Percolation in Finite Matching Lattices

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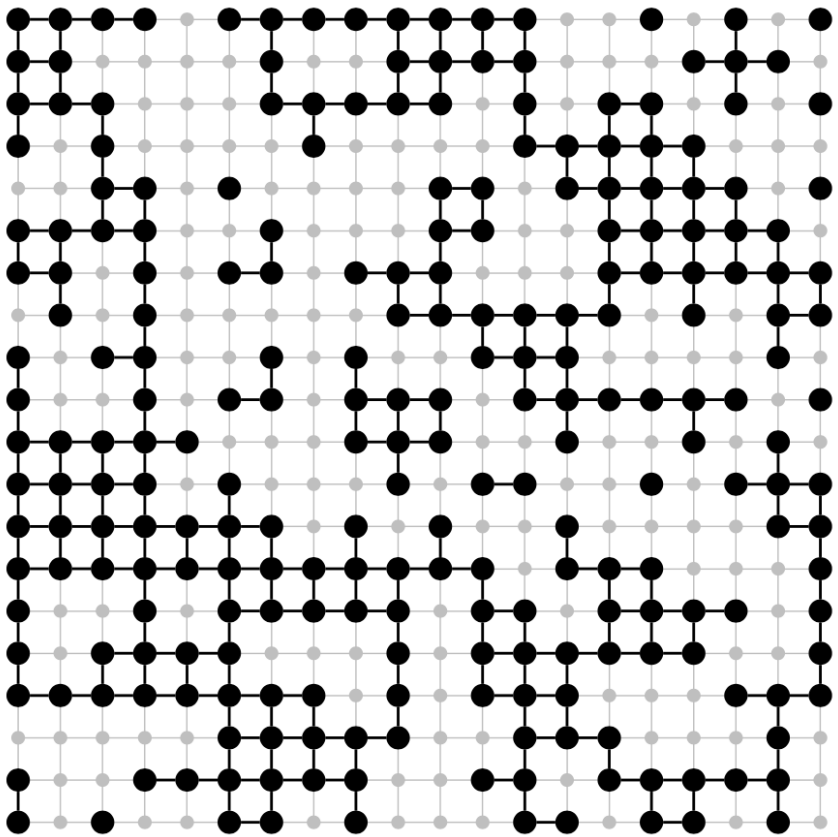
We consider

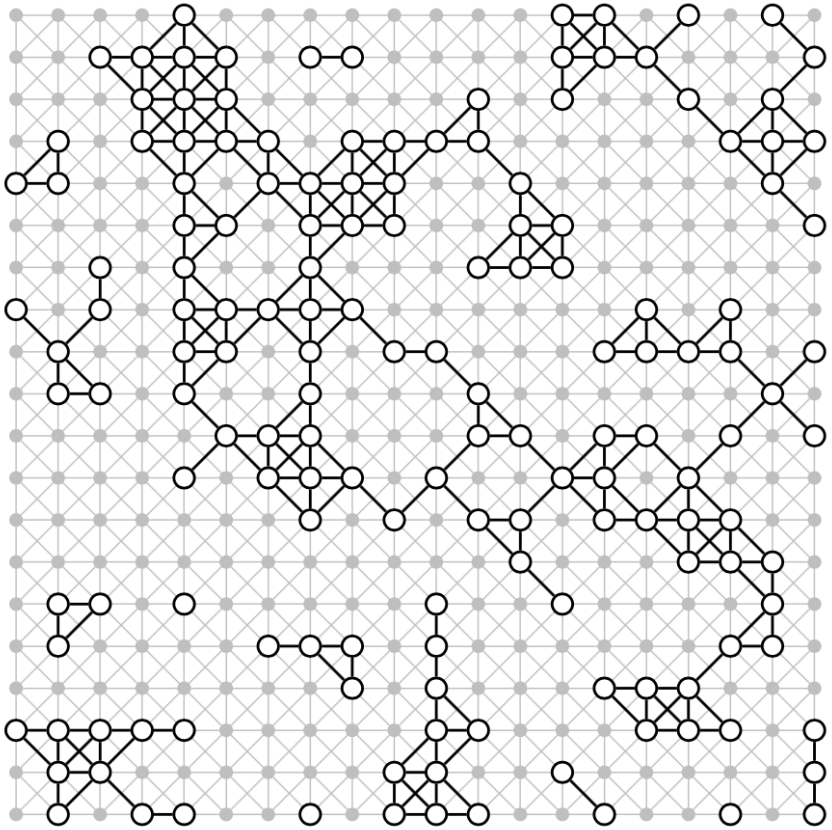
$n(p)$ = the number of clusters per site in a percolation system at occupation probability p

$$n(p) = \sum_{s=1}^{\infty} n_s(p)$$

where $n_s(p)$ is the number of clusters of s sites, per site of the lattice. It is analogous to the *free energy* for percolation.

Lattice and matching lattice clusters





Sykes and Essam matching formula

$$n(p) - n^*(1-p) = \phi(p)$$

* = matching or dual lattice

$\phi(p)$ is the matching polynomial which follows from Euler's formula

$$\phi(p) = \frac{\langle F_0 \rangle + \langle V \rangle - \langle E \rangle}{N_s}$$ site percolation,

$$\phi(p) = 1 - \langle E \rangle / N_s$$ bond percolation,

$$\phi = P_3 - P_0$$ general hypergraph

F_0 = elemental faces , V=vertices, E=edges, N_s= L^2 no. sites

P_0 = prob. none connect, P_3 = prob. 3 vertices connect

Matching polynomials

- Site percolation on any fully triangulated lattice: $\langle V \rangle = p L^2$, $\langle E \rangle = 3 p^2 L^2$, and $\langle F_0 \rangle = 2p^3L^2$

$$\phi(p) = p - 3p^2 + 2p^3$$

- Site percolation on a square lattice. $\langle V \rangle = p L^2$, $\langle E \rangle = 2 p^2 L^2$, and $\langle F_0 \rangle = p^4 L^2$

$$\phi(p) = p - 2p^2 + p^4$$

Finite system

F = # faces, V = # vertices, E = # edges. Euler:

- F + V – E = 1 for each cluster

Each face of enclosed area > 1 corresponds to a matching lattice cluster. This yields

$$N_L - \hat{N}_L - (V - E + F_0) = \begin{cases} 1 & C \text{ cross-wrapping,} \\ -1 & \hat{C} \text{ cross-wrapping,} \\ 0 & \text{otherwise.} \end{cases}$$

Averaging yields the exact expression

(Mertens Ziff, PRE, in press)

$$n(p) - n^*(1-p) - \phi(p) = \frac{R^{\times}(p) - R^{*,\times}(1-p)}{L^2}$$

where $R^{\times}(p)$ is the cross-configuration probability and $R^{*,\times}(1-p)$ is the cross-configuration probability on the dual lattice.

