

Local Central Limit Theorem for Determinantal Point Processes

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Received: 30 March 2014 / Accepted: 8 July 2014 / Published online: 22 July 2014
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Abstract We prove a local central limit theorem (LCLT) for the number of points $N(J)$ in a region J in $\mathbb{R}^d$ specified by a determinantal point process with an Hermitian kernel. The only assumption is that the variance of $N(J)$ tends to infinity as $|J| \rightarrow \infty$. This extends a previous result giving a weaker central limit theorem for these systems. Our result relies on the fact that the Lee–Yang zeros of the generating function for $\{E(k; J)\}$ —the probabilities of there being exactly k points in J —all lie on the negative real z -axis. In particular, the result applies to the scaled bulk eigenvalue distribution for the Gaussian Unitary Ensemble (GUE) and that of the Ginibre ensemble. For the GUE we can also treat the properly scaled edge eigenvalue distribution. Using identities between gap probabilities, the LCLT can be extended to bulk eigenvalues of the Gaussian Symplectic Ensemble. A LCLT is also established for the probability density function of the k -th largest eigenvalue at the soft edge, and of the spacing between k -th neighbors in the bulk.

Keywords Central limit theorem · Local central limit theorem · Lee–Yang zeros · Random matrices

1 Introduction

Determinantal point processes are prominent structures in the theory of random matrices as well as many other contexts [36]. These are processes for which the k -point correlation function can be written as a $k \times k$ determinant,

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$$\rho_{(k)}(x_1, \dots, x_k) = \det[K(x_j, x_l)]_{j,l=1,\dots,k}, \quad (1.1)$$

where $K(x, y)$ —referred to as the correlation kernel—is independent of k . A necessary and sufficient condition for (1.1) to represent a point process in J , when K (viewed as the kernel for an integral operator supported on J) is Hermitian, is that all its eigenvalues be discrete and lie between zero and one (see e.g. [18]). Such K 's are the only one we shall consider here.

One of the best known examples of a determinantal point process is given by the eigenvalues of the random matrices specified by the Gaussian unitary ensemble (GUE): a Gaussian probability measure on the space of complex $N \times N$ Hermitian matrices which is unitary invariant and thus unchanged by conjugation by unitary matrices (see e.g. [12, 33]). By scaling the eigenvalues so that the mean density is unity and taking $N \rightarrow \infty$, one obtains a translation invariant determinantal point process specified by the so-called sine kernel $K(x, y) = \sin \pi(x - y)/\pi(x - y)$. The GUE also admits a soft edge scaling in the neighborhood of the largest eigenvalue, which now involves changing the origin so that it is centered near the largest eigenvalues, then scaling so the expected spacing between eigenvalues in this neighbourhood is of order unity in the limit $N \rightarrow \infty$. This is called a soft edge scaling since $x = 0$ is a soft wall—eigenvalues do occur in the region $x > 0$ but their density falls off super-exponentially. The resulting point process defined by the eigenvalues is determinantal with the explicit form of the correlation kernel given by the Airy kernel $K(x, y) = (\text{Ai}(x)\text{Ai}'(y) - \text{Ai}(y)\text{Ai}'(x))/(x - y)$.

The eigenvalues of the Ginibre ensemble of non-Hermitian matrices with standard complex entries give an example of a determinantal point process with a complex Hermitian kernel: in the limit $N \rightarrow \infty$ this is given by $K(w, z) = \frac{1}{\pi} e^{-(|w|^2 + |z|^2)/2} e^{w\bar{z}}$ (see e.g. [28]), where z and w are complex.

