Liouville Quantum Multifractality

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Based on joint work with

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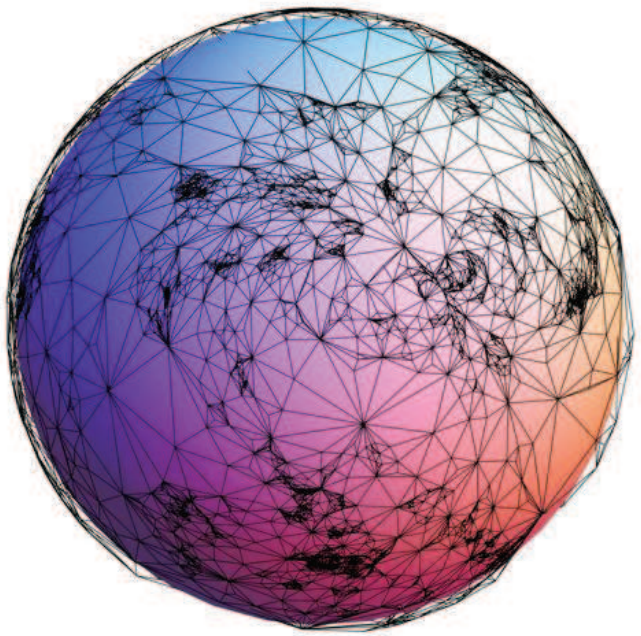
Random Planar Map & Conformal Map



[Courtesy of N. Curien]

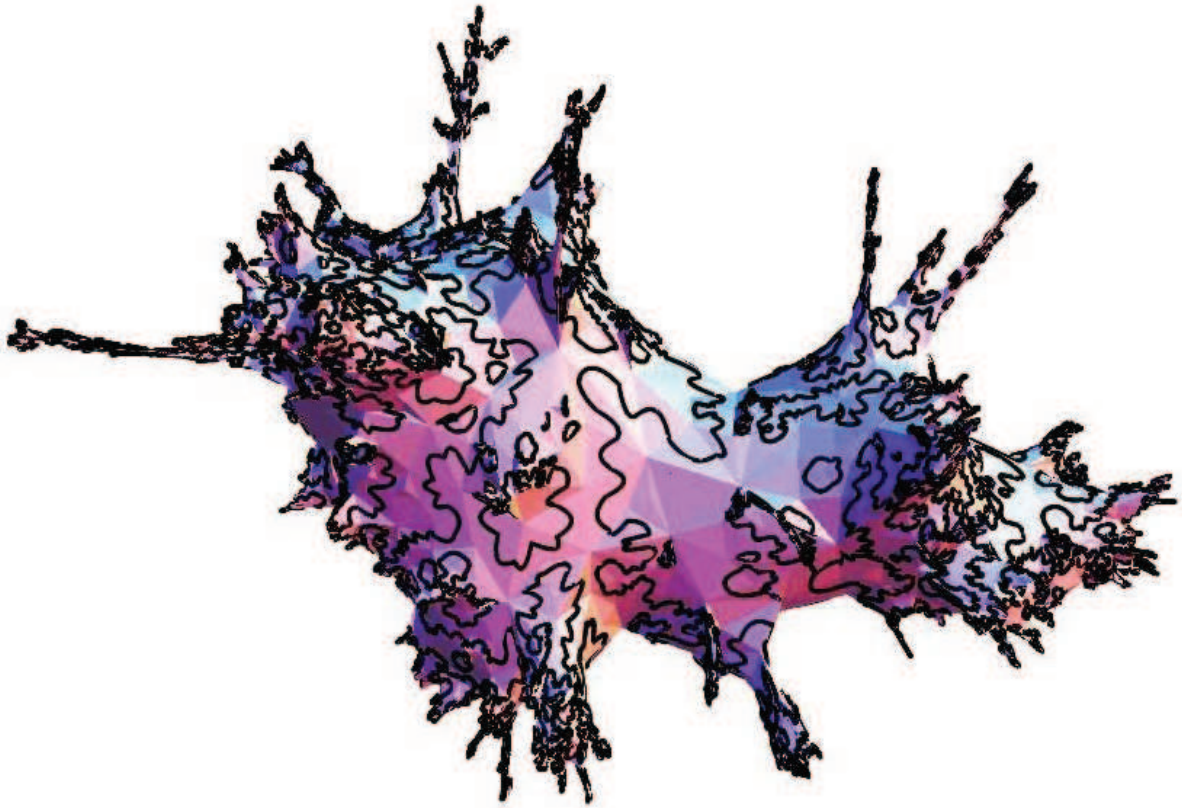
Left: A random triangulation of the sphere. Right: Conformal map to the sphere.

