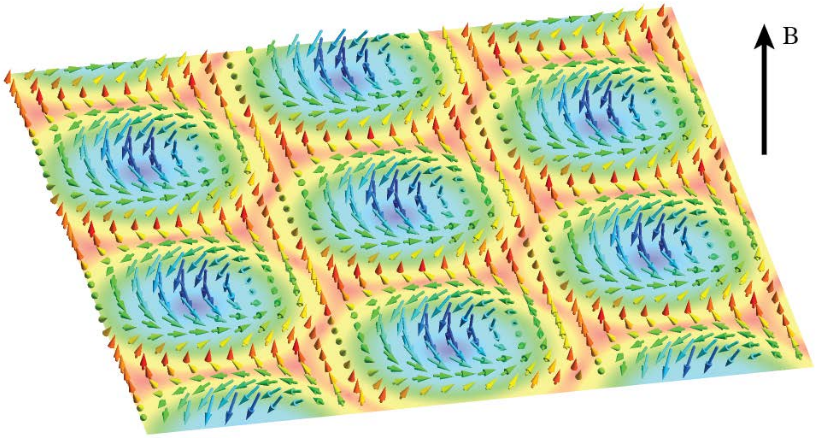
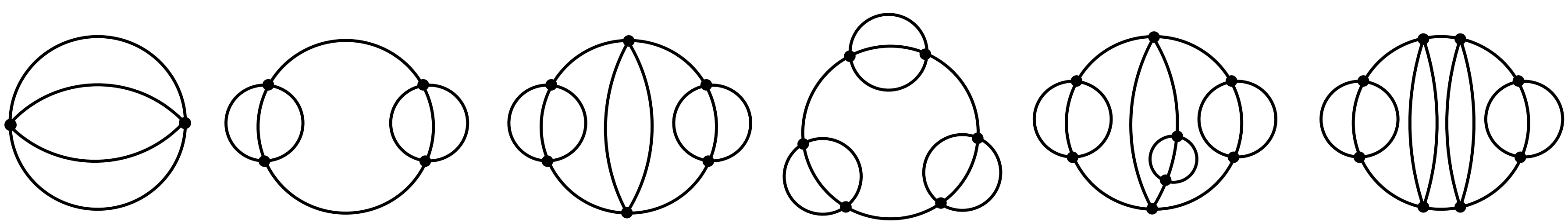


Image by Christoph Schuette

- Remarkably, "Skyrmion crystal" has become a frequently mentioned term in condensed matter physics. These are magnetic "baby Skyrmions" described by maps S^2 -> S^2 in the classical Heisenberg model.

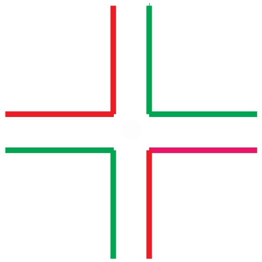
Three Large N Limits

- O(N) Vector: solvable because the bubble diagrams can be summed. 
- Matrix (‘t Hooft) Limit: planar diagrams. Solvable only in special cases.
- Tensor of rank three and higher. When interactions are specially chosen, dominated by the “melonic” diagrams. Bonzom, Gurau, Riello, Rivasseau; Carrozza, Tanasa; Witten; IK, Tarnopolsky



