



Nonperturbative Time Dependent Solution of a Simple Ionization Model

Ovidiu Costin¹, Rodica D. Costin¹, Joel L. Lebowitz^{2,3}

¹ The Ohio State University, 231 W 18th Ave, Columbus, OH 43210, USA.

E-mail: costin.9@osu.edu; costin.10@osu.edu

² Department of Mathematics, Rutgers University, Hill Center - Busch Campus, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA. E-mail: lebowitz@math.rutgers.edu

³ Department of Physics, Rutgers University, Hill Center - Busch Campus, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA

Received: 23 July 2017 / Accepted: 15 December 2017
© Springer-Verlag GmbH Germany, part of Springer Nature 2018

Abstract: We present a non-perturbative solution of the Schrödinger equation $i\psi_t(t, x) = -\psi_{xx}(t, x) - 2(1 + \alpha \sin \omega t)\delta(x)\psi(t, x)$, written in units in which $\hbar = 2m = 1$, describing the ionization of a model atom by a parametric oscillating potential. This model has been studied extensively by many authors, including us. It has surprisingly many features in common with those observed in the ionization of real atoms and emission by solids, subjected to microwave or laser radiation. Here we use new mathematical methods to go beyond previous investigations and to provide a complete and rigorous analysis of this system. We obtain the Borel-resummed transseries (multi-instanton expansion) valid for all values of α , ω , t for the wave function, ionization probability, and energy distribution of the emitted electrons, the latter not studied previously for this model. We show that for large t and small α the energy distribution has sharp peaks at energies which are multiples of ω , corresponding to photon capture. We obtain small α expansions that converge for all t , unlike those of standard perturbation theory. We expect that our analysis will serve as a basis for treating more realistic systems revealing a form of universality in different emission processes.

1. Introduction

The ionization of atoms and the emission of electrons from a metal, induced by an oscillating field, such as one produced by a laser, continues to be a problem of great theoretical and practical interest, see [2, 5, 15, 21] and the references therein. This phenomena goes under the name of photo-emission. It was first explained by Einstein in 1905; an electron absorbs " n photons" acquiring their energy, $n\hbar\omega$, which permits it to escape the potential barrier confining it. While the complete physics of these phenomena would involve quantization of the electromagnetic field and its interaction with matter, i.e. photons and relativity, the basic understanding is contained already in the semiclassical limit where the electromagnetic field is not quantized, expected to be valid when the density of photons is large [4]; for a mathematical derivation of this limit via Floquet states

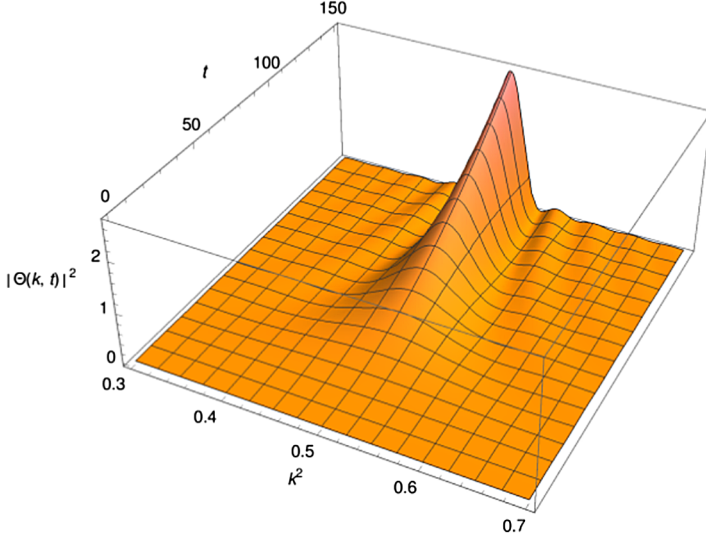
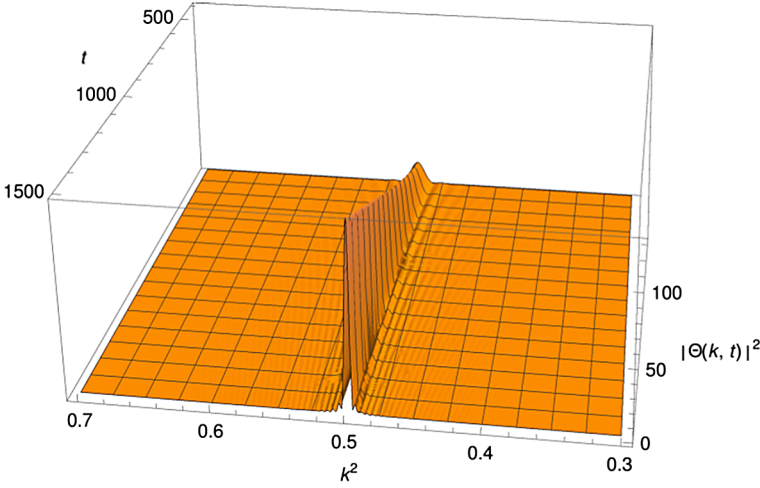
(A) $t \leq 150$ (B) $500 \leq t \leq 1500$