Or

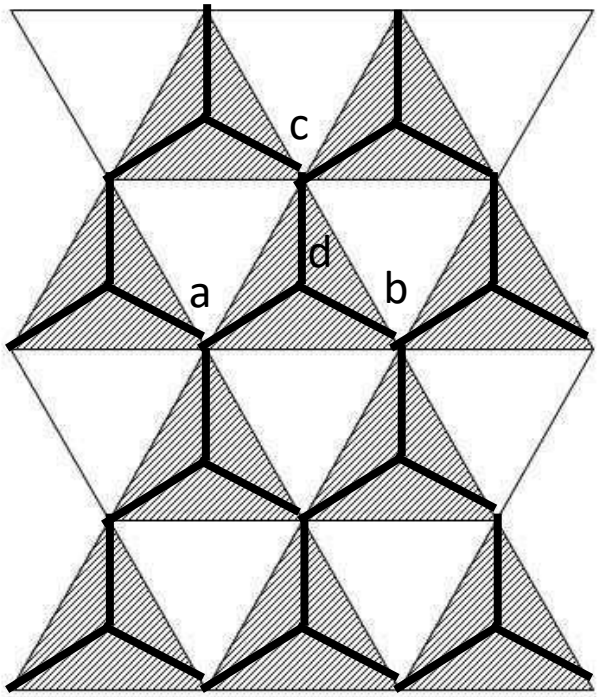
$$n'(p) - n'^*(1-p) - \phi(p) = 0$$

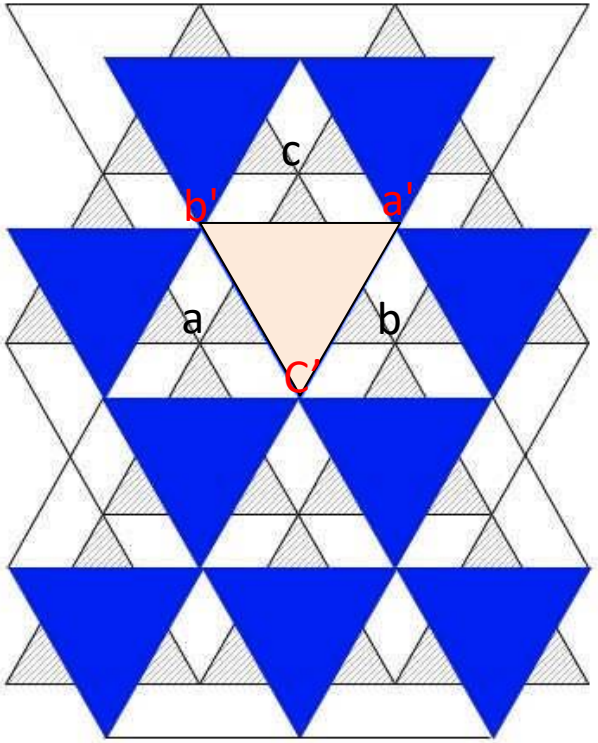
Where prime means *do not* count crossing-wrapping clusters or simply *any* crossing clusters.

$$n(p) - n^*(1-p) - \phi(p) = \frac{R^{\times}(p) - R^{*,\times}(1-p)}{L^2}$$

- As L goes to ∞, both sides go to zero very quickly. The condition on the right is equivalent to Jacobsen and Scullard's condition for finding p_c.

This is generalization of P(all) = P(none) condition for any self-similar hypergraph of 3-edges (triangles)





The triangle can contain *any* collection of bonds, including correlated ones. (Scullard, Ziff)

Behavior of $n(p)$ near the threshold p_c for an infinite system:

$$n(p) = A_0 + B_0(p - p_c) + C_0(p - p_c)^2 + \mathcal{A}^{\pm}|p - p_c|^{2-\alpha} + \ldots$$

where $\alpha = -2/3$ in 2d. The singularity is very weak and difficult to observe directly.

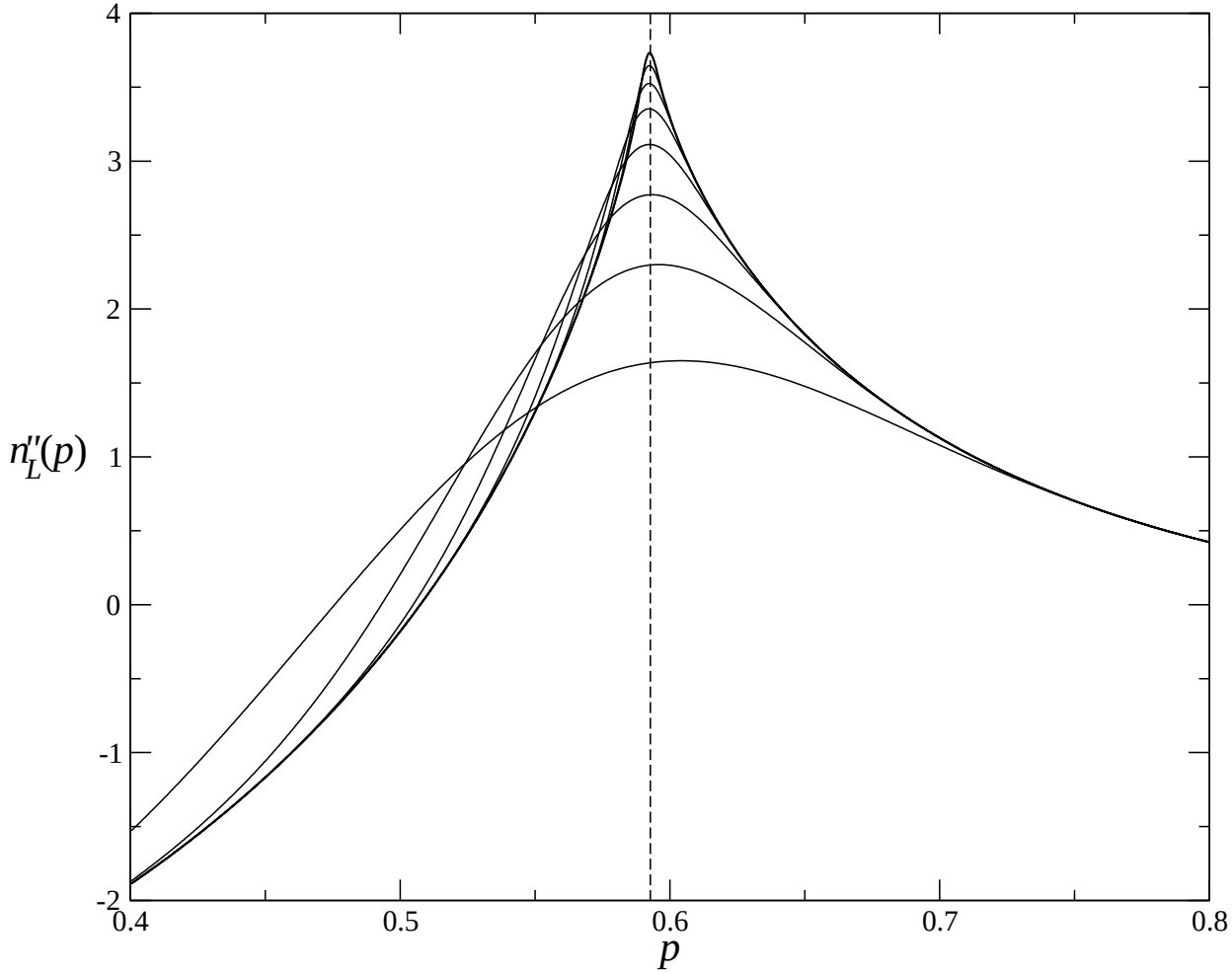
Finite system:

$$n(p) = A_0 + B_0(p - p_c) + C_0(p - p_c)^2 + L^{-d} f(z) + \ldots$$

$$z = b(p - p_c)L^{1/\nu}$$

Where $f(z)$ is a *shape dependent* universal function

n’’(p) for site percolation, using the Newman-Ziff algorithm to calculate the derivative from Monte Carlo, for L = 8, 16, … 1024, demonstrating the cusp singularity.