In the continuum scaling limit: **Liouville Quantum Gravity**

A.M. Polyakov '81

(See also J. Teschner's talk)

Random Planar Maps & Statistical Models



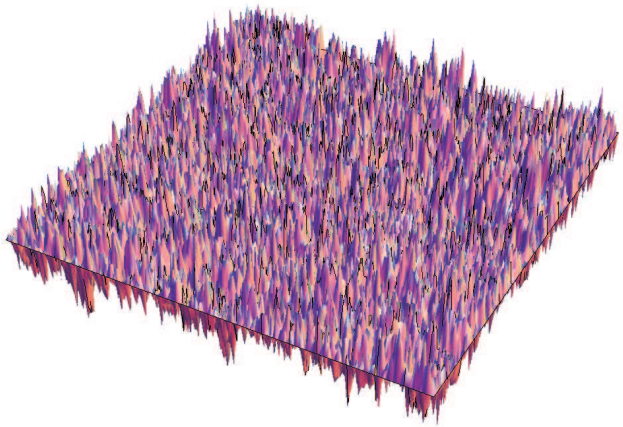
Percolation hulls [Courtesy of N. Curien].

Liouville QG Random Measure

$$\mu = \text{“}e^{\gamma h} dz\text{”}$$

$\diamondsuit$

Gaussian Free Field (GFF)



[Courtesy of J. Miller]

Distribution h with *Gaussian weight* $\exp\left[-\frac{1}{2}(h,h)_\nabla\right]$, and **Dirichlet inner product** in domain D

$$\begin{aligned}(f_1, f_2)_\nabla &:= (2\pi)^{-1}\int_D \nabla f_1(z)\cdot\nabla f_2(z)dz \\ &= \mathrm{Cov}\big((h, f_1)_\nabla, (h, f_2)_\nabla\big)\end{aligned}$$

Liouville Quantum Measure

$$\mu_\gamma := \lim_{\varepsilon \to 0} \exp\left[\gamma h_\varepsilon(z)\right] \varepsilon^{\gamma^2/2} dz,$$

where $h_\varepsilon(z)$ is the GFF average on a circle of radius ε, converges weakly for $\gamma < 2$ to a random measure, denoted by $e^{\gamma h(z)} dz$.

[Høegh-Krohn '71; Kahane '85; D. & Sheffield '11]

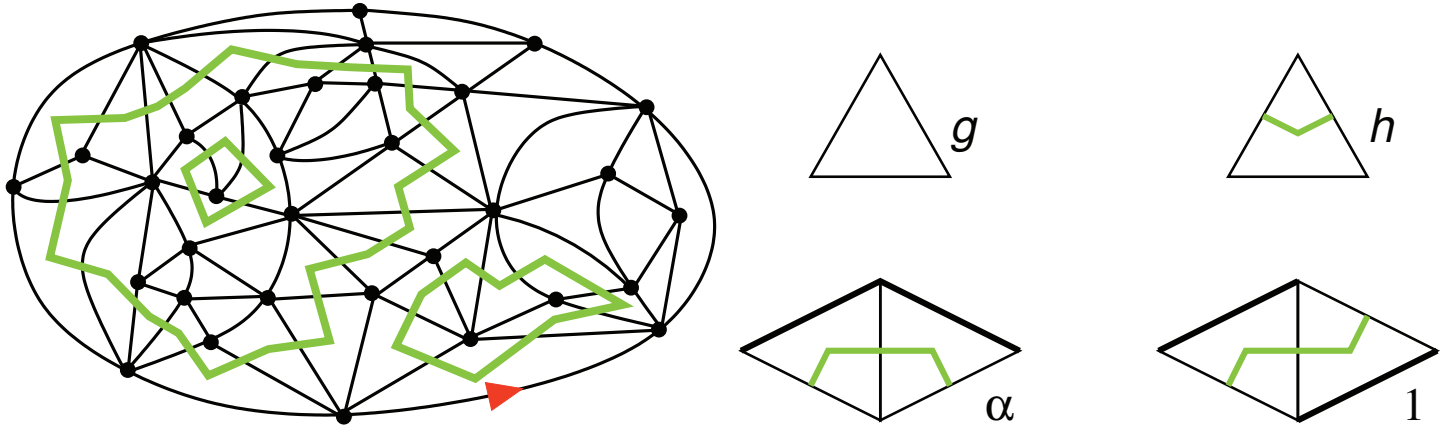
For $\gamma = 2$, the **renormalized** *one,*

$$\sqrt{\log(1/\varepsilon)} \left[\exp\left[\gamma h_\varepsilon(z)\right] \varepsilon^{\gamma^2/2}\right]_{\gamma=2} dz,$$

converges, as $\varepsilon \to 0$, to a positive non-atomic random measure.