Fig. 1. Θ as a function of k^2 and time, for $\alpha = 1/20$ and $\omega = 3/2$, calculated from the leading order in (19)

see [19]. One then considers the solution of the non-relativistic Schrödinger equation in an oscillating field giving rise to a potential with period $2\pi/\omega$, [2, 5]. Resonant energy absorption at multiples of ω then yields effects qualitatively similar to those of photons, in some regimes, see Fig. 1.

In units in which $\hbar = 2m = 1$ the Schrödinger equation has the form

$$i \frac{\partial \psi}{\partial t} = [H_0 + V(t, x)] \psi \quad (1)$$

Here H_0 describes the time-independent system assumed to have both discrete and continuous spectrum, and the laser field is modeled by a time periodic potential, $V(t, x) = V(t + 2\pi/\omega, x)$. Typically, the latter is represented as a vector potential or a dipole field, e.g. $V(t, x) = E \cdot x \sin \omega t$, [2,5].

Starting in a bound state of the reference hamiltonian H_0 , $\psi(x, 0) = u_b(x)$ corresponding to the energy $-E_b$ and expanding in generalized eigenstates, assuming u_b is the only effective bound state, the evolution is given by

$$\psi(x, t) = \theta(t)e^{iE_b t}u_b(x) + \int_{\mathbb{R}^d} \Theta(k, t)u(k, x)e^{-ik^2 t} dk \quad (2)$$

Physically, $|\theta|^2$ gives the probability of finding the particle in the eigenstate $u_b(x)$ and $|\Theta(k, t)|^2$ is the probability density of the ionized electron in “quasi-free” states (continuous spectrum) with energies k^2 . It follows from the unitarity of the evolution that

$$|\theta(t)|^2 + \int_{\mathbb{R}^d} |\Theta(k, t)|^2 dk = 1$$

Accordingly, if $\theta(t) \rightarrow 0$ as $t \rightarrow \infty$, we say that the system ionizes completely.

When $\omega > E_b$, a first order approximation [2,5] in the strength of V (used very judiciously) gives emission into states with energy $k^2 + E_b = \omega$. Clever physics arguments also yield Fermi’s golden rule of exponential decay from the initial bound state [2,5]. These only hold approximately and only over some “intermediate” time scales as discussed in the sequel.

To deal with the case of transitions caused by large fields V , one needs to go to high order perturbation theory, which is complicated [2,5]. In fact, as we will explain, standard perturbation theory only produces a finite number of correct perturbative orders. To deal with larger fields, one uses various “strong field” approximations due to Keldysh and others [17]. For literature on strong field approximations, see [1,3,14,18]. There, one uses scattering states $\tilde{u}(k, x)$ strongly modified (Volkov states) by the oscillating field. We shall not consider that here, but instead focus on getting a complete rigorous solution of (1) for a toy model which nevertheless exhibits many features of more realistic situations, see [6]. We can then study carefully how photons show up in this semiclassical limit.