Expanding $f(z)$ around $z = 0$ we find

$$n_L(p) \sim A + B(p - p_c) + C(p - p_c)^2 + \ldots$$

with

$$A = A_0 + A_1 L^{-d} + \ldots$$

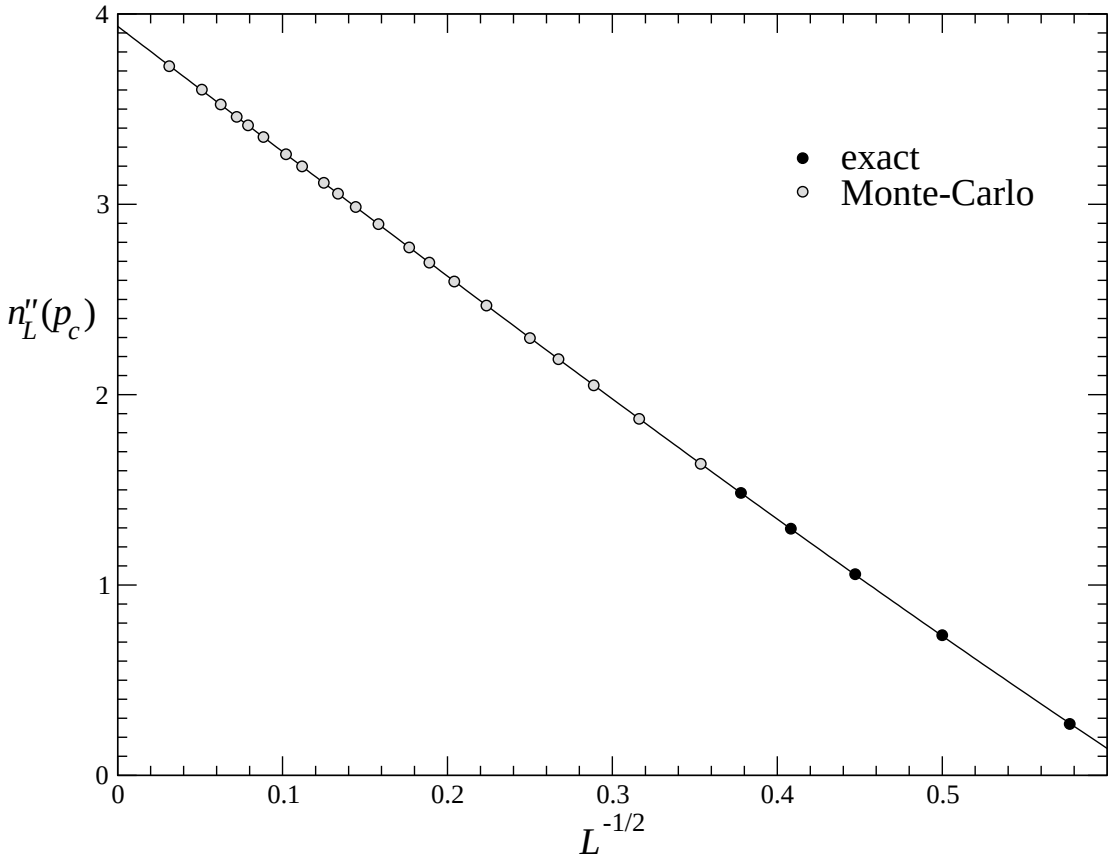
$$B = B_0 + B_1 L^{-d+1/\nu} + \ldots$$

$$C = C_0 + C_1 L^{-d+2/\nu} + \ldots$$

$$A_1 = f(0),\ B_1 = b f'(0),\ C_1 = b^2 f''(0).$$

Thus, $R = A_1 C_1 / B_1^2$ is universal (shape-dependent)

Plot of $C_0 = n''(p_c)$ vs $L^{-1/2}$. Square-site



Some exact results:

$A_0 = n(p_c)$ = number of clusters per site.

Square-bond (bond clusters per site) (Temperley, Lieb 71)

$$A_0 = \frac{24\sqrt{3} - 41}{32} = 0.017788105676 65...$$

Triangle-bond (components) (Baxter, Temperley, Ashley 79)

$$A_0 = \frac{35}{4} - \frac{3}{p_c^{btr}} = 0.111844275...$$

$$p_c^{btr} = 2\sin \pi/18 = 0.347296355...$$

Kagome-site (clusters per site)

$$A_0 = \frac{1}{3}\left(n_c^{tr} - (p_c^{btr})^3\right) = 0.023318403094\ldots$$

Series result (Iwan Jensen)

Site percolation on the triangular lattice: 69th order series expansion, using substitution $u = p(1-p)$ of Domb and Pearce, find

$$A_0 = n(p_c) = 0.017625277368(2)$$

An exact expression for this?

A_1 = excess number of clusters

$= \lim_{L \to \infty} (N_L - L^2 n(p_c))$

Universal at the critical point, depending only upon the shape of the system. (Ziff, Finch, Adamchik)

For a torus with a twist, a formula follows from conformal field theory (Kleban, Ziff). For example

$A_1 = 0.883576308...$ Square torus

$A_1 = 0.878290117...$ 60° rhombus torus

(rectangle with aspect ratio √3/2 and twist ½)

Matching polynomial for site percolation on a square lattice:

For site percolation on a square lattice,

$$\phi(p) = p - 2p^2 + p^4$$

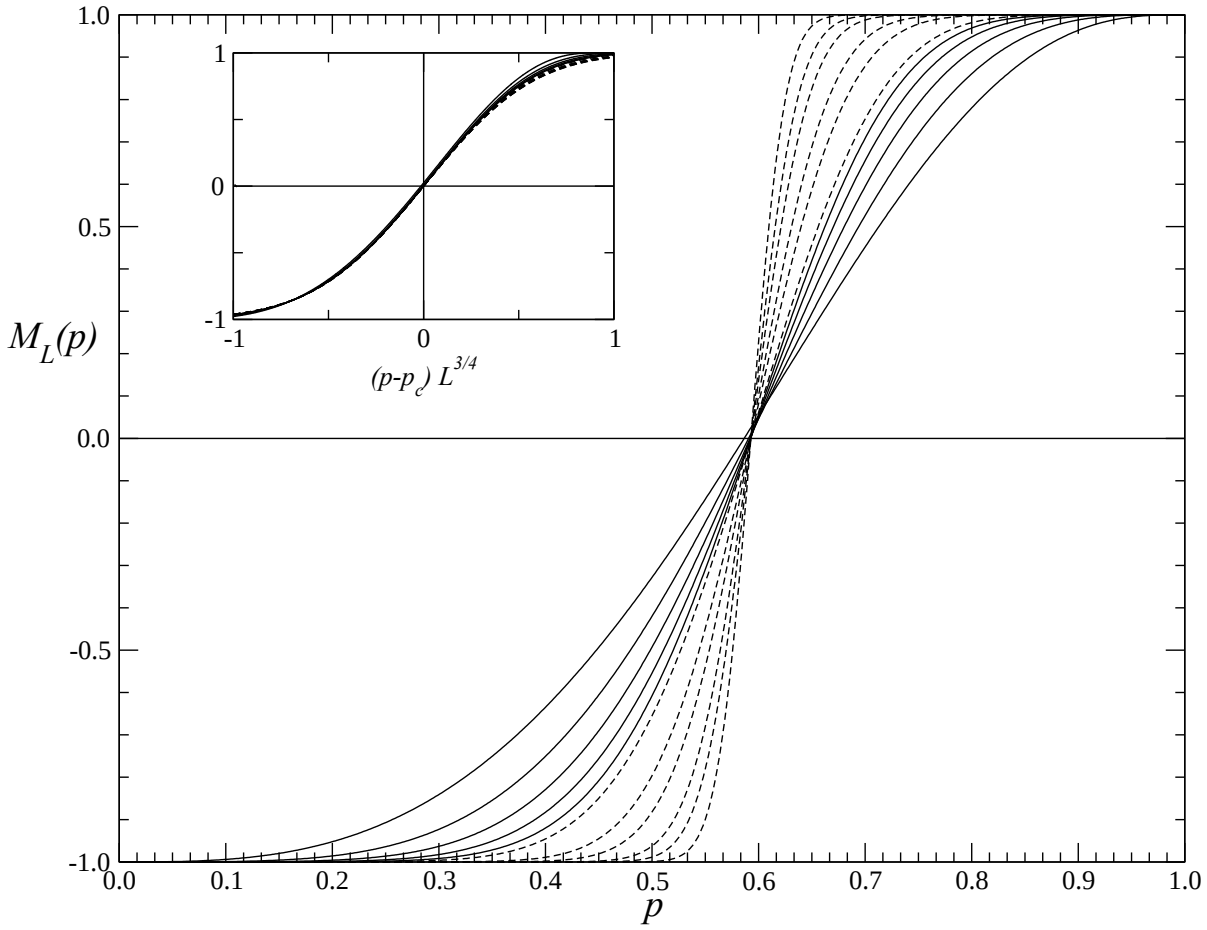
Then Sykes-Essam $n(p) - n^*(1-p) = \phi(p)$ implies

$$A_0^{\mathrm{SQ}} - A_0^{\mathrm{NNSQ}} = p_c - 2p_c^2 + p_c^4 = 0.013495622626 04(1),$$
$$B_0^{\mathrm{SQ}} + B_0^{\mathrm{NNSQ}} = 1 - 4p_c + 4p_c^3 = -0.537943928141750(5),$$
$$C_0^{\mathrm{SQ}} - C_0^{\mathrm{NNSQ}} = -4 + 12p_c^2 = 0.216174568755 5(3)$$

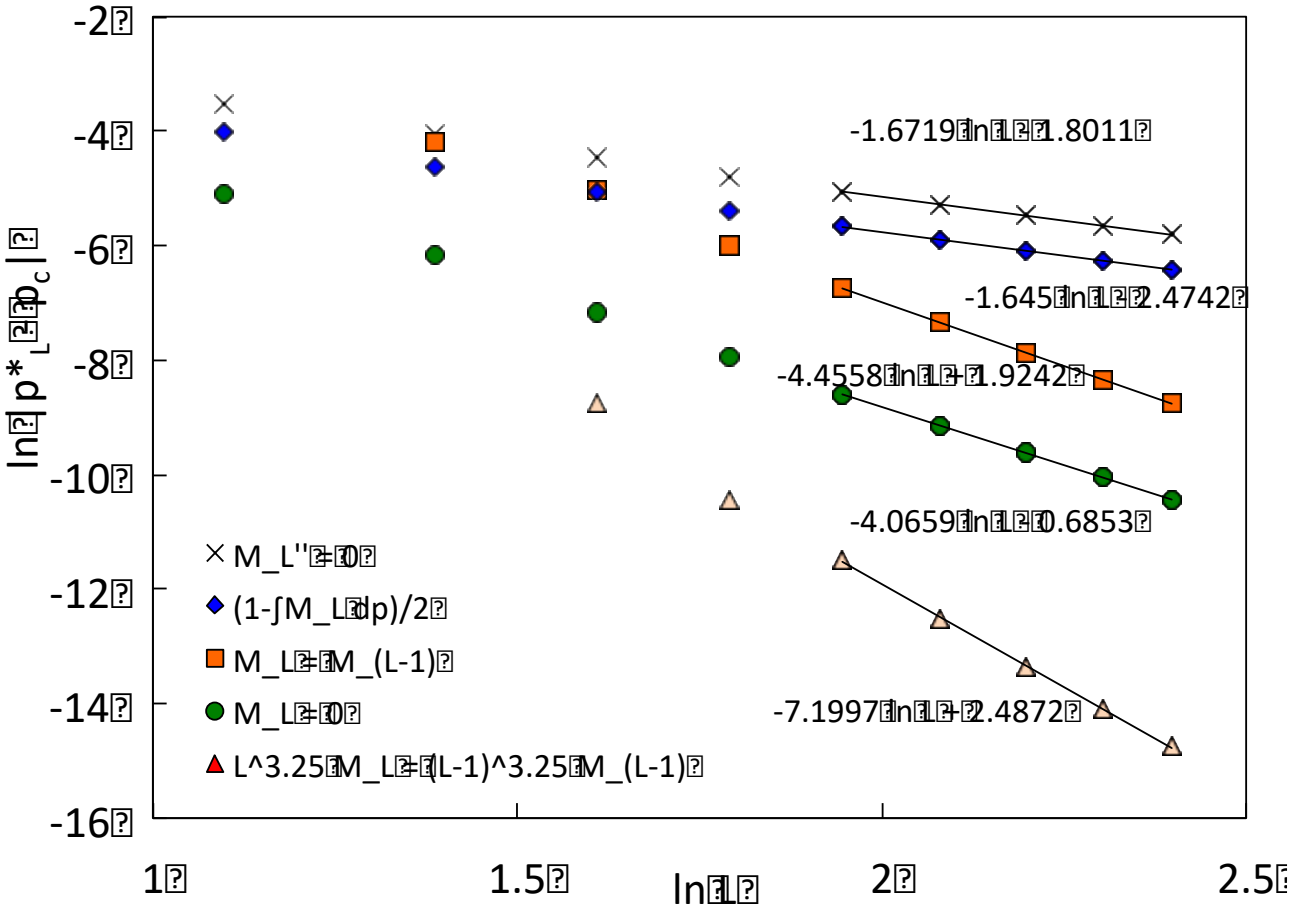
Using Jacobsen's p_c = ≈ 0.592746050792 10(2).

$$
\begin{aligned}
M_L(p) &= n(p) - n^*(p) - L^2\phi(p) \\
&= f(z) - f(-z) \\
&= 2B_1 z + 2D_1 z^3 + \text{ finite-size corrections}
\end{aligned}
$$

Note A_0, B_0, C_0… cancel out. The singularity in f(z) also cancels out.