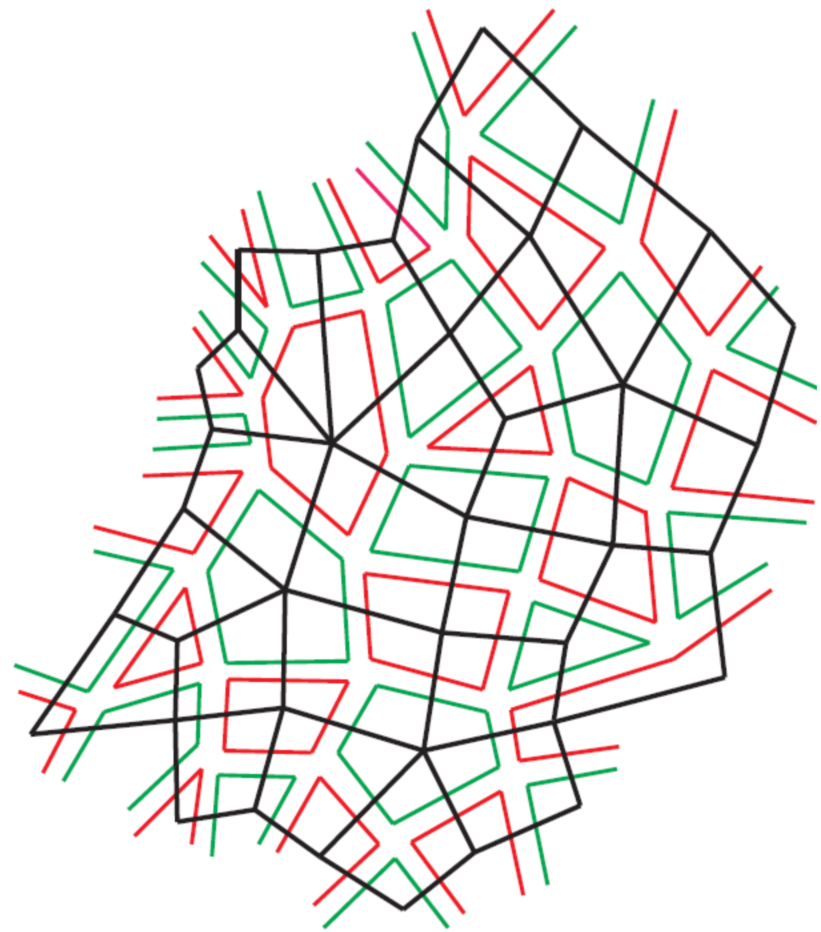
O(N) x O(N) Matrix Model

- Theory of real matrices ϕ^{ab} with distinguishable indices, i.e. in the bi-fundamental representation of $O(N)_a x O(N)_b$ symmetry.
- The interaction is at least quartic: $g\, \mathrm{tr}\, \phi\phi^T\phi\phi^T$
- Propagators are represented by colored double lines, and the interaction vertex is



- In d=0 or 1 special limits describe two-dimensional quantum gravity.

- In the large N limit where gN is held fixed we find planar Feynman graphs, and each index loop may be red or green.
- The dual graphs shown in black may be thought of as random surfaces tiled with squares whose vertices have alternating colors (red, green, red, green).



From Bi- to Tri-Fundamentals

- For a 3-tensor with distinguishable indices the propagator has index structure

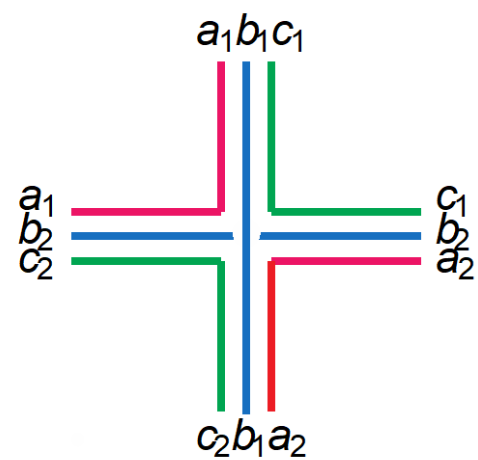
$$\langle \phi^{abc} \phi^{a'b'c'} \rangle = \delta^{aa'} \delta^{bb'} \delta^{cc'}$$

- It may be represented graphically by 3 colored wires 

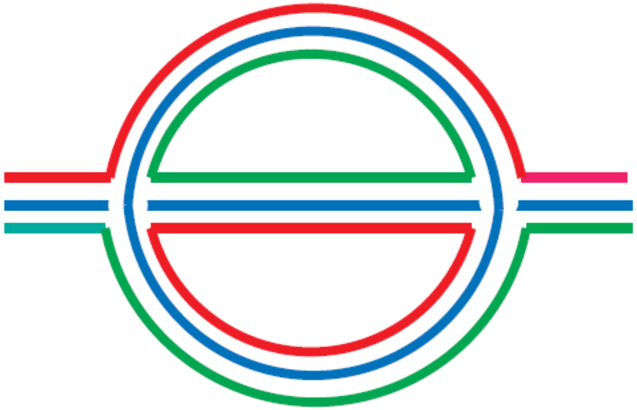
- Tetrahedral interaction with $O(N)_a \times O(N)_b \times O(N)_c$ symmetry

Carrozza, Tanasa; IK, Tarnopolsky

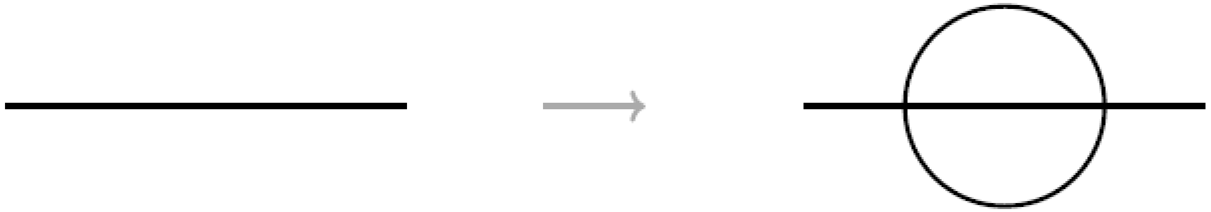
$$\frac{1}{4} g \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_1}$$