[D., Rhodes, Sheffield, Vargas '14]

$O(n)$ model on a Random Planar Map

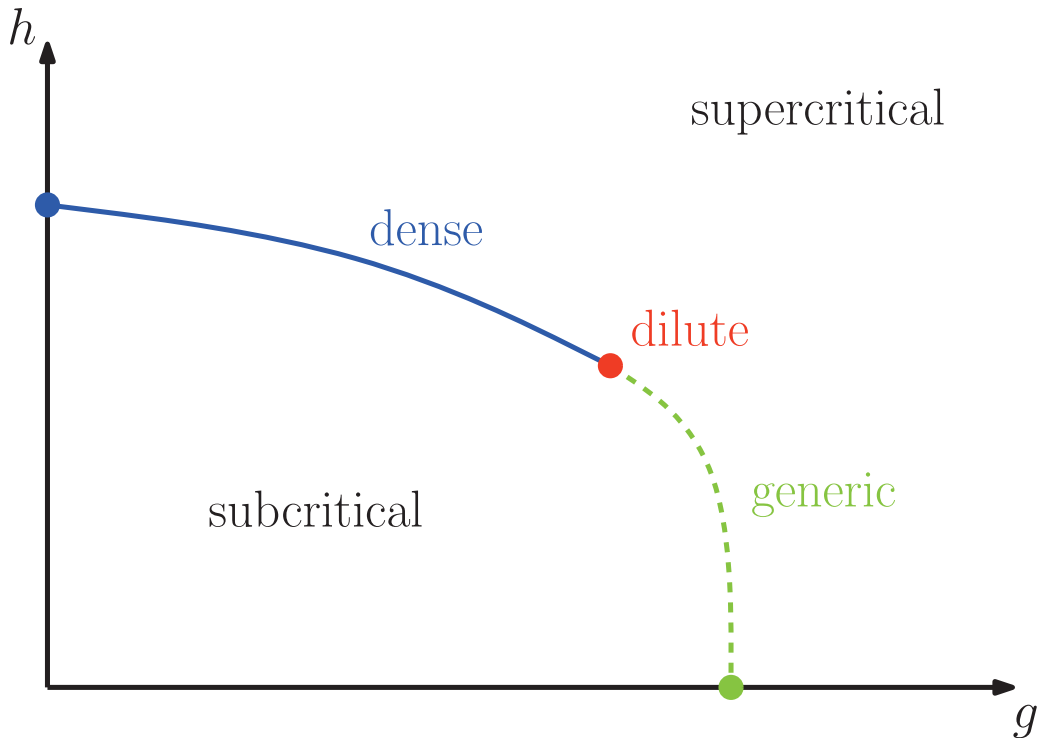


A disk triangulation $\mathcal{C}$, and its local weights ($\alpha = 1$ here).

$$Z_\ell = \sum_{\mathcal{C}} u^{V(\mathcal{C})} w(\mathcal{C}), \;\; w(C) = n^{\mathcal{L}} g^{T_1} h^{T_2},$$

the sum runs over all configurations $\mathcal{C}$ of a **disk** of fixed perimeter ℓ; u is an auxiliary weight per **vertex**, $V(\mathcal{C})$ denotes the number of vertices of the map of $\mathcal{C}$ (*volume*); T_1, T_2 are the numbers of empty or occupied triangles; the number of **loops** $\mathcal{L}$ of $\mathcal{C}$ is weighted by $n \in [0,2]$.