The model we study is a one dimensional system with reference Hamiltonian H_0 , whose mathematical properties are analyzed in [10], is

$$H_0 = -\frac{\partial^2}{\partial x^2} - 2\delta(x), \quad x \in \mathbb{R}, \quad (3)$$

It has a single bound state

$$u_b(x) = e^{-|x|}$$

with energy $-E_b = -1$ and its generalized eigenfunctions are

$$u(k, x) = \frac{1}{\sqrt{2\pi}} \left(e^{ikx} - \frac{e^{i|kx|}}{1 + ik} \right), \quad x, k \in \mathbb{R} \quad (4)$$

Beginning at $t = 0$, when $\psi(x, 0) = u_b(x)$, we add a parametric harmonic perturbation to the base potential. For $t \geq 0$ we have

$$H = -\frac{\partial^2}{\partial x^2} - 2\delta(x) - 2\alpha \sin \omega t \delta(x) = H_0 + V(t, x) \quad (5)$$

(where we take for definiteness $\alpha, \omega > 0$) and look for solutions of the associated Schrödinger equation in the form (2). The full behavior of $\psi(x, t)$ is very complicated despite the simplicity of the model. We expect the main feature of the evolution of $\psi(x, t)$ to be universal for ionization by an oscillatory field.

As already noted this model has been studied extensively before. We refer the reader in particular to [6] where it was shown that, for all α and ω , $\theta(t) \rightarrow 0$, i.e., we have complete ionization. We also investigated there both analytically and numerically the behavior of $\theta(t)$ as a function of ω and showed qualitative agreement with experiments on the ionization of hydrogen-like atoms by strong radio frequency fields. In [8] we studied general periodic potentials and found the condition on the Fourier coefficients for complete ionization. There are (exceptional) situations where one does not get complete ionization. In [9] we showed ionization when the external forcing is an oscillating electric field. A large field approximation for this latter setting can be found in [11].

In this paper we introduce new methods which allow us to complete the analysis of this model for all t, α : we obtain a rapidly convergent representation (in the form of a Borel summed transseries, or “multi-instanton expansion”) for the solution $\psi(x, t)$ valid for all t, ω and α and we find the distribution $|\Theta(k, t)|^2$ of energies of the emitted electrons as a function of t, α, ω . The latter, which was not done before, is where the “photonic” picture shows up most clearly. We will investigate this connection more explicitly in a separate article [7].

There are strong peaks of $|\Theta(k, t)|^2$ which for small α and $\omega \in (\frac{1}{n}, \frac{1}{n-1})$ are centered near $k^2 = n\omega - 1$, see Fig. 1 for $\omega = 3/2$. The main peak corresponds to the absorption of one photon and approaches a Dirac distribution centered at $k^2 = 1/2$ in the limit $t \rightarrow \infty$ followed by $\alpha \rightarrow 0$. Clearly, the discreteness of the emission spectrum in the above limit is a consequence of the periodicity of the classical oscillating field and does not require the concept of photons, see also [18], Footnote 1. We find that there are other (smaller) peaks emanating from the bottom of the continuous spectrum. For small α these are centered near $k^2 = n\omega$, see Theorem 4, (iv). We also obtain a perturbation expansion of the wave function for small α in a form which is uniformly convergent for any $t \in \mathbb{R}^+$, and which, in principle, can be carried out explicitly to any order.

It follows from our analysis that the predictions of the usual perturbation theory hold when $t = o(\alpha^{-2} |\ln \alpha|)$, beyond which the behavior of the physical quantities is qualitatively different.

1.1. The Laplace transform and the energy representation. It was shown in [6] that

$$\theta(t) = 1 + 2i \int_0^t \varphi(s) ds \quad (6)$$

and

$$\Theta(k, t) = \sqrt{\frac{2}{\pi}} \frac{|k|}{1 - i|k|} \int_0^t \varphi(s) e^{i(1+k^2)s} ds \quad (7)$$

where φ satisfies the integral equation

$$\varphi(t) = \alpha \sin \omega t \left(1 + \int_0^t \varphi(s) \eta(t-s) ds \right)$$

with

$$\eta(s) = \frac{2i}{\pi} \int_0^\infty \frac{u^2 e^{-is(1+u^2)}}{1+u^2} du = \frac{\sqrt{i} e^{-is}}{\sqrt{\pi} \sqrt{s}} - i \operatorname{erfc}(\sqrt{is})$$