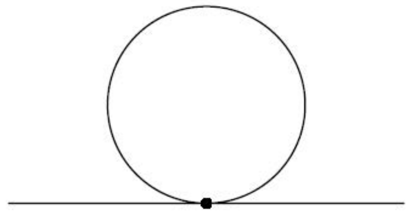
- Leading correction to the propagator has 3 index loops

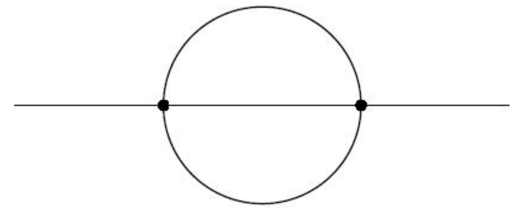


- Requiring that this “melon” insertion is of order 1 means that $\lambda = gN^{3/2}$ must be held fixed in the large N limit.
- Melonic graphs obtained by iterating



Snails vs. Melons

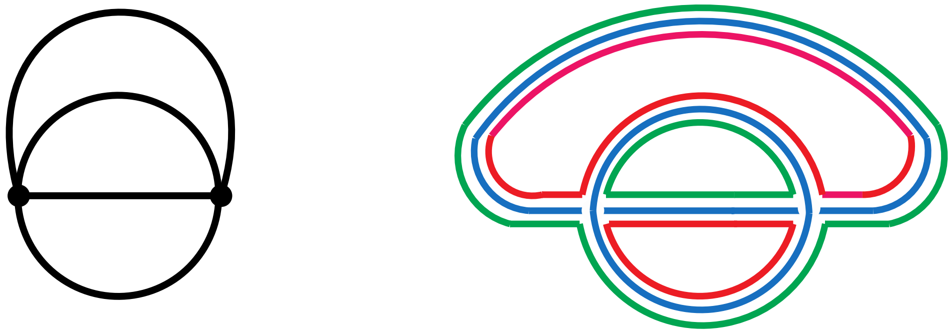




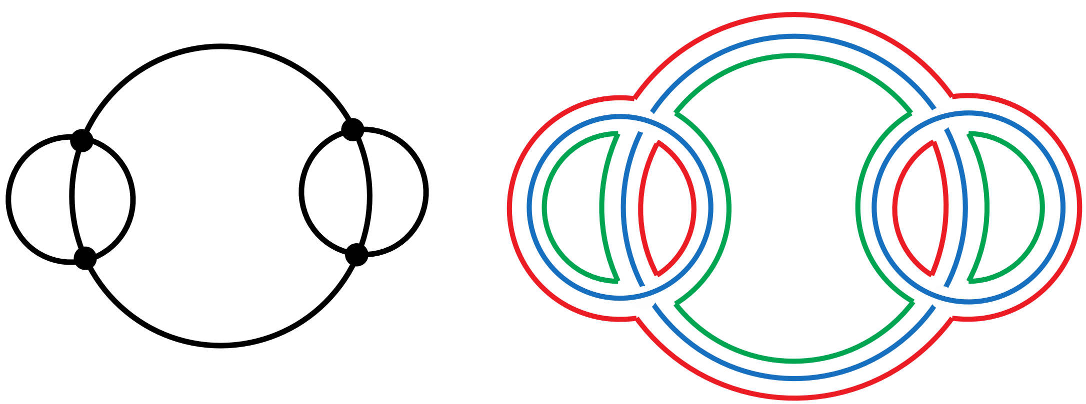
- In large N vector models snail diagrams dominate.
- In matrix models both contribute.
- In tensor models with tetrahedral interactions the melons dominate.

Cables and Wires

- The Feynman graphs of the quartic field theory may be resolved in terms of the colored wires (triple lines)



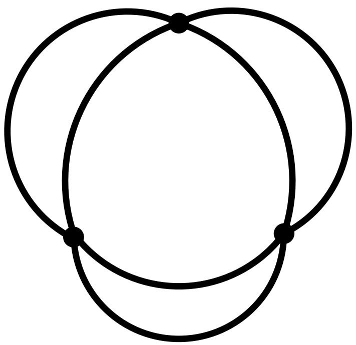
$$g^2 N^6 \sim N^3 \lambda^2$$

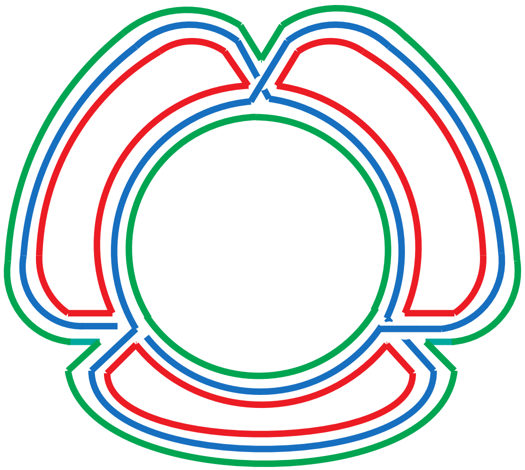


$$g^4 N^9 \sim N^3 \lambda^4$$

Non-Melonic Graphs

- Most Feynman graphs in the quartic field theory are not melonic are therefore subdominant in the new large N limit, e.g.