Convergence of estimates of p_c based upon $M_L(p)$.



John Essam, King’s College, Royal Holloway, April 13, 2016

Holes in percolation clusters

- Hulls (Cardy – Ziff)
- Depth
- Network (Kleban, Huber, Ziff)
- Single-generation Holes (Deng, Hu, Ziff)

Zipf's law form for 2d critical clusters

- If you rank-order all cluster areas, largest to smallest, then for large *n*, the enclosed hull area *An* of the *n*-th largest cluster equals:

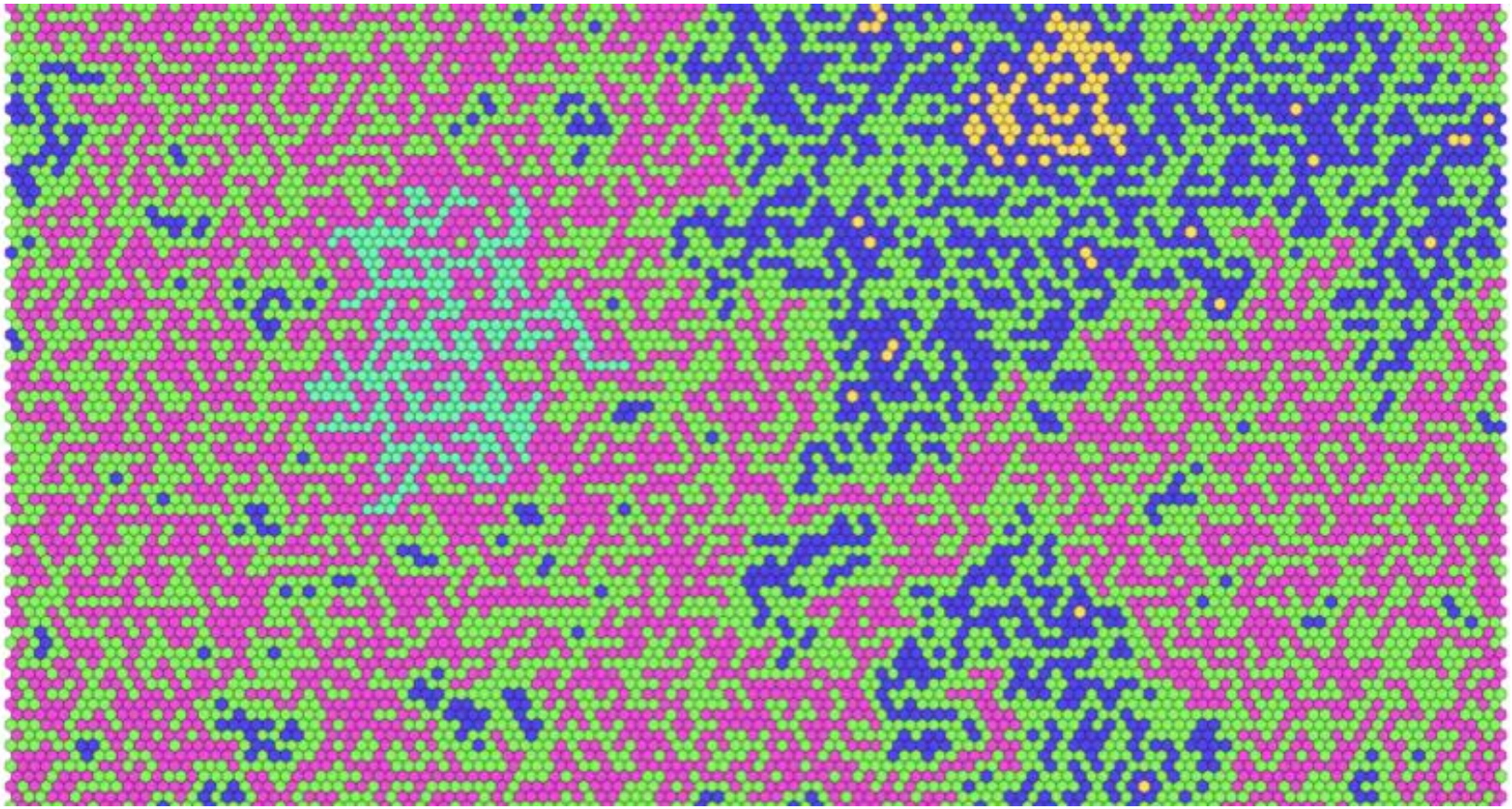
$$\frac{A_n}{L^2} \sim \frac{C}{n}$$

- Where *C* = 1/(8 π $\sqrt{3}$) = 0.0229... for percolation and L = system dimension. (Ziff, Lorenz & Kleban 1999, Cardy & Ziff, 2003)

- Equivalently, the number of clusters whose enclosed area is greater than A scales as C L^2 / A.

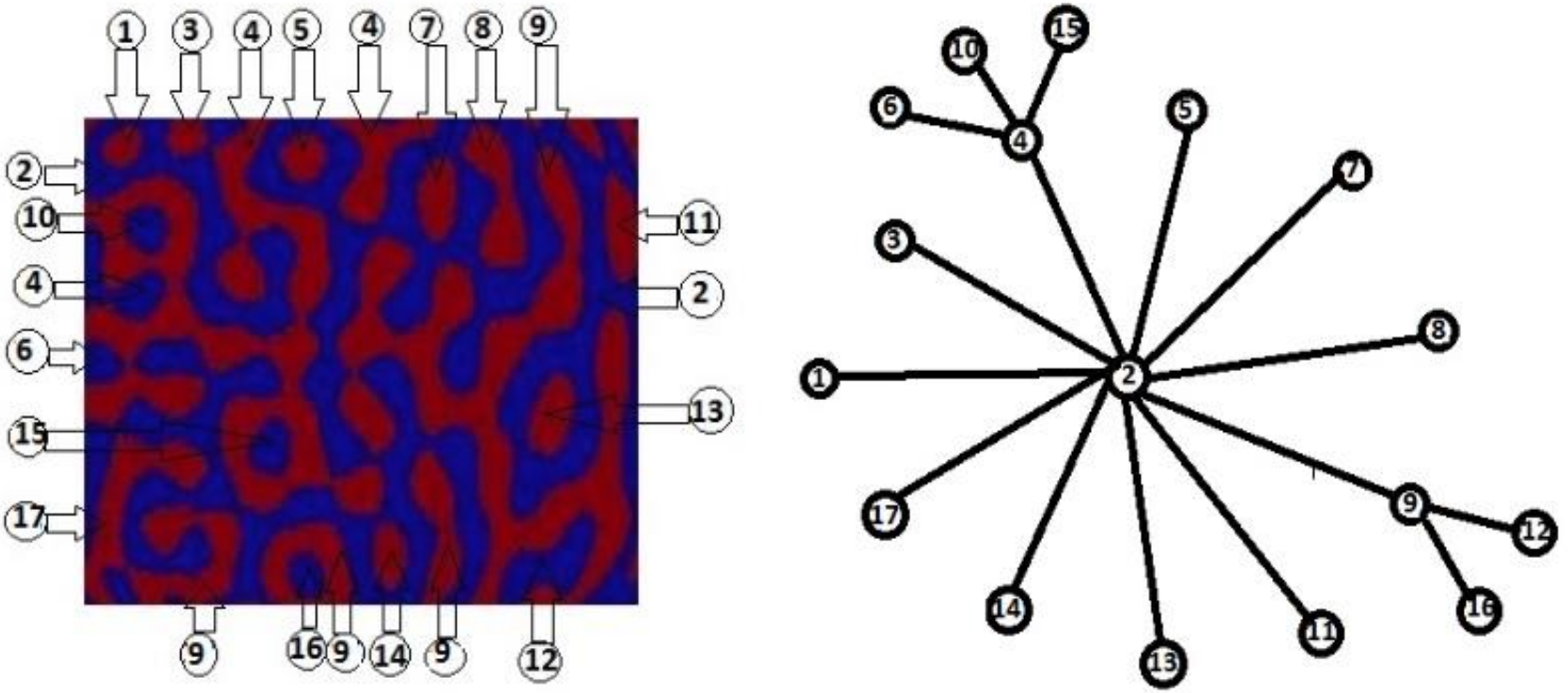
Depth is how many boundaries you have to cross -- grows as log(L)