Phase diagram and critical loci



*Qualitative phase diagram of the $O(n)$ loop model ($n \in [0,2]$) on a random map. For $u = 1$, a line of critical points separates the subcritical and supercritical phase. Critical points may be in three different universality classes: generic (**pure gravity**), **dilute** and **dense**.*

Random Map Nesting Theorem *[Borot, Bouttier, D. '16]*

Fix (g,h) *and* $n \in (0,2)$ *such that the model reaches a* **dilute** *or* **dense** *critical point for the vertex weight* $u = 1$. In the ensemble of random pointed disks of volume V and perimeter L, the probability distribution of the number $\mathcal{N}$ of **separating loops** between the marked point and the boundary behaves as:

$$\mathbb{P}\left[\mathcal{N} = \frac{c \ln V}{\pi} p \,\middle|\, V, L = \ell\right] \dot{\sim} (\ln V)^{-\frac{1}{2}} V^{-\frac{c}{\pi} J(p)} \quad \text{(sphere)},$$

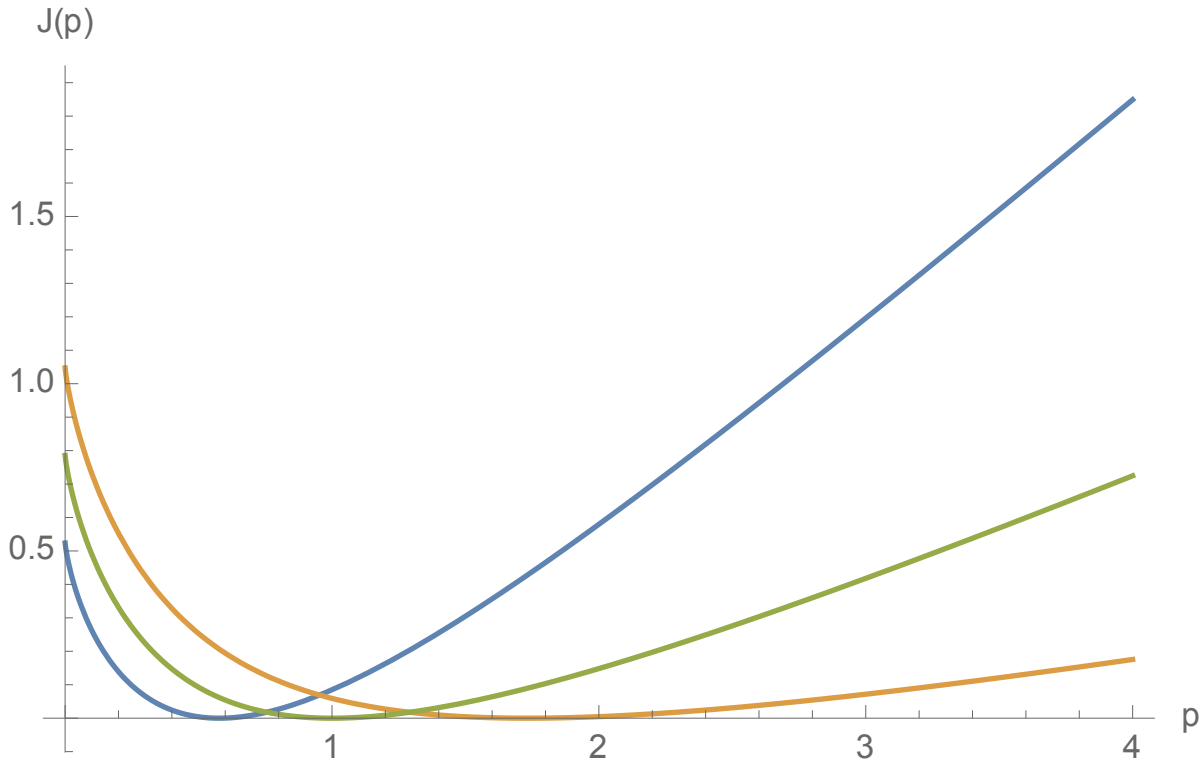
$$\mathbb{P}\left[\mathcal{N} = \frac{c \ln V}{2\pi} p \,\middle|\, V, L = V^{\frac{c}{2}} \ell\right] \dot{\sim} (\ln V)^{-\frac{1}{2}} V^{-\frac{c}{2\pi} J(p)} \quad \text{(disk)},$$

where $\ell > 0$ is fixed, and $\ln V \gg p$, and:

$$J(p) = p \ln\left(\frac{2}{n} \frac{p}{\sqrt{1+p^2}}\right) + \operatorname{arccot}(p) - \arccos(n/2).$$

with $c = 1$ (**dilute**), $c = 1/[1 - \frac{1}{\pi} \arccos\left(\frac{n}{2}\right)]$ (**dense**), which decreases from 2 to 1 as n increases from 0 to 2.