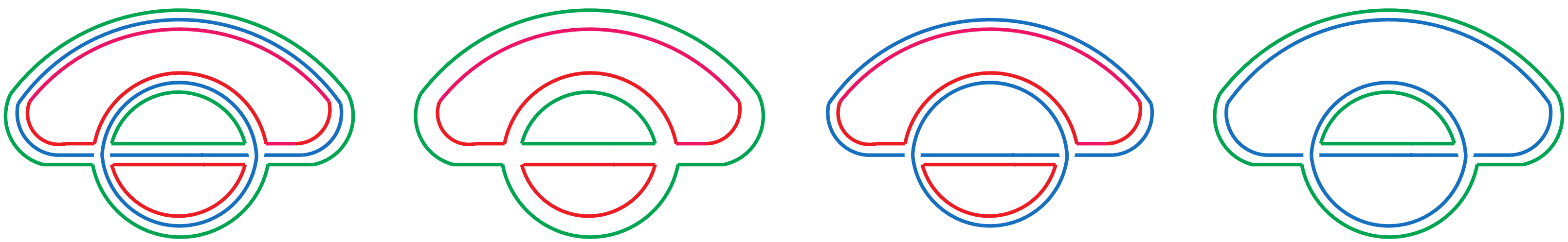




- Scales as $g^3 N^6 \sim N^3 \lambda^3 N^{-3/2}$
- None of the graphs with an odd number of vertices are melonic.

Large N Scaling

- "Forgetting" one color we get a double-line graph.



- The number of loops in a double-line graph is

$f = \chi + e - v$ where χ is the Euler characteristic, e is the number of edges, and v is the number of vertices, $e = 2v$

- If we erase the blue lines we get $f_{rg} = \chi_{rg} + v$

- Adding up such formulas, we find

$$f_{bg} + f_{rg} + f_{br} = 2(f_b + f_g + f_r) = \chi_{bg} + \chi_{br} + \chi_{rg} + 3v$$

- The total number of index loops is

$$f_{\text{total}} = f_b + f_g + f_r = \frac{3v}{2} + 3 - g_{bg} - g_{br} - g_{rg}$$

- The genus of a graph is $g = 1 - \chi/2$
- Since $g \geqslant 0$, for a "maximal graph" which dominates at large N all its subgraphs must have genus zero: $f_{\text{total}} = 3 + 3v/2$
- Scales as $N^3 (gN^{3/2})^v$
- In the 3-tensor models $\lambda = gN^{3/2}$ must be held fixed in the large N limit.

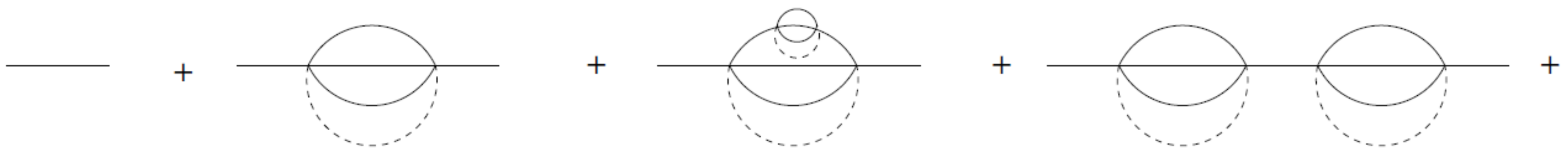
The Sachdev-Ye-Kitaev Model

- Quantum mechanics of a large number N_{SYK} of anti-commuting variables with action

$$I = \int \mathrm{d}t \left(\frac{\mathrm{i}}{2} \sum_i \psi_i \frac{\mathrm{d}}{\mathrm{d}t} \psi_i - \mathrm{i}^{q/2} j_{i_1 i_2 \ldots i_q} \psi_{i_1} \psi_{i_2} \ldots \psi_{i_q} \right)$$