It can be checked, [6], that the Laplace transform of φ

$$\Phi(p) := \mathcal{L}\varphi(p) = \int_0^\infty \varphi(s) e^{-ps} ds \quad (8)$$

is analytic in the right half plane and satisfies the functional equation¹

$$\Phi(p) = \frac{i\alpha}{2} \frac{\Phi(p-i\omega)}{i\sqrt{ip+\omega-1}+1} - \frac{i\alpha}{2} \frac{\Phi(p+i\omega)}{i\sqrt{ip-\omega-1}+1} + \frac{\alpha\omega}{\omega^2+p^2} \quad (9)$$

(The square root is understood to be positive on $\mathbb{R}^+$, and analytically continued on its Riemann surface.²)

2. Main Results

2.1. Results for general α, ω .

Theorem 1. For all $\alpha > 0$ and $\omega > 0$,

- (i) $(1+|p|)^2 \Phi(p)$ is bounded and Φ is analytic in the closed right half plane, except for

$$p = -i\beta_n, \quad \beta_n = 1 + n\omega, \quad n \in \mathbb{Z} \quad (10)$$

where it is analytic in $\sqrt{p+i\beta_n}$;

- (ii) in the open left half plane Φ has exactly one array of simple poles located at

$$p_n := p_n(\alpha, \omega) = p_0(\alpha, \omega) - in\omega, \quad n \in \mathbb{Z} \quad (11)$$

and the residues $R_n = \operatorname{res}(\Phi, p_n)$ can be calculated using continued fractions, see Sect. 3.3, and satisfy

$$R_n = O(|n|^{-1/2}); \quad |n| \rightarrow \infty \quad (12)$$

Away from the line of poles, $(1+|p|)^2 \Phi(p)$ is bounded in the left half plane. The functions p_n and R_n are analytic in α in a neighborhood of $[0, \infty)$;

- (iii) $\Re p_0(\alpha; \omega) < 0$; p_0 satisfies an equation of the form $A(p, \alpha; \omega) = B(p, \alpha; \omega)$ where A, B are meromorphic functions (given by convergent continued fraction representations, see (37), (38)).

The following is the *non-perturbative* (arbitrary coupling) form of the decay of the bound state.

¹ A very similar functional equation can be obtained directly from the Schrödinger equation for $\mathcal{L}\psi(0, p)$.

² In previous papers we used r instead of α and a different branch of the square root; with these changes the formulas agree.

Theorem 2. (i) The function θ in (6) has a Borel summed **transseries** representation (also known as a **multi-instanton expansion**, [22]) convergent for all $t > 0$

$$\theta(t) = -2 \sum_{n \in \mathbb{Z}} \frac{R_n}{p_n} e^{p_n t} + \sum_{n \in \mathbb{Z}} e^{-i\beta_n t} \int_0^\infty e^{-st} F_n(s; \alpha) ds \quad (13)$$

with $\beta_n \in \mathbb{R}$, $p_n \in \mathbb{C}$, $\Re p_n < 0$, $R_n \in \mathbb{C}$ as in (10)–(12). The F_n are analytic in α in a neighborhood of $[0, \infty)$, analytic in s if $\Re s > 0$ and in $\sqrt{s}$ for s near 0. For large $|s|$, $F_n = O(s^{-3})$. The first sum converges factorially and the second at least as fast as $1/n^3$.

(ii) Similarly, the function Θ is a Borel summed transseries

$$\begin{aligned} \Theta(k, t) = & \sqrt{\frac{2}{\pi}} \frac{|k|}{1 - i|k|} \left[\Phi(-i(1 + k^2)) + i \sum_{n \in \mathbb{Z}} \frac{R_n e^{(p_n + i + ik^2)t}}{p_n + i(1 + k^2)} \right. \\ & \left. + \sum_{n \in \mathbb{Z}} e^{(-i\beta_n + i + ik^2)t} \int_0^\infty e^{-st} \frac{F_n(s, \alpha)}{-i\beta_n + i + ik^2 - s} ds \right] \end{aligned} \quad (14)$$

where the F_n are the same as in (13).