It is the purpose of the present paper is to give a local central limit theorem (LCLT) for the probabilities $E(k; J)$ —conditioned gap probabilities—that there are exactly k points in J , where k is close to the expected number of points in J , in the limit $|J| \rightarrow \infty$, for the class of determinantal point processes introduced in the first paragraph. We begin in Sect. 2 by recalling the central limit theorem (CLT) of Costin and Lebowitz [7] for number fluctuations in determinantal point processes, and then giving an alternative derivation which uses only the location of the eigenvalues of the underlying integral operator, or equivalently the zeros of the generating function for $\{E(k; J)\}$. It is possible to interpret the generating function as a grand partition function, so in terminology familiar in the theory of lattice gases the corresponding zeros may be referred to as Lee-Yang zeros. We will then prove the LCLT by using a theorem of Newton to establish log-concavity of $\{E(k; J)\}$, which is a known sufficient condition for a CLT to imply a LCLT. In Sect. 3 the LCLT theorem is applied to specify the distribution of $E(k; J)$ for the scaled limits, both bulk and soft edge, of the GUE, and for the Ginibre ensemble of non-Hermitian complex random matrices (see e.g. [28] for precise definitions). We extend the results for the GUE to the GSE and in part also to the GOE (Gaussian symplectic and orthogonal ensemble) by making use of inter-relation formulas from [29]. We furthermore obtain a LCLT for the distribution of the k -th largest eigenvalue at the soft edge, and the distribution of the spacing between k -th neighbors in the bulk.

2 A Local Limit Theorem

Our setting is a determinantal point process in $\mathbb{R}^d$. We denote by $N(J)$ the random variable for the number of points in $J \subset \mathbb{R}^d$. We set $\mu_J = \text{mean}(N(J))$ and $\sigma_J^2 = \text{Var}(N(J))$, and

we denote by $E(k; J)$ the probability that there are exactly k points in J . We remark that in terms of the correlation functions, with $\rho_{(2)}^T$ denoting the truncated (connected) two-point correlation

$$\mu_J = \int_J \rho_{(1)}(x) dx, \quad \sigma_J^2 = \int_J dx_1 \int_J dx_2 \left(\rho_{(2)}^T(x_1, x_2) + \rho_{(1)}(x_1) \delta(x_1 - x_2) \right). \quad (2.1)$$

Costin and Lebowitz [7] studied $N(J)$ for the particular determinantal point process corresponding to the eigenvalues of the GUE in the limit $N \rightarrow \infty$, scaled so that $\mu_J = |J|$ (bulk scaling limit) and thus specified by the sine kernel. They proved the CLT

$$\lim_{|J| \rightarrow \infty} \frac{(N(J) - \mu_J)}{\sigma_J} \stackrel{d}{=} \eta, \quad (2.2)$$

where η is a standard Gaussian random variable. This was done by showing that as a consequence of the property that $\sigma_J \rightarrow \infty$ as $|J| \rightarrow \infty$, all cumulants of the characteristic function beyond the second vanish for $|J| \rightarrow \infty$. In fact the proof makes no explicit use of the particular determinantal point process under consideration, requiring only that the corresponding kernel be locally trace class and self-adjoint, and that the variance tends to infinity, and so (2.2) is a universal property of determinantal point processes in this setting (see also [37]).

Alternative ways of proving a CLT for determinantal processes were given by Shirai and Takahashi [35] and by Hough et al. [18]. They used probabilistic methods to show that $N(J)$ can be viewed as the sum of independent but not identically distributed Bernoulli random variables $x_l \in \{0, 1\}$ with $\Pr(x_l = 1) = \lambda_l(J)$. It follows then from standard arguments in Teller (see [11, Sect. XVI.5, Theorem 2]) that we have a CLT whenever $\sigma_J^2 = \sum_{j=0}^{\infty} \lambda_j(J)(1 - \lambda_j(J)) \rightarrow \infty$ for $J = J_s$ with $s \rightarrow \infty$. In view of the relationship between $N(J)$ and $\{E(k; J)\}$ this CLT can be written

$$\lim_{s \rightarrow \infty} \sup_{x \in (-\infty, \infty)} \left| \sum_{k \leq \sigma_{J_s} x + \mu_{J_s}} E(k; J_s) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt \right| = 0. \quad (2.3)$$

Here we have introduced a parameter s in specifying the region J so as to be able to consider the natural interval $J = (-s, \infty)$ in the soft edge scaling case, which has $|J| = \infty$. For bulk scaling we can take $J_s = (0, s)$.