Large Deviations Function



$J(p)$ for $n = 1$ (blue), $n = \sqrt{2}$ (Ising, green) and $n = \sqrt{3}$ (3-Potts, orange).

The Conformal Loop Ensemble (CLE)

[Sheffield '09, Sheffield & Werner '12]

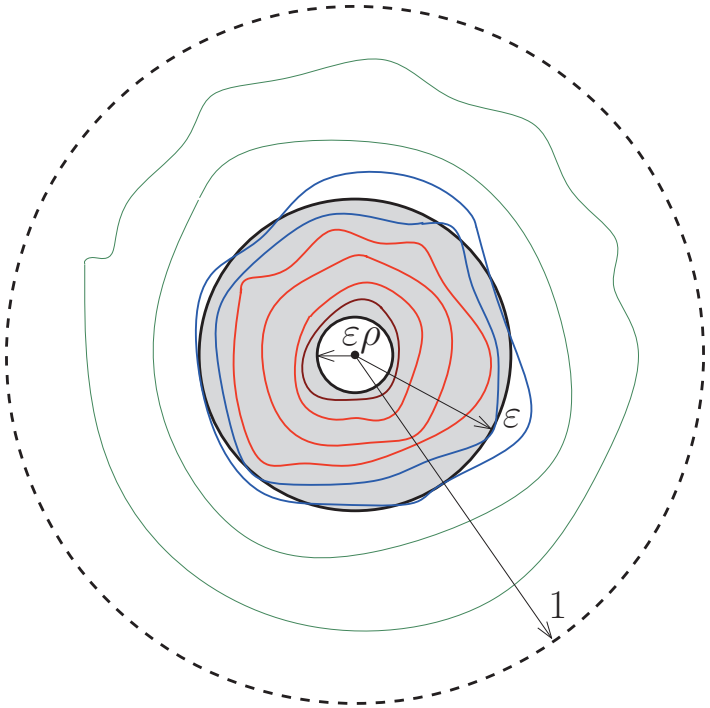
The critical $O(n)$-model on a regular planar lattice is expected to converge in the continuum limit to $\mathrm{SLE}_\kappa/\mathrm{CLE}_\kappa$, for

$$n = -2\cos\left(4\pi/\kappa\right), \qquad n \in (0,2], \qquad \begin{cases} \kappa \in (8/3,4], & \text{dilute phase} \\ \kappa \in [4,8), & \text{dense phase,} \end{cases}$$

(Loop-erased random walk & spanning trees *[Lawler, Schramm, Werner]*, Ising & percolation *[Smirnov]*, GFF contour lines *[Schramm-Sheffield]*.) The same is expected in the critical dilute or dense phase on a **random planar map**, the random area measure being in the scaling limit the **Liouville quantum measure** for

$$\gamma = \min\{\sqrt{\kappa}, 4/\sqrt{\kappa}\}.$$

Nesting in the Conformal Loop Ensemble (CLE)



$\mathcal{N}_z(\varepsilon)$ *is the* **number of nested loops** *of a* CLE_κ, $\kappa \in (8/3, 8)$ *surrounding the ball* $B(z, \varepsilon)$ *in the unit disk.*

Extreme nesting in CLE *[Miller, Watson & Wilson '14]*

Let $\mathcal{N}_z(\varepsilon)$ be the number of loops of a CLE_κ, $\kappa \in (8/3, 8)$ surrounding the ball $B(z, \varepsilon)$, and Φ_ν the set of points z where

$$\lim_{\varepsilon \to 0} \mathcal{N}_z(\varepsilon) / \ln(1/\varepsilon) = \nu.$$

$$\dim_{\mathcal{H}} \Phi_\nu = 2 - \gamma_\kappa(\nu)$$

$$\gamma_\kappa(\nu) = \nu \Lambda_\kappa^*(1/\nu), \nu \geqslant 0; \quad \Lambda_\kappa^*(x) := \sup_{\lambda \in \mathbb{R}} (\lambda x - \Lambda_\kappa(\lambda))$$

$$\Lambda_\kappa(\lambda) = \ln \left(\frac{-\cos(4\pi/\kappa)}{\cos\left(\pi \sqrt{\left(1 - \frac{4}{\kappa}\right)^2 + \frac{8\lambda}{\kappa}}\right)} \right)$$