Corollary 3. For all $\alpha, \omega > 0$ we have

$$\lim_{t \rightarrow \infty} \theta(t) = 0, \quad \lim_{t \rightarrow \infty} \Theta(k, t) = \sqrt{\frac{2}{\pi}} \frac{|k|}{1 - i|k|} \Phi(-i(1 + k^2)) \quad (15)$$

2.2. Perturbation theory: results for small α . In this section we assume that $\omega^{-1} \notin \mathbb{N}$. See Note 5 regarding $\omega^{-1} \in \mathbb{N}$.

Notation: In the rest of the paper “ o_α ” denote functions analytic and vanishing at $\alpha = 0$.

Theorem 4. Assume $\omega^{-1} \notin \mathbb{N}$. Let $p_0 = p_0(\alpha, \omega)$ as in Theorem 1 (ii).

(i) For $\omega > 1$, we have, for α small enough,

$$p_0 = \alpha^2 \frac{-\sqrt{\omega - 1} + i\sqrt{1 + \omega}}{2\omega} (1 + o_\alpha) \quad (16)$$

With m the least integer for which $m\omega > 1$ we have

$$\Re p_0 = -\frac{1}{m\omega} \frac{\sqrt{m\omega - 1}}{\prod_{k < m} (1 - \sqrt{1 - k\omega})^2} \frac{\alpha^{2m}}{2^{2m+1}} (1 + o_\alpha) \quad (17)$$

(ii) The residues (see (12) for arbitrary α) satisfy $R_0 = \frac{p_0}{2}$ for $\omega > 1$, and for $\omega < 1$

$$R_0 = \frac{im\alpha^{m-1} p_0}{2^m \prod_{k < m} (1 - \sqrt{1 - k\omega})} (1 + o_\alpha) \quad (18)$$

Furthermore, as $n \rightarrow \infty$, $R_n/R_0 = O(\alpha^{2|n|}/\Gamma(n/2))$ and $F_n = O(\alpha^{2n+2}/n^3)$.

(iii) As a function of α , $\theta(t)$ is analytic for small $\alpha \in \mathbb{C}$ and real-analytic for $\alpha \in \mathbb{R}$. With m as in (i), on the scale $t \in (0, o(\alpha^{-2m} \ln \alpha))$ we have³

$$|\theta(t)|^2 = e^{-2\Re p_0 t} (1 + o(1))$$

As $t \rightarrow \infty$,

$$|\theta(t)|^2 = O(\alpha^4 t^{-3})$$

(iv) The distribution of energies satisfies

$$\begin{aligned} \Theta(k, t) = & \sqrt{\frac{2}{\pi}} \frac{|k|}{1 - i|k|} \frac{\alpha}{2} \left[\frac{1}{1 + k^2 + \omega} - \frac{1}{-ip_0 + 1 + k^2 - \omega} \right. \\ & \left. + e^{(p_0 + i + ik^2)t} \left(\frac{e^{i\omega t}}{-ip_0 + 1 + k^2 + \omega} - \frac{e^{-i\omega t}}{-ip_0 + 1 + k^2 - \omega} \right) \right] (1 + o_a) \end{aligned}$$

(Note that $p_0 = O(\alpha^2)$ and that for k such that $1 + k^2 - \omega = O(\alpha^2)$ we have $\Theta(k, t) = O(\alpha^{-1})$).

Note 5. If for some $m \in \mathbb{N}$ we have $m\omega^{-1} = 1 + O(\alpha^{2m})$, which means that poles are close to branch points, there is a smooth transition region where R_0 and $\Re p_0$ change from $O(\alpha^{2m+2})$ to $O(\alpha^{2m})$. We will not analyze this intricate transition in the present paper.

Corollary 6. The functions Θ and θ have fully convergent perturbation expansions in small α , in sup norm away from $t = 0$, provided we keep the power series of p_n in the exponent in (13) and (14).

Proof. This follows from uniform convergence and analyticity of each term in (13) and (14). For the expansion to be uniformly rapidly convergent the exponentials should not be further expanded as power series. $\square$

3. Proofs

3.1. Organization of the paper and main ideas. We are interested in obtaining rapidly convergent expansions for $\theta(t)$ and $\Theta(k, t)$ for all α and ω . To achieve this we study in great detail the singularity structure of $\Phi(p)$. We prove in particular that $\Phi(p)$ has exactly one array of evenly spaced poles, for $\Re p < 0$, and one array of branch points, for $\Re p = 0$. Their location and residues determine, via the inverse Laplace transform, the transseries representation of Θ and θ . To show this rigorously for all α we first establish these facts for small α using compact operator techniques; we then extend them for arbitrary α by devising a periodic operator isospectral with the one of interest, whose pole structure can be analyzed by appropriate complex analysis tools.