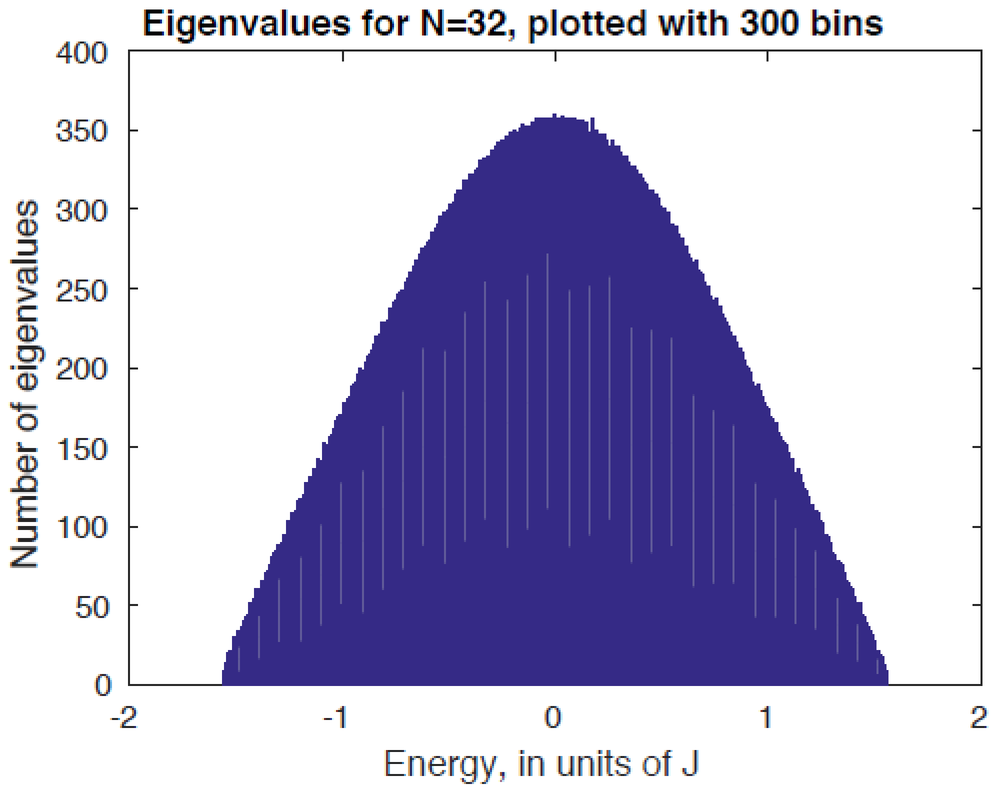
- Random couplings j have a Gaussian distribution with zero mean.
- The model flows to strong coupling and becomes nearly conformal. Sachdev, Ye; Georges, Parcollet, Sachdev; Kitaev; Polchinski, Rosenhaus; Maldacena, Stanford; Jevicki, Suzuki, Yoon; Kitaev, Suh

- The simplest dynamical case is q=4.
- Exactly solvable in the large N_{SYK} limit because only the melonic Feynman diagrams contribute



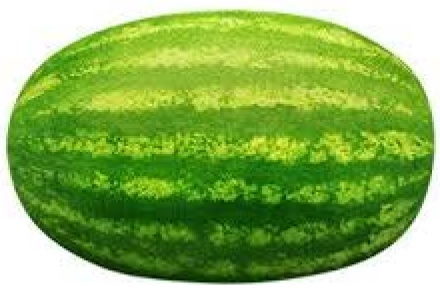
- Solid lines are fermion propagators, while dashed lines mean disorder average.
- The exact solution shows resemblance with physics of certain two-dimensional black holes.
  Kitaev; Almheiri, Polchinski; Sachdev; Maldacena, Stanford, Yang; Engelsoy, Mertens, Verlinde; Jensen; Kitaev, Suh; …

- Spectrum for a single realization of N_{SYK}=32 model with q=4. Maldacena, Stanford
- No exact degeneracies, but the gaps are exponentially small. Large low T entropy.



Random No More!

- E. Witten, “An SYK-Like Model Without Disorder,” arXiv: 1610.09758.
- Appeared on the evening of Halloween: October 31, 2016.


- It is sometimes tempting to change the term “melonic diagrams” to “pumpkinlike diagrams.”

The O(N)^3 Model

- A pruned version: there are N^3 Majorana fermions IK, Tarnopolsky

$$\{\psi^{abc}, \psi^{a'b'c'}\} = \delta^{aa'}\delta^{bb'}\delta^{cc'}$$

$$H = \frac{g}{4}\psi^{abc}\psi^{ab'c'}\psi^{a'bc'}\psi^{a'b'c} - \frac{g}{16}N^4$$