This is site percolation on the triangular lattice, where we consider both the black and white site clusters.



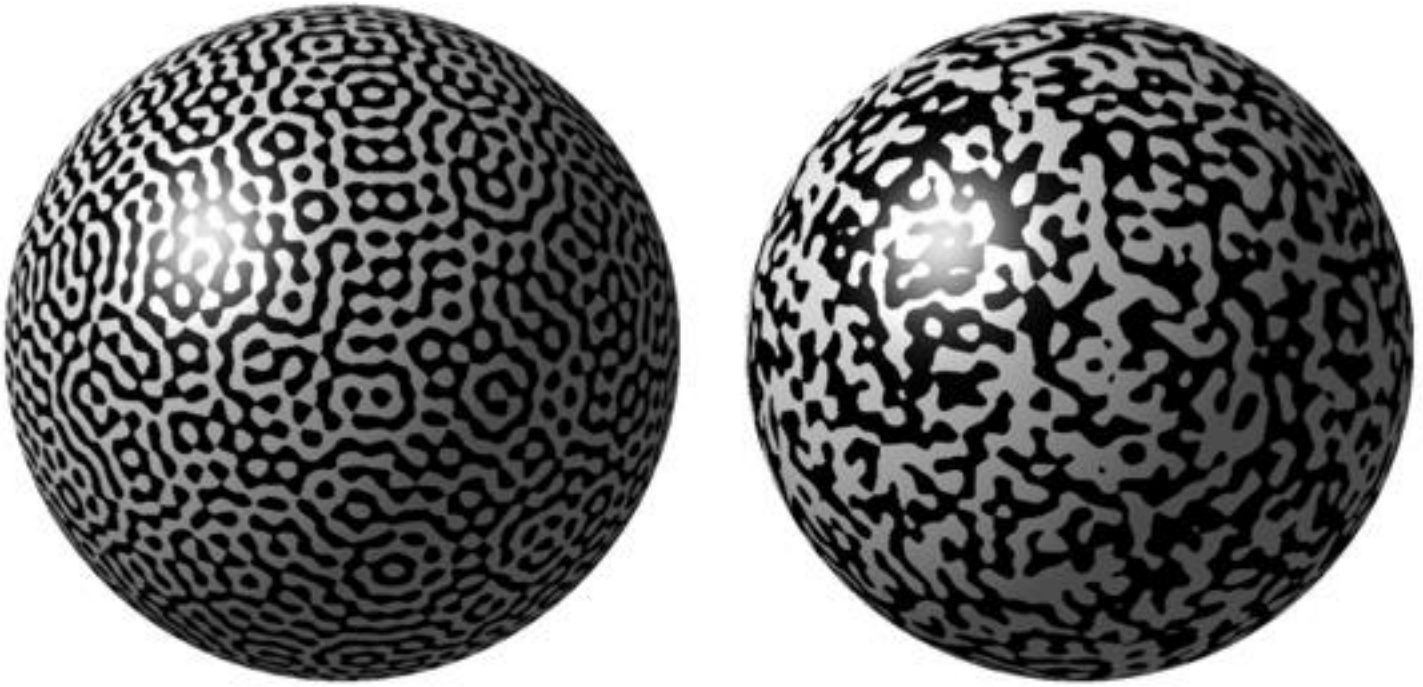
TOPOLOGIES OF NODAL SETS OF RANDOM BAND LIMITED FUNCTIONS

PETER SARNAK AND IGOR WIGMAN

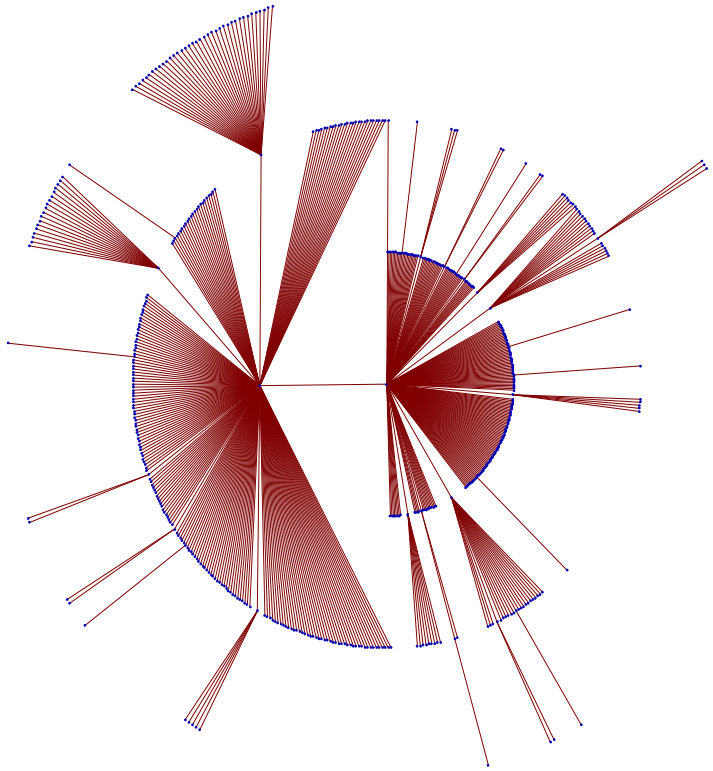


Proposed percolation of a random function, with tree showing neighboring clusters

Percolation on random nodal lines on a sphere (Sarnak and Wigman)