We give below a stronger result based on the location of the Lee–Yang zeros of the generating function (grand partition function)

$$\Xi(z; J) = \sum_{k=0}^{\infty} z^k E(k; J). \quad (2.4)$$

Generally [12, Eq. (9.4)]

$$\Xi(1 - \xi; J) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \xi^k \int_J dx_1 \cdots \int_J dx_k \rho_{(k)}(x_1, \dots, x_k). \quad (2.5)$$

In the case of a determinantal point process, and thus $\rho_{(k)}$ given by (1.1), it is well known (see e.g. [39]) that the sum on the RHS is the expanded form of the determinant of the Fredholm integral operator with kernel K supported on J , and thus can be written

$$\Xi(1 - \xi; J) = \prod_{l=0}^{\infty} (1 - \xi \lambda_l(J)), \quad (2.6)$$

where the $\lambda_l(J)$ are eigenvalues of the integral operator K supported on J . Equivalently, with C independent of z ,

$$\Xi(z; J) = C \prod_{l=0}^{\infty} (1 + z\mu_l(J)), \quad \mu_l(J) = \frac{\lambda_l(J)}{1 - \lambda_l(J)}. \quad (2.7)$$

Restricting attention to kernels such that the integral operator is locally trace class and self-adjoint implies that

$$1 > \lambda_0(J) \geq \lambda_1(J) \geq \dots \geq \lambda_n(J) \geq \dots \geq 0, \quad (2.8)$$

or equivalently $\infty > \mu_0(J) \geq \mu_1(J) \geq \mu_2(J) \geq \dots$. The factorisation (2.6) now tells us that the zeros of $\Xi(z; J)$ all lie on the negative real axis.

This latter fact allows a limiting form for the $\{E(k; J)\}$ to be determined, which is a stronger statement than (2.3). It is specified by the following LCLT.

Theorem 1 *Consider a determinantal point process labelled by a parameter s , and consider a region $J = J_s$. Suppose that the eigenvalues of the integral operator corresponding to the correlation kernel $K(x, y)$ supported on J_s are discrete and between 0 and 1 as in (2.8), and that $\sigma_{J_s} \rightarrow \infty$ as $s \rightarrow \infty$. We then have that the $E(k; J)$ satisfy the LCLT*

$$\lim_{s \rightarrow \infty} \sup_{x \in (-\infty, \infty)} \left| \sigma_{J_s} E([\sigma_{J_s} x + \mu_{J_s}]; J_s) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| = 0. \quad (2.9)$$

A condition for the passage from a central to a local limit theorem has been given by Bender [3]. All that is required is that all the zeros of $\Xi(z; J)$ are on the negative real axis. The proof that this is sufficient goes via the fact that the restriction of zeros to the negative z -axis implies, by Newton's theorem (log-concavity of the sequence of elementary symmetric functions of degree k in the variables $\{\mu_k(J)\}$; see e.g. [31]), that the $E(k; J)$ are log concave, i.e. $\log E(k+1; J) - 2 \log E(k; J) + \log E(k-1; J) \leq 0$. It is this latter property which is shown in [3] to be a sufficient condition for the passage from a central to a local limit theorem. As noted above, the assumption (2.8) implies that the $\{\mu_k(J)\}$ are all positive real and thus that the zeros of $\Xi(z; J)$ are all on the negative real axis, thus establishing the validity of Theorem 1.