Moment generating function of the loop log-conformal radius *[Cardy & Ziff '02; Kenyon & Wilson '04; Schramm, Sheffield & Wilson '09]*

Large Deviations

Euclidean case: for a ball of radius ε

$$\mathbb{P}\big(\mathcal{N}_z \approx \nu \ln(1/\varepsilon) \,\big|\, \varepsilon\big) = \mathbb{P}\big(\mathcal{N}_z \approx \nu t \,|\, t\big) \asymp \varepsilon^{\gamma_\kappa(\nu)} = \exp[-t\gamma_\kappa(\nu)].$$

Liouville Quantum Gravity:

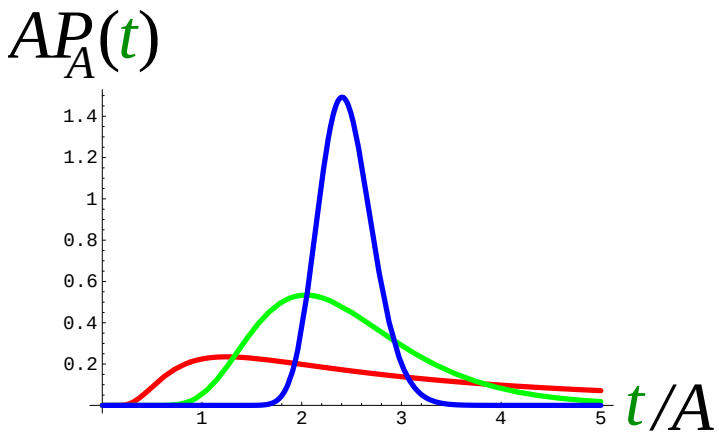
$$t := -\ln\varepsilon;\ A := -\gamma^{-1}\ln\delta, \quad \delta := \int_{B(z,\varepsilon)} \mu_\gamma \quad \text{(quantum ball)}$$

Conditioned on δ*, hence* A*, perform the convolution*

$$\mathbb{P}_{\mathcal{Q}}(\mathcal{N}_z \,|\, A) := \int_0^\infty \mathbb{P}\big(\mathcal{N}_z \,|\, t\big) P(t|A)\, dt,$$

where $P(t|A)$ *is the probability distribution of the random Euclidean log-radius* t*,* **given** *the quantum log-radius* A*.*

Probability Distribution $(\gamma = \sqrt{8/3})$ $[A = 2;\ 20;\ 200]$



$$t = -\ln\varepsilon,\ \ A = -\gamma^{-1}\ln\delta,\ \ \delta = \int_{B(z,\varepsilon)} \mu_\gamma$$

$$P(t\,|\,A) = \frac{A}{\sqrt{2\pi t^3}} \exp\left[-\frac{1}{2t}\left(A - a_\gamma t\right)^2\right]$$

$$a_\gamma := 2/\gamma - \gamma/2$$

Large Deviations

$$t = -\ln\varepsilon,\ A = -\gamma^{-1}\log\delta \ \text{(quantum ball)},$$

$$\mathcal{N} \approx -\nu\ln\varepsilon = \nu t,\ \mathcal{N} \approx -p\ln\delta = \gamma p A$$

It implies $\nu = \gamma p A/t$.

$$\mathbb{P}_{\mathcal{Q}}(\mathcal{N}_Z \approx \gamma p A\,|\,A) := \int_0^\infty \mathbb{P}\big(\mathcal{N}_Z \approx \gamma p A = \nu t\,|\,t\big)P(t|A)dt$$

The asymptotic result above then yields, for $A \to +\infty$,

$$\mathbb{P}_{\mathcal{Q}}(\mathcal{N}_Z \approx \gamma p A\,|\,A) \asymp \int_0^\infty \frac{dt\,A}{\sqrt{2\pi t^3}}\exp\left(-\frac{(A-a_\gamma t)^2}{2t} - \gamma_\kappa(\nu)t\right)$$

$$\asymp \exp[-A\Theta(p)] \ \ \text{(saddle point at } \nu t \text{ constant)}$$

$\Theta(p)$ is the large deviation function for the loop number around a δ-quantum ball scaling as $p\log(1/\delta)$.