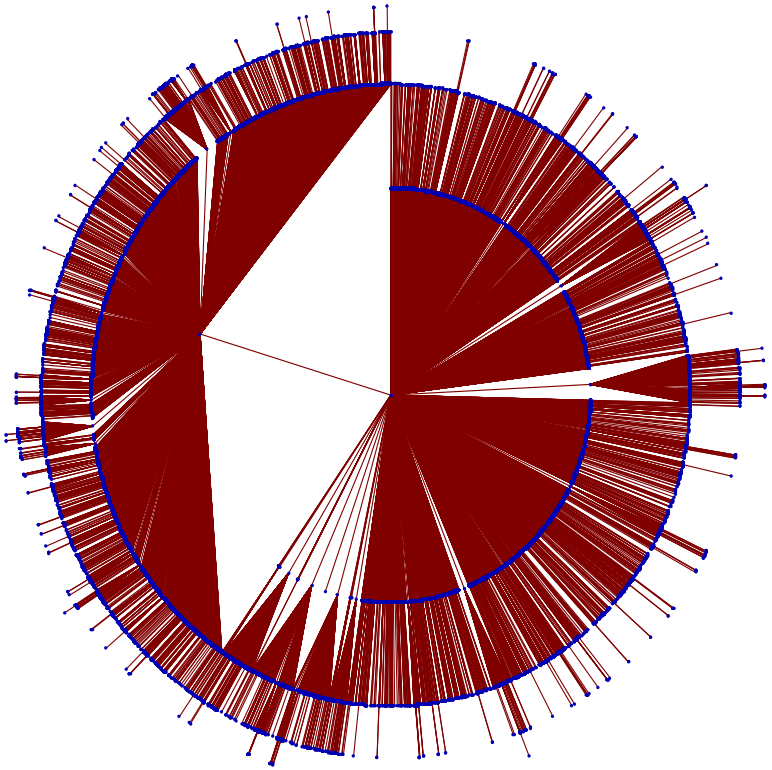


The neighbor tree for percolation on the triangular lattice.

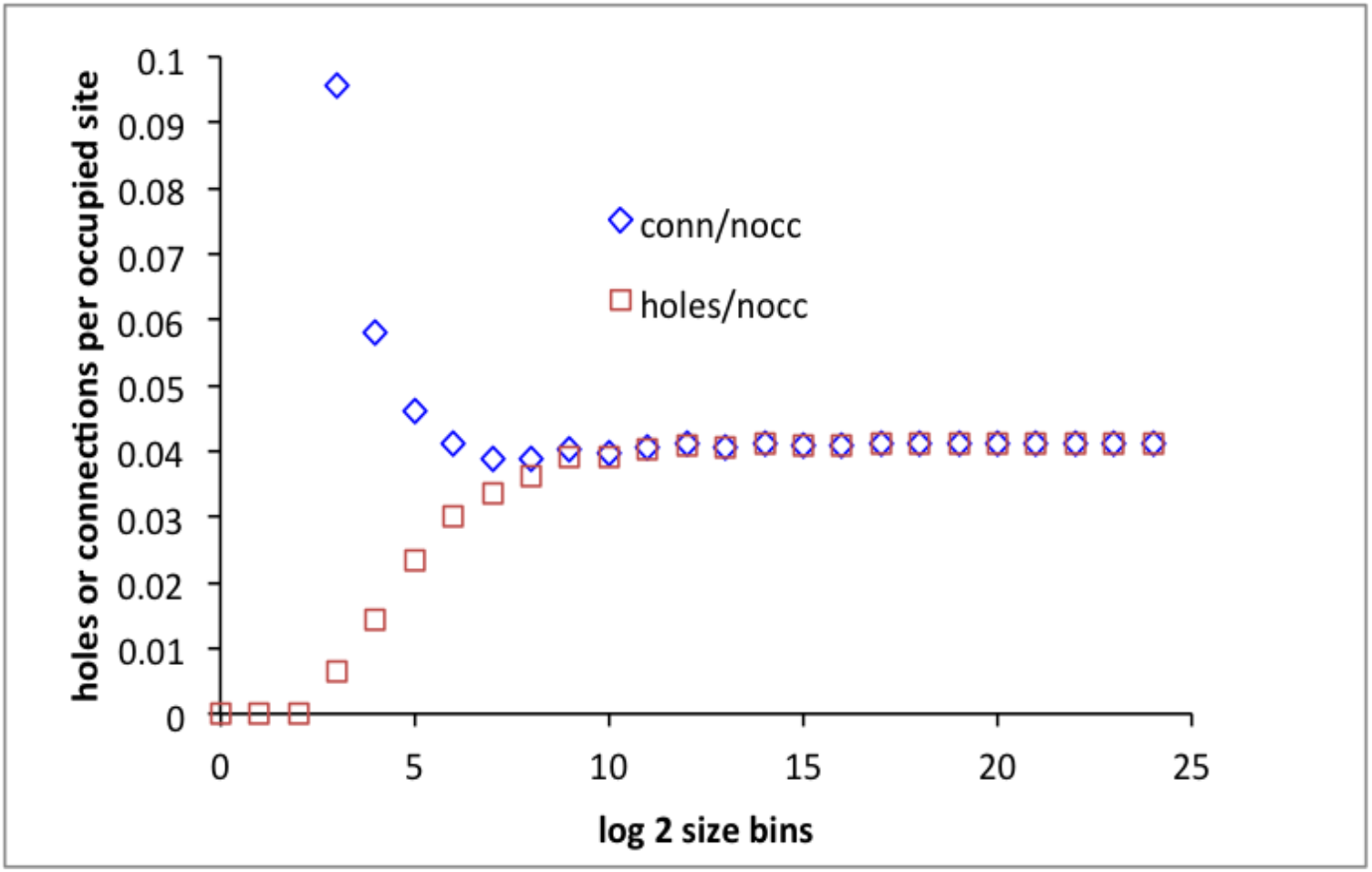


The two biggest clusters are the largest white and largest black clusters in the system. Most bordering clusters are small.

Another example for percolation



Number of holes in a cluster -- about 4% the number of sites (triangular lattice)



Triangular lattice 8192x8192 site perc.

k	no. clusters	total conn.	total n_{occ}	conn/n_{occ}	holes/n_{occ}
0	87	87	87	1.000000	0.000000
1	118	118	296	0.398649	0.000000
2	125	125	657	0.190259	0.000000
3	178	191	1999	0.095548	0.006503
4	160	213	3665	0.058117	0.014461
5	167	339	7343	0.046166	0.023424
6	139	518	12574	0.041196	0.030142
7	161	1162	29857	0.038919	0.033526
8	162	2297	59246	0.038771	0.036036
9	154	4614	114402	0.040331	0.038985
10	142	8533	215281	0.039637	0.038977
11	116	13822	340106	0.040640	0.040299
12	130	30253	737841	0.041002	0.040826
13	141	67672	1663503	0.040680	0.040596
14	105	102861	2505662	0.041051	0.041010
15	127	249988	6110111	0.040914	0.040893
16	130	514538	12554749	0.040984	0.040973
17	127	985047	23968732	0.041097	0.041092
18	93	1381148	33613961	0.041089	0.041086
19	95	3022199	73572521	0.041078	0.041077
20	97	5800992	141239389	0.041072	0.041071
21	87	10870856	264640586	0.041078	0.041077
22	94	24429918	594543566	0.041090	0.041090
23	1294	771142476	18766506087	0.041091	0.041091
24	1071	784755557	19097799457	0.041091	0.041091
sum:	5300				

What is this constant?