3 Random Matrix Applications

3.1 Bulk GUE

Perhaps the best known example of a determinantal point process is the bulk scaled GUE. In this limit the correlations are given by (1.1) with the sine kernel, and we take $J = (0, s)$. For this kernel it is a classical result [10] that for large $|J|$, $\sigma_J^2 \sim (1/\pi^2) \log |J| + C/\pi^2 + (1 + \log 2\pi)/\pi^2$, where C denotes Euler's constant. In particular this diverges for $|J| \rightarrow \infty$, or equivalently for $s \rightarrow \infty$ with $J = (0, s)$, so Theorem 1 applies. The sine kernel is one of a whole class of kernels for which high precision computation of the $E(k; J)$ is available using Bornemann's Matlab software [4] based on the Fredholm determinant formula (2.6). Thus for a given finite $|J|$ we can compute the deviation of the exact value from the limiting Gaussian form. This is done in Table 1. Note that this deviation is small, differing only in the third nonzero digit for values of k within 3 standard deviations of the mean.

Table 1 Tabulation of $E(k; J)$ for the bulk GUE, with $|J| = 10$, and the corresponding Gaussian form. In the latter $\mu_J = 10$ and $\sigma_J = 0.761$

k	7	8	9	10	11	12	13
Exact	1.49×10^{-4}	0.0161	0.2238	0.5202	0.2234	0.0163	1.61×10^{-4}
Gaussian	2.2×10^{-4}	0.0166	0.221	0.524	0.221	0.0166	2.2×10^{-4}

3.2 Soft Edge GUE

The GUE also admits a soft edge scaling $\lambda \mapsto \sqrt{2N} + \lambda/(\sqrt{2}N^{1/6})$, $N \rightarrow \infty$. This has the effect of moving the origin to the neighborhood of the largest eigenvalue, and making the mean spacing in this neighborhood of order unity. The corresponding correlations are then given by (1.1) with $K(x, y)$ equal to the Airy kernel as specified in the second paragraph. The corresponding probability of there being k eigenvalues in J is denoted $E^{\text{soft}}(k; J)$, with the natural choice of J being $(-s, \infty)$. The Airy kernel is real symmetric and asymptotically $\sigma_J^2 = (1/2\pi^2) \log s^{3/2}$ [13, Eq. (2.30)], so according to Theorem 1 $E^{\text{soft}}(k; J)$ must obey the LCLT (2.9). Since the density of the eigenvalues at the soft edge has the asymptotic form $|\lambda|^{1/2}/\pi$ for $\lambda \rightarrow -\infty$ [12, Eq. (7.69)] one has $\mu_J \sim 2s^{3/2}/(3\pi) + O(1)$, which together with the asymptotic form of σ_J^2 is data to be substituted into (2.9). If we take $s = (15\pi)^{2/3}$ so that $\mu_J \approx 10$, Bornemann's package gives $E^{\text{soft}}(10; (-(15\pi)^{2/3}, \infty)) = 0.6405$, while the Gaussian form with $\mu_J = 9.99$, $\sigma_J^2 = 0.377$ as computed from (2.1) for this choice of J gives $E^{\text{soft}}(10; (-(15\pi)^{2/3}, \infty)) \approx 0.649$. This shows that the limiting Gaussian form is quite accurate even for small σ_{J_s} .

3.3 Conditioning with Fixed Eigenvalues

For the scaled GUE and some other point processes in one-dimension one can define the probability densities $\{p^{\text{soft}}(k; (-s, \infty))\}$ for there being a particle at $-s$ and exactly k particles in $(-s, \infty)$ (for this to make sense the soft wall at $x = 0$ must be such that the expected number of particles in $x > 0$ is finite). Equivalently this is the probability density function for the distribution of the $(k+1)$ -th largest eigenvalue. One can also define $\{p^{\text{bulk}}(k; s)\}$ for there being exactly k particles between two particles at separation s in the bulk. Equivalently $\{p^{\text{bulk}}(k; s)\}$ is the probability density function for the k th neighbor spacing in the bulk. In the case of the scaled GUE these probability densities also fit into the setting of Theorem 1 and thus satisfy a LCLT, as we will now demonstrate.