The proof of Theorem 1 (i) is found in Sect. 3.2, (ii) and (iii) in Sect. 3.5, (12) in Sect. 3.7.1. The functional equation (9) is rewritten as a parameter dependent equation on $\ell^2(\mathbb{Z})$ and analyzed with compact operator techniques.

Section Sect. 3.3 contains results and notations used further in the paper.

Theorem 4 (i) is proved in Sect. 3.4, (ii) in Sect. 3.8 and (iii), (iv) in Sect. 3.10. For small α the position of the poles is found from a continued fraction representation described in Sect. 3.3. The information about the poles for larger α relies on the analysis of a periodic compact operator isospectral to the main one and zero-counting techniques. Theorem 2 is proved in Sect. 3.9.1.

³ This is the Fermi Golden Rule of exponential decay of the bound state valid for small amplitudes and moderately large times.

3.2. *Proof of Theorem 1(i).* Denoting

$$p = -iq, \quad \Phi(p) = g(q) \quad (19)$$

(9) becomes

$$g(q) = \alpha h(q + \omega)g(q + \omega) - \alpha h(q - \omega)g(q - \omega) + f(q) \quad (20)$$

where

$$h(q) := \frac{1}{2} \frac{1}{\sqrt{q-1}-i}, \quad f(q) := -\frac{\alpha\omega}{q^2 - \omega^2}$$

It turns out that the pole of h at $q = 0$ has no bearing on the regularity of the solutions, as the equation can be regularized in a number of ways. One is presented in detail in [8]. A simpler way is presented in Sect. 3.2.1.

It is convenient to discretize (20). With the notation

$$q =: q_n = \sigma + n\omega \quad \text{with } \Re\sigma \in [0, \omega), \quad n \in \mathbb{Z} \quad (21)$$

and setting $h_n = h(q_n)$, $f_n = f(q_n)$, we obtain the difference equations with parameters σ, ω

$$g_n = \alpha h_{n+1}g_{n+1} - \alpha h_{n-1}g_{n-1} + f_n \quad (22)$$

or, in operator notation,

$$\mathbf{g} = K_0(\sigma, \alpha)\mathbf{g} + \mathbf{f} \quad (23)$$

3.2.1. *Regularization of the operator.* We rewrite (22). Let

$$d_n = \frac{1}{4}(\sigma + n\omega - i\beta)^{-1} \quad \text{and } b_n = d_n/(h_{n+1}h_{n-1}); \quad (\beta > 0) \quad (24)$$

Then,

$$g_n = (1 - b_n)g_n + \frac{\alpha d_n}{h_{n-1}}g_{n+1} - \frac{\alpha d_n}{h_{n+1}}g_{n-1} + b_n f_n \quad (25)$$

or,

$$\mathbf{g} = K(\sigma, \alpha)\mathbf{g} + B\mathbf{f} \quad (26)$$

where $[K(\sigma, \alpha)\mathbf{g}]_n = (1 - b_n)g_n + \frac{\alpha d_n}{h_{n-1}}g_{n+1} - \frac{\alpha d_n}{h_{n+1}}g_{n-1}$ and $(Bf)_n = b_n f_n$. Now K , $\mathbf{f}$ and B are pole-free in the closed lower half plane (analyticity of the solution in the upper half plane is known, see the beginning of Sect. 1.1). Note that $\mathbf{f}$ is a multiple of α .

Extension. It is convenient to remove the restriction in (21) on σ , and allow $\sigma \in \mathbb{C}$.

Remark 7. Note that $K = I - B + BK_0$ therefore $(I - K_0)^{-1} = (I - K)^{-1}B$, so equations (23) and (26) are equivalent wherever B is invertible, that is, for $\sigma \notin 1 + \omega\mathbb{Z}$. We will rely on this equivalence to choose the more convenient one for a particular purpose.