Liouville Quantum Large Deviations

In the plane, the Legendre transform gave

$$\gamma_\kappa(\nu) = \lambda - \nu\Lambda_\kappa(\lambda), \quad \frac{1}{\nu} = \frac{\partial\Lambda_\kappa(\lambda)}{\partial\lambda}.$$

In **Liouville Quantum Gravity**

$$\Theta(p) = U_\gamma^{-1}(\lambda/2) - p\Lambda_\kappa(\lambda), \quad \frac{1}{p} = \frac{\partial\Lambda_\kappa(\lambda)}{\partial U_\gamma^{-1}(\lambda/2)},$$

where $U_\gamma^{-1}(\lambda/2) := \left(\sqrt{a_\gamma^2 + 2\lambda} - a_\gamma\right)/\gamma$ is the inverse **KPZ** function, with

$$\gamma = \min\left\{\sqrt{\kappa}, 4/\sqrt{\kappa}\right\}, \quad a_\gamma = 2/\gamma - \gamma/2.$$

$\square$

Nesting in Liouville Quantum Gravity

Theorem *[Borot, Bouttier, D. '16]*

The quantum nesting probability of a CLE_κ *in a simply connected domain, for the number* $\mathcal{N}_z$ *of loops surrounding a ball centered at* z *and conditioned to have a given Liouville quantum area* δ*, has the large deviations form,*

$$\mathbb{P}_Q\left(\mathcal{N}_z \approx \frac{cp}{2\pi}\ln(1/\delta)\,\middle|\,\delta\right) \asymp \delta^{\frac{c}{2\pi}J(p)}, \qquad \delta \to 0,$$

where c *and* J *are the same as in the combinatorial result obtained for the critical* $O(n)$ *model in the scaling limit of large random maps.*

CLE on the Riemann sphere *[Kemppainen & Werner '14]*

Theorem *[Borot, Bouttier, D. '16]*

The nesting probability in $\mathrm{CLE}_\kappa(\widehat{\mathbb{C}})$ *between two balls of radius* ε_1 *and* ε_2 *and centered at two distinct punctures, has the large deviations form,*

$$\mathbb{P}^{\widehat{\mathbb{C}}}\left[\mathcal{N}(\varepsilon_1,\varepsilon_2) \approx \nu \ln(1/(\varepsilon_1\varepsilon_2))\right] \asymp (\varepsilon_1\varepsilon_2)^{\gamma_\kappa(\nu)}, \quad \nu \geqslant 0, \quad \varepsilon_1,\varepsilon_2 \to 0,$$

where $\gamma_\kappa(\nu)$ *is the large deviations function of the disk topology.*

Corollary

The nesting probability in $\mathrm{CLE}_\kappa(\widehat{\mathbb{C}})$ *between two balls of same radius* ε *and centered at two distinct punctures, has the large deviations form,*

$$\mathbb{P}^{\widehat{\mathbb{C}}}\big(\mathcal{N}(\varepsilon,\varepsilon) \approx \nu \ln(1/\varepsilon)\big) \asymp \varepsilon^{\widehat{\gamma}_\kappa(\nu)}, \quad \nu \geqslant 0, \quad \varepsilon \to 0,$$

where $\widehat{\gamma}_\kappa(\nu)$ *is related to the disk large deviations function by*

$$\widehat{\gamma}_\kappa(2\nu) = 2\gamma_\kappa(\nu).$$

Riemann sphere

Theorem *[Borot, Bouttier, D. '16]*

On the Riemann sphere $\widehat{\mathbb{C}}$, the large deviations function $\widehat{\Theta}$ which governs the quantum nesting probability between two non-overlapping δ-quantum balls,

$$\mathbb{P}_{\mathcal{Q}}^{\widehat{\mathbb{C}}}(\mathcal{N} \approx p \ln(1/\delta) \,|\, \delta) \asymp \delta^{\widehat{\Theta}(p)}, \qquad \delta \to 0,$$

is related to the similar function Θ for the disk topology by

$$\widehat{\Theta}(2p) = 2\Theta(p),$$

so that

$$\mathbb{P}_{\mathcal{Q}}^{\widehat{\mathbb{C}}}\left(\mathcal{N} \approx \frac{cp}{\pi} \ln(1/\delta) \,\middle|\, \delta \to 0\right) \asymp \delta^{\frac{c}{\pi} J(p)},$$

where c and J are the same as before.

Perfect matching of **LQG** results for CLE_κ with those for the $O(n)$ **model** on a **random planar map**, with the correspondence $\delta \leftrightarrow 1/V$, with $\delta \to 0$, $V \to +\infty$.

HAPPY BIRTHDAYS,
JOHN & SUSAN!