Consider first $p^{\text{soft}}(k; (-s, \infty))$ in the case of the soft edge scaled GUE. We can view this as a gap probability in the soft edge GUE, conditioned to have an eigenvalue at $-s$. In such a setting, it is known [15, 19] that the correlation kernel, to be denoted K_s^{soft} , can be written in terms of the usual soft edge scaled GUE kernel according to

$$K_s^{\text{soft}}(x, y) = K^{\text{soft}}(x, y) - \frac{K^{\text{soft}}(x, -s)K^{\text{soft}}(-s, y)}{K^{\text{soft}}(-s, -s)}. \quad (3.1)$$

With $\Xi_s^{\text{soft}}(z; J) := \sum_{k=0}^{\infty} z^k E_s(k; J)$, the conditions for the validity of the LCLT are met provided $\sigma_{J_s} \rightarrow \infty$ as $s \rightarrow \infty$. In the special case $J_s = (-s, \infty)$, and with $\rho^{\text{soft}}(x)$ denoting the soft edge eigenvalue density, it follows from the definitions that $E_s(k; J_s) =$

$p^{\text{soft}}(k; (-s, \infty)) / \rho^{\text{soft}}(-s)$, and so with μ_{J_s} and $\sigma_{J_s}^2$ taking the asymptotic values specified in the second paragraph of this section we obtain

$$\lim_{s \rightarrow \infty} \sigma_{J_s} p^{\text{soft}}([\sigma_{J_s} x + \mu_{J_s}]; (-s, \infty)) / \rho_{(1)}^{\text{soft}}(-s) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \quad (3.2)$$

If we set $\ell = [\sigma_{J_s} x + \mu_{J_s}]$, then for large $-s$, $-s \sim \mu_\ell - \sigma_\ell x$ with $\mu_\ell = (3\pi\ell/2)^{2/3}$ and $\sigma_\ell = \sigma_{J_s} / \rho_{(1)}^{\text{soft}}(-s)$, telling us that (3.2) can be rewritten as the statement

$$\lim_{\ell \rightarrow \infty} \sigma_\ell p^{\text{soft}}(\ell; (-\mu_\ell + \sigma_\ell x, \infty)) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad (3.3)$$

which is a LCLT with respect to the continuous variable in the probability density. This latter limit formula is consistent with an analogous result for the fluctuations of the distribution of the k -th largest eigenvalue in the finite N GUE [17], and extended to the GOE and GSE in [32] (see also [5]).

The reasoning required to establish a LCLT for $\{p^{\text{bulk}}(k; s)\}$ is analogous. Denote by $K_0^{\text{bulk}}(x, y)$ the correlation kernel for the determinantal process specified by the sine kernel, but conditioned to have an eigenvalue at the origin. This is given by a certain Bessel kernel (see [12, Eq. (7.48)] with $a = 1$). With $\rho_{(n)}^{0,s}$ ($\rho_{(n)}^0$) denoting n -point correlation functions for the bulk state conditioned to have eigenvalues at 0 and s (at 0) we have

$$\rho_{(n+1)}^0(x_1, \dots, x_n, s) / \rho_{(1)}^0(s) = \rho_{(n)}^{0,s}(x_1, \dots, x_n),$$

so proceeding as in the derivation of (3.1) gives

$$K_{0,s}^{\text{bulk}}(x, y) = K_0^{\text{bulk}}(x, y) - \frac{K_0^{\text{bulk}}(x, s) K_0^{\text{bulk}}(s, y)}{K_0^{\text{bulk}}(s, s)}.$$

The variance for large s is determined by $K_0^{\text{bulk}}(x, y)$, and its variance for large s is determined by $K^{\text{bulk}}(x, y)$, telling us in particular that the variance diverges logarithmically in this limit. It follows that $E_{0,s}(k; J)$ satisfies a local limit theorem. But $E_{0,s}(k; J) = p^{\text{bulk}}(k; s) / \rho_{(2)}^{\text{bulk}}(0, s)$ thus giving a LCTL for the latter ratio, and furthermore $\rho_{(2)}^{\text{bulk}}(0, s) \rightarrow 1$ as $s \rightarrow \infty$ so the LCLT applies to $p^{\text{bulk}}(k; s)$ itself. Moreover, the analogous change of variables in going from (3.2) to (3.3) implies that this can equivalently be regarded as a LCLT for the continuous variable in the probability density with $\sigma_k = \sigma_J$. Heuristics and graphical evidence for such a limit theorem dates back to the early literature on random matrix theory [6, Appendix N, Fig. 9].