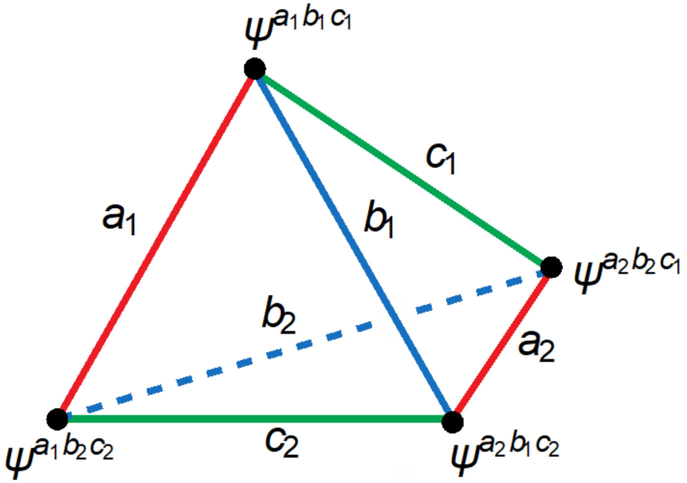
- Has $O(N)_a$x$O(N)_b$x$O(N)_c$ symmetry under

$$\psi^{abc} \to M_1^{aa'} M_2^{bb'} M_3^{cc'} \psi^{a'b'c'}, \qquad M_1, M_2, M_3 \in O(N)$$

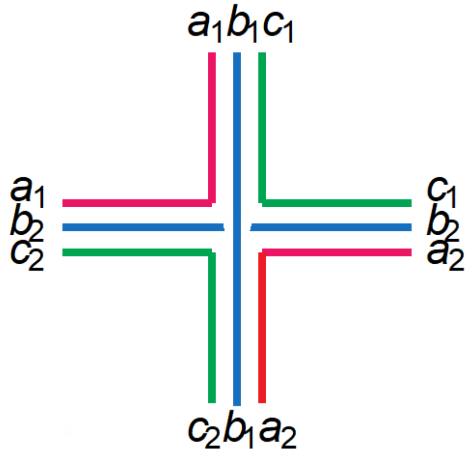
- The SO(N) symmetry charges are

$$Q_1^{aa'} = \frac{i}{2}[\psi^{abc}, \psi^{a'bc}]\,, \qquad Q_2^{bb'} = \frac{i}{2}[\psi^{abc}, \psi^{ab'c}]\,, \qquad Q_3^{cc'} = \frac{i}{2}[\psi^{abc}, \psi^{abc'}]$$

- The 3-tensors may be associated with indistinguishable vertices of a tetrahedron.

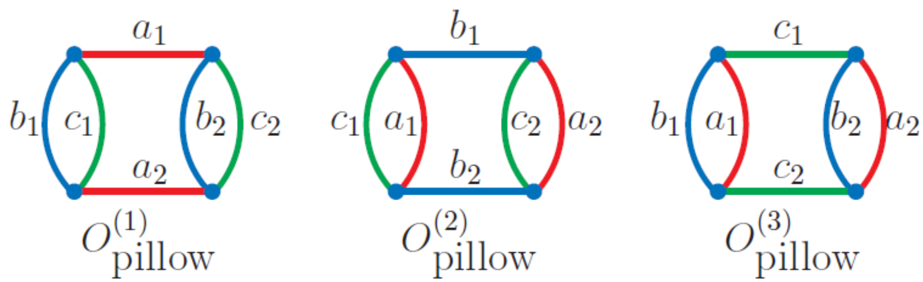


- This is equivalent to



- The triple-line Feynman graphs are produced using the propagator

- The tetrahedral term is the unique dynamical quartic interaction with $O(N)^3$ symmetry.
- The other possible terms are quadratic Casimirs of the three SO(N) groups.



$$O^{(1)}_{\text{pillow}} = \sum_{a_1<a_2} Q_1^{a_1a_2} Q_1^{a_1a_2} , \qquad O^{(2)}_{\text{pillow}} = \sum_{b_1<b_2} Q_2^{b_1b_2} Q_2^{b_1b_2} , \qquad O^{(3)}_{\text{pillow}} = \sum_{c_1<c_2} Q_3^{c_1c_2} Q_3^{c_1c_2}$$

- In the model where $SO(N)^3$ is gauged, they vanish.

$O(N)^3$ vs. SYK Model

- Using composite indices $I_k = (a_k b_k c_k)$

$$H = \frac{1}{4!} J_{I_1 I_2 I_3 I_4} \psi^{I_1} \psi^{I_2} \psi^{I_3} \psi^{I_4}$$

The couplings take values $0, \pm 1$

$$J_{I_1 I_2 I_3 I_4} = \delta_{a_1 a_2} \delta_{a_3 a_4} \delta_{b_1 b_3} \delta_{b_2 b_4} \delta_{c_1 c_4} \delta_{c_2 c_3} - \delta_{a_1 a_2} \delta_{a_3 a_4} \delta_{b_2 b_3} \delta_{b_1 b_4} \delta_{c_2 c_4} \delta_{c_1 c_3} + 22 \text{ terms}$$