Hole size distribution

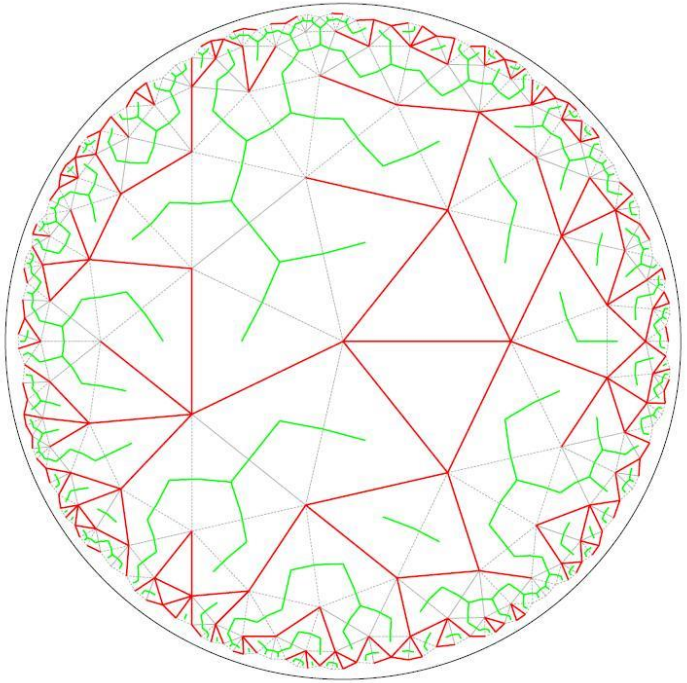
- Hu, Deng, Ziff

$$\tau_h = 1 + \frac{d_f}{d} = \frac{187}{96} \approx 1.948$$

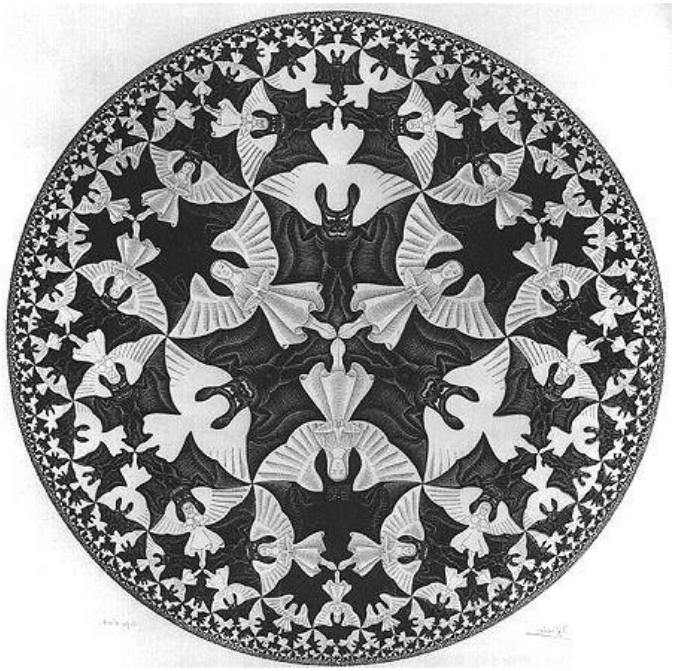
- Leads to explosive percolation – fraction of holes goes from zero to finite value at p_c.
- Explains “Enclaves” of M. Sheinman, A. Sharma, J. Alvarado, G. H. Koenderink, and F. C. MacKintosh

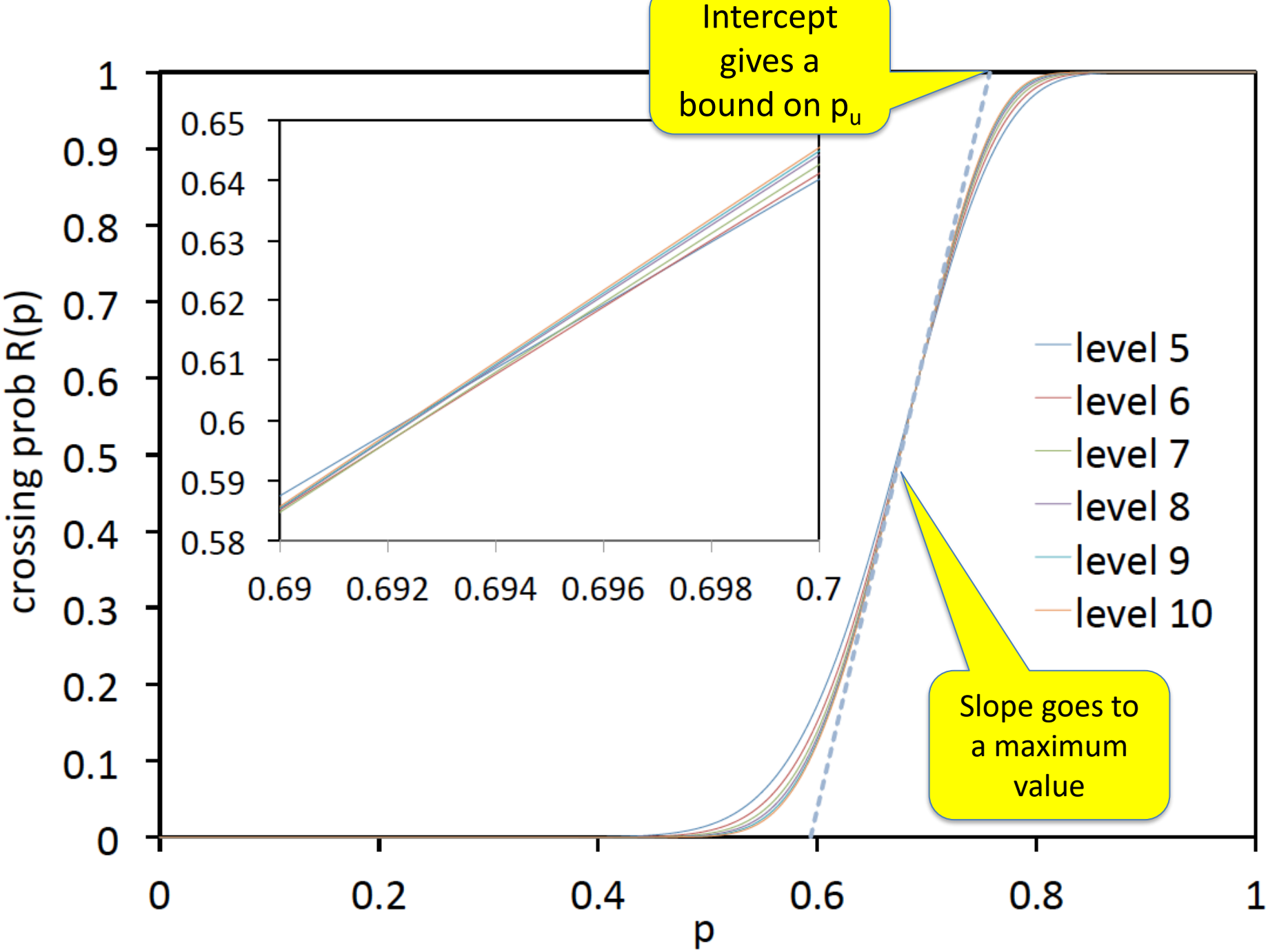
Percolation on hyperbolic graphs (Print upstairs in the Hill Center)

- Dual hyperbolic lattices:



- Escher





For these non-amenable graphs, there are two percolation thresholds!