3.4 Ginibre Ensemble

The eigenvalues of random matrices also provide examples of a determinantal point process in the plane (see e.g. [12, Ch. 15]). In an appropriate scaled $N \rightarrow \infty$ limit these all give rise to the complex Hermitian kernel for the Ginibre ensemble of non-Hermitian standard complex Gaussian matrices, mentioned in the second paragraph of the Introduction. From [27] we know that whenever J can be generated from a fixed region J_0 by dilation, $\sigma_J^2 \sim -(|\partial J|/\pi) \int_{\mathbb{R}^2} d\vec{r} |\vec{r}| \rho_{(2)}^T(\vec{r}, \vec{0}) = |\partial J|/(2\pi^{3/2})$, where $|\partial J|$ denotes the length of the perimeter of J , and the final equality follows from the explicit formula $\rho_{(2)}^T(\vec{r}, \vec{0}) = -e^{-|\vec{r}|^2}$ as implied by the correlation kernel. In particular σ_J diverges as $|J| \rightarrow \infty$ so the LCLT (2.9) must hold. In addition to the asymptotic value of σ_J^2 as noted, we furthermore have $\mu_J = |J|/\pi^2$ as data in the LCLT. The fast decay of the truncated correlations allows the number density CLT to be studied using different methods [27], and furthermore extended to

the case of multiple neighboring regions [24]. However, we don't know of any alternative way to derive the LCLT. In the case that J is a disk, the eigenvalues in (2.6) are known explicitly (see e.g. [12, Prop. 15.5.3]), however this does not persist for other shaped regions, nor is there an efficient numerical scheme to compute the corresponding Fredholm determinant. We remark that in the case of lattice gases there are techniques which allow a LCLT to be established without consideration of the Lee–Yang zeros [8].

3.5 GOE and GSE

We now turn our attention to the bulk scaled GOE and GSE. The statistical states formed by the eigenvalues are examples of Pfaffian point processes (see e.g. [12, Ch. 6]). A formula analogous to (2.6) applies for the square of the generating function,

$$\left(\Xi(1 - \xi; J)\right)^2 = \det(I - \xi J^{-1}A), \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

where A is a real 2×2 antisymmetric integral operator. As the 2×2 matrix integral operator $J^{-1}A$ is not self adjoint, we have no immediate information as to the location of the zeros of the generating function. We note however that whenever the zeros of $\Xi(z; J)$ come in complex conjugate pairs whose real parts are non-positive, and $\sigma_J \rightarrow \infty$ then the process satisfies a CLT [25].

Independent of the location of the zeros of the generating function for the GOE and GSE it was shown in [7] that the CLT (2.2) for the bulk scaled GUE implies a CLT for the bulk scaled GOE and GSE (the GOE, GUE and GSE correspond to $\beta = 1, 2, 4$ in the Dyson–Mehta scheme and will be so referred to below). This was done by using the facts that superimposing two GOE spectra at random and integrating every second eigenvalue gives a GUE distributed spectrum, and that integrating out every second eigenvalue of the GSE gives the GOE [9, 14, 30]. We have not been able to deduce from these relations a LCLT for the bulk GOE or GSE. But there are other inter-relations between bulk scaled random matrix ensembles which are suitable for this purpose [29],

$$\begin{aligned} E_1(2n; (0, 2s)) + E_1^{\text{bulk}}(2n \pm 1; (0, 2s)) &= E^\pm(n; (0, s)) \\ E_4(n; (0, s)) &= \frac{1}{2} \left(E^+(n; (0, 2s)) + E^-(n; (0, 2s)) \right). \end{aligned}$$