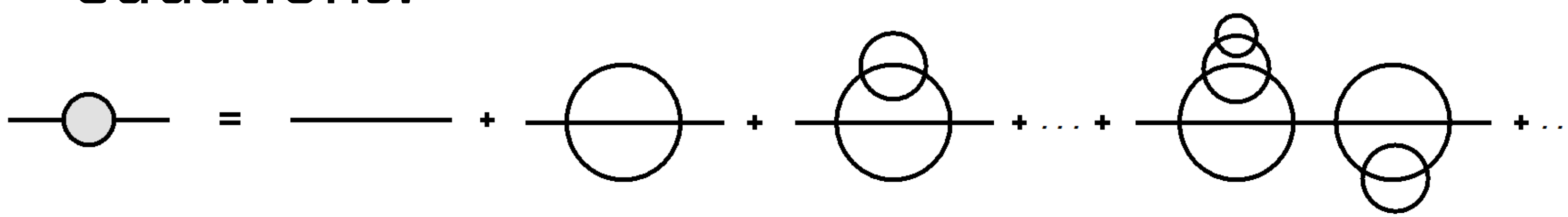
- The number of distinct terms is

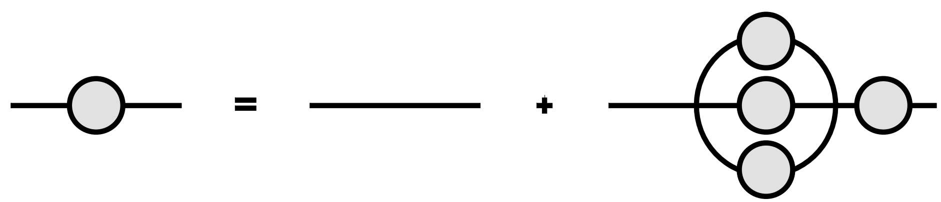
$$\frac{1}{4!} \sum_{\{I_k\}} J^2_{I_1 I_2 I_3 I_4} = \frac{1}{4} N^3 (N-1)^2 (N+2)$$

- Much smaller than in SYK model with $N_{\mathrm{SYK}} = N^3$

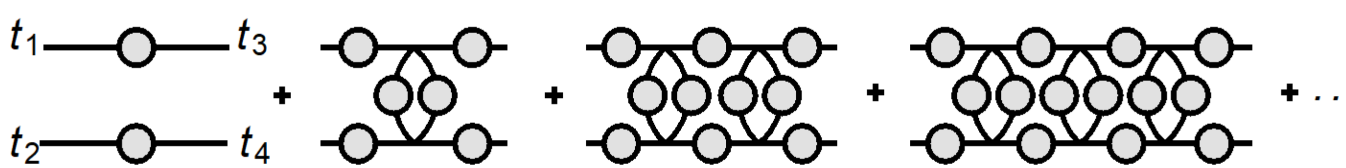
$$\frac{1}{24} N^3 (N^3 - 1)(N^3 - 2)(N^3 - 3)$$

- The tensor Hamiltonian is much sparser.
- Nevertheless, both are solvable in the large N limit using the same Schwinger-Dyson equations.





$$G(t_1 - t_2) = -\left(\frac{1}{4\pi g^2 N^3}\right)^{1/4} \frac{\mathrm{sgn}(t_1 - t_2)}{|t_1 - t_2|^{1/2}}$$



Conclusions

- Large N methods are ubiquitous in theoretical physics, ranging from the Skyrme to SYK and Tensor models.
- The $O(N)^3$ fermionic tensor quantum mechanics seems to be the closest counterpart of the basic SYK model for Majorana fermions.
- Solution of S-D equations indicates a (nearly) conformal phase with real scaling dimensions.
- At finite N there is a discrete spectrum. More work is needed on approach to large N limit.

Further Results

- Large N renormalizable QFT with tensor degrees of freedom and $O(N)^3$ symmetry.
- A frequent feature: formally complex scaling dimensions. They indicate an instability of the conformal phase.
- Example: two coupled tensor or SYK models. For one sign of coupling there is spontaneous symmetry breaking in the large N limit.

$$H = \frac{1}{4!} J_{ijkl} \left(\chi_1^i \chi_1^j \chi_1^k \chi_1^l + \chi_2^i \chi_2^j \chi_2^k \chi_2^l + 6\alpha \chi_1^i \chi_1^j \chi_2^k \chi_2^l \right)$$

Tetrahedral Bosonic Tensor Model

- Action with a potential that is not positive definite IK, Tarnopolsky; Giombi, IK, Tarnopolsky

$$S = \int d^d x \left(\frac{1}{2} \partial_\mu \phi^{abc} \partial^\mu \phi^{abc} + \frac{1}{4} g \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_1} \right)$$