On the RHSs the superscripts $\pm$ refer to the determinantal point process with kernels $\frac{1}{2}(K_{\sin}(x, y) \pm K_{\sin}(x, -y))$, with $K_{\sin}$ referring to the sine kernel, while the subscripts on the LHS refer to the value of β . Theorem 1 applies to $E^\pm(n; (0, 2s))$ with $\mu = s$ and $\sigma_\pm^2 \sim \frac{1}{2}\sigma^2|_{\beta=2}$, so we see immediately that the bulk scaled GSE satisfies the LCLT (2.9) with $\sigma^2|_{\beta=4} \sim \sigma^2|_{\beta=2}/2$, and that the sum $E_1(2n; (0, 2s)) + E_1(2n \pm 1; (0, 2s))$ satisfies a LCLT. In particular this latter result implies $E_1(2n \pm 1; (0, 2s))$ are asymptotically equal. Combining this with an anticipated but as yet unproven unimodal property of $\{E_1(n; (0, s))\}$ would then imply $E_1(2n; (0, 2s))$ and $E_1(2n \pm 1; (0, 2s))$ are asymptotically equal, and the expected LCLT for the individual $E_1(n; (0, s))$ would follow.

Using Bornemann's package we note that as for the bulk scaled GUE, the finite J probabilities for the bulk scaled GOE and GSE are well approximated by a LCLT. This we have done in Table 2 for $|J| = 10$. For the corresponding values of σ_J^2 , we have made use of values accurate up to and including the constant: $\sigma_J^2 \sim (2/(\pi^2\beta)) \log |J| + B_\beta$ [10].

Table 2 Tabulation of $E(k; J)$ for the bulk GOE and GSE, with $|J| = 10$, and the corresponding Gaussian form. In the latter $\mu_J = 10$ and $\sigma_J = 0.908$ for the GOE, and $\sigma_J = 0.387$ for the GSE

k	7	8	9	10	11	12	13
Exact $\beta = 1$	0.0027	0.0464	0.2427	0.4169	0.2416	0.0467	0.0029
Gaussian	0.0029	0.0463	0.2413	0.4185	0.2413	0.0463	0.0029
Exact $\beta = 4$	8.65×10^{-7}	0.0028	0.1819	0.6307	0.1818	0.0028	9.7×10^{-7}
Gaussian	5.7×10^{-6}	0.036	0.176	0.641	0.176	0.0036	5.7×10^{-6}

4 Concluding Remarks

While we have provided a rigorous demonstration of a LCLT for the bulk scaled GUE, GSE and GOE (the latter for $E_1(2n; (0, 2s)) + E_1(2n \pm 1; (0, 2s))$), more generally one expects LCLT's to hold for the β generalization of the Gaussian ensembles [12, §1.9] for general $\beta > 0$. Explicit examples of such LCLT are stated as conjectures in [13, Conj. 6]. For the finite N circular β -ensemble, and with $J = (0, \phi)$ a segment of the unit circle, a CLT for $N(J)$ in the limit $N \rightarrow \infty$ has been established by Killip [23]. As we have seen, a sufficient condition for a LCLT, which of course implies a CLT, is that the zeros of (2.4) are all negative real, together with $\sigma(J_s) \rightarrow \infty$ as $s \rightarrow \infty$. However, as only the case $\beta = 2$ is determinantal, we have no way of establishing such a property, or even providing numerical evidence, for general $\beta > 0$. An exception is the cases $\beta = 1$ and $\beta = 4$ which, as commented in the subsection above relating to the GOE and GSE, have a Pfaffian structure. Making use of this Pfaffian structure, recent numerical studies have been carried out in [21] which indicate that the zeros of (2.4) for the finite N circular symplectic ensemble are all on the negative real axis.

After scaling, the spectrum of the Gaussian ensembles of Hermitian matrices has in a statistical sense just the two types of distinct behaviours—bulk and soft edge. If we consider instead ensemble of so-called Wishart matrices—matrices $X^\dagger X$ where X is a complex Ginibre matrix, then the constraint that the eigenvalues be positive gives rise to a new statistical behaviour for eigenvalue near the origin referred to as the hard edge regime (see e.g. [12, §7.2.1]). When X is a rectangular complex Ginibre matrix of size $n \times N$ ($n \geq N$) the hard edge regime is a determinantal point process specified by the Bessel kernel

$$K(x, y) = \left(J_a(x^{1/2})y^{1/2}J'_a(y^{1/2}) - x^{1/2}J'_a(x^{1/2})J_a(y^{1/2}) \right) / (x - y), \quad a = n - N.$$

This kernel is real symmetric so (2.8) holds, and we know furthermore that for $J_s = (0, s)$ and s large, $\sigma_{J_s}^2 \sim \frac{1}{\pi^2} \log s$ [13, Eq. (2.30)] and this in particular diverges. The conditions of Theorem 1 are therefore met, and thus the corresponding conditioned gap probabilities satisfy a LCLT.

Our application of Theorem 1 has been focussed on random matrices. But there are other well known examples of determinantal processes in statistical physics and mathematics obeying the conditions of the theorem and which thus must then exhibit the same LCLT (see e.g. [18]). A prominent example, conditional upon the validity of the Montgomery–Odlyzko law, is the Riemann zeros for large modulus [22]. The Montgomery–Odlyzko law states that certain statistical properties of the latter, upon appropriate scaling, coincide with the bulk scaled GUE and this if valid form a determinantal point process. A proof that these zeros satisfy a LCLT is an open question. The weaker statement of a CLT was proved by Fuji [16]; see [34, Th. 1.1] for its form in notation similar to that used in the present paper. The result

of [16] assumes the validity of the Riemann hypothesis. This assumption has been removed in the recent work [26].

Spin polarized free fermions in dimension d provide examples of determinantal point processes in higher dimensions [38]. With $k_F = 2\sqrt{\pi}(\Gamma(1 + d/2))^{1/d}$, the corresponding bulk scaled (unit density) kernel is computed to equal $c_F J_{d/2}(k_F \|\vec{x} - \vec{y}\|) / (k_F \|\vec{x} - \vec{y}\|)^{d/2}$ where $J_\nu(x)$ denotes the usual Bessel function and $c_F = 2^{d/2} \Gamma(1 + d/2)$. When $d = 1$ this corresponds to the sine kernel. From the explicit form of the kernel substituted in (2.1), it is shown in [38] that for J a sphere of radius R , σ_j^2 / R^{d-1} is proportional to $\log R$ in the limit $R \rightarrow \infty$, and in particular σ_j^2 diverges in this limit so Theorem 1 applies.

It should also be mentioned that in random matrix theory one encounters determinantal point processes in which the correlation kernel is real and non-symmetric. A simple example is the rank one perturbation of the GUE at the soft edge, with parameter tuned so that it corresponds to the critical regime for the separation of the largest eigenvalue (see e.g. [12, Eq. (7.41)] for the precise form of the kernel). Although Theorem 1 does not apply directly, since the eigenvalues of the rank one perturbed matrix strictly interlace those of the unperturbed one (Cauchy interlacing theorem) we see that the LCLT is inherited from the corresponding LCLT for the soft edge GUE, as is seen by considering for example (3.3). More recently a family of real non-symmetric kernels involving the Meijer G -function has been found in the context of studying the hard edge scaling for products of rectangular complex Gaussian matrices [20], which has recently been shown to be an example of a determinantal point process [1], [2]. A proof of a LCLT for in this setting is an open question, as is the weaker statement of a CLT.

Acknowledgments The work of PJF was supported by the Australian Research Council. The work of JLL was supported by NSF Grant DMR1104500. JLL thanks B. Pittel, D. Ruelle and particularly E. Speer for very helpful information about LCLT. We thank T. Spencer and H.-T. Yau for the invitation to participate in the IAS Princeton program ‘Non-equilibrium dynamics and random matrices’, thus facilitating the present collaboration, and we thank H. Spohn and P. Sarnak for comments on various drafts.

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