- Schwinger-Dyson equation for 2pt function
  Patashinsky, Pokrovsky

$$G^{-1}(p) = -\lambda^2 \int \frac{d^d k d^d q}{(2\pi)^{2d}} G(q) G(k) G(p+q+k)$$

- Has solution

$$G(p) = \lambda^{-1/2} \left(\frac{(4\pi)^d d \Gamma(\frac{3d}{4})}{4\Gamma(1-\frac{d}{4})} \right)^{1/4} \frac{1}{(p^2)^{\frac{d}{4}}}$$

Spectrum of two-particle spin zero operators

- Schwinger-Dyson equation

$$\int d^dx_3 d^dx_4 K(x_1, x_2; x_3, x_4) v_h(x_3, x_4) = g(h) v_h(x_1, x_2)$$

$$K(x_1, x_2; x_3, x_4) = 3\lambda^2 G(x_{13}) G(x_{24}) G(x_{34})^2$$

$$v_h(x_1, x_2) = \frac{1}{\left[(x_1 - x_2)^2\right]^{\frac{1}{2}\left(\frac{d}{2} - h\right)}}$$

$$g_{\text{bos}}(h) = -\frac{3\Gamma\left(\frac{3d}{4}\right)\Gamma\left(\frac{d}{4} - \frac{h}{2}\right)\Gamma\left(\frac{h}{2} - \frac{d}{4}\right)}{\Gamma\left(-\frac{d}{4}\right)\Gamma\left(\frac{3d}{4} - \frac{h}{2}\right)\Gamma\left(\frac{d}{4} + \frac{h}{2}\right)}$$

- In d<4 the first solution is complex $\frac{d}{2} + i\alpha(d)$

Complex Fixed Point in 4-ε Dimensions

- The tetrahedron

$$O_t(x) = \phi^{a_1b_1c_1}\phi^{a_1b_2c_2}\phi^{a_2b_1c_2}\phi^{a_2b_2c_1}$$

mixes at finite N with the pillow and double-sum operators

$$O_p(x) = \frac{1}{3}\left(\phi^{a_1b_1c_1}\phi^{a_1b_1c_2}\phi^{a_2b_2c_2}\phi^{a_2b_2c_1} + \phi^{a_1b_1c_1}\phi^{a_2b_1c_1}\phi^{a_2b_2c_2}\phi^{a_1b_2c_2} + \phi^{a_1b_1c_1}\phi^{a_1b_2c_1}\phi^{a_2b_1c_2}\phi^{a_2b_2c_2}\right),$$

$$O_{ds}(x) = \phi^{a_1b_1c_1}\phi^{a_1b_1c_1}\phi^{a_2b_2c_2}\phi^{a_2b_2c_2}$$

- The renormalizable action is

$$S = \int d^dx\left(\frac{1}{2}\partial_\mu\phi^{abc}\partial^\mu\phi^{abc} + \frac{1}{4}\left(g_1O_t(x) + g_2O_p(x) + g_3O_{ds}(x)\right)\right)$$

- The large N scaling is

$$g_1 = \frac{(4\pi)^2 \tilde{g}_1}{N^{3/2}}, \quad g_2 = \frac{(4\pi)^2 \tilde{g}_2}{N^2}, \quad g_3 = \frac{(4\pi)^2 \tilde{g}_3}{N^3}$$

- The 2-loop beta functions and fixed points:

$$\tilde{\beta}_t = -\epsilon \tilde{g}_1 + 2\tilde{g}_1^3 ,$$

$$\tilde{\beta}_p = -\epsilon \tilde{g}_2 + \left(6\tilde{g}_1^2 + \frac{2}{3}\tilde{g}_2^2\right) - 2\tilde{g}_1^2 \tilde{g}_2 ,$$

$$\tilde{\beta}_{ds} = -\epsilon \tilde{g}_3 + \left(\frac{4}{3}\tilde{g}_2^2 + 4\tilde{g}_2\tilde{g}_3 + 2\tilde{g}_3^2\right) - 2\tilde{g}_1^2(4\tilde{g}_2 + 5\tilde{g}_3)$$

$$\tilde{g}_1^* = (\epsilon/2)^{1/2}, \quad \tilde{g}_2^* = \pm 3i(\epsilon/2)^{1/2}, \quad \tilde{g}_3^* = \mp i(3 \pm \sqrt{3})(\epsilon/2)^{1/2}$$

- The scaling dimension of $\phi^{abc}\phi^{abc}$ is

$$\Delta_O = d - 2 + 2(\tilde{g}_2^* + \tilde{g}_3^*) = 2 \pm i\sqrt{6\epsilon} + \mathcal{O}(\epsilon)$$