Proposition 8. (i) *The operator $K(\sigma, \alpha)$ is compact in $\ell^2(\mathbb{Z})$. It is linear-affine in α , and analytic in σ except for a square root branch point at $1 - \lfloor \omega^{-1} \rfloor \omega$.*
(ii) *For $\sigma \neq 0$, $K_0(\sigma, \alpha)$ has the properties of K listed above.*

Proof. (i) For compactness, note that $K(\sigma, \alpha)$ is a composition of two shifts and multiplications by diagonal operators whose elements vanish in the limit $|n| \rightarrow \infty$ (all its coefficients, see (25), are $O(|n|^{-1/2})$). Noting that $h(q)$ has a pole at $q = 0$ and a branch point at $q = 1$, the analyticity properties are manifest. The proof of (ii) is similar. $\square$

Theorem 1(i) now follows from the results above, Proposition 9 below, and an argument similar to (and simpler than) the one in Sect. 2.

Proposition 9. *The homogeneous equation*

$$\mathbf{g} = K(\sigma, \alpha)\mathbf{g} \quad (27)$$

has no nontrivial ℓ^2 solution if $\Im\sigma \geq 0$. By the Fredholm alternative (26) has a unique solution which has the same analyticity properties as K .

In particular, $(I - K(\sigma, \alpha))^{-1}$ is analytic if $\Im\sigma > 0$ and on each segment $(1+n\omega, 1+(n+1)\omega)$, $n \in \mathbb{Z}$; at $\sigma = \beta_n = 1+n\omega$ it is analytic in $\sqrt{\sigma - \beta_n}$.

The proof is given in [6]; for completeness, we sketch the argument in the Appendix.

3.3. Further properties of the homogeneous equation. The general theory of recurrence relations [13] shows that the homogeneous part of (22) has two linearly independent solutions, one that grows like $(\alpha/2)^{-n}(n!)^{1/2}$ and one that decays like $(\alpha/2)^n/(n!)^{1/2}$ for $n \rightarrow \infty$, and two similar solutions for $n \rightarrow -\infty$; the one that decays at $+\infty$ is different from the one that decays at $-\infty$, unless 1 is in the ℓ^2 spectrum of $K_0(\sigma, \alpha)$. Since we need more details, we reprove the relevant claims. The main results are given in Corollaries 14, 15.

In this section it is convenient to work with the continuous equations (20). Its homogeneous part is

$$g(q) = \alpha h(q + \omega)g(q + \omega) - \alpha h(q - \omega)g(q - \omega) \quad (28)$$

Lemma 12 shows the existence of a solution of (28) which goes to zero as $n \rightarrow +\infty$ for α not too large, with tight uniform estimates for all $q \in \mathbb{R}$, and of a similar solution for $n \rightarrow -\infty$. Lemma 13 shows existence of such solutions for any $\alpha > 0$, providing estimates only for $|q|$ large enough.

Looking for a solution that decays for large q we define

$$\rho(q) := \rho(q; \alpha) = g(q)/g(q - \omega)$$

and obtain from (28)

$$\rho(q) = \mathcal{N}(\rho) \quad (29)$$

where $\mathcal{N}$ is the nonlinear operator

$$\mathcal{N}(\rho)(q) = -\frac{\alpha h(q - \omega)}{1 - \alpha h(q + \omega)\rho(q + \omega)}$$

Similarly, looking for solutions which decay for $q \rightarrow -\infty$ the ratio

$$\Omega(q) := \Omega(q; \alpha) = g(q - \omega)/g(q)$$

satisfies

$$\Omega(q) = \mathcal{M}(\Omega) \quad (30)$$

where

$$\mathcal{M}(\Omega)(q) = \frac{\alpha h(q)}{1 + \alpha h(q - 2\omega) \Omega(q - \omega)}$$

Notations 10. As usual, a domain in $\mathbb{C}$ is an open, connected subset. $\mathbb{H}_\ell$ denotes the open lower half plane in $\mathbb{C}$.

Let

$$A > 0, \quad \alpha_A = A/(1 + A^2)$$

and denote

$$J_N = \{q \mid \Re(q) \geq N\omega, \Im(q) \in [0, \varepsilon]\}$$

(for a suitably small ε). Consider the Banach space $\mathcal{C}(J_N)$ of functions continuous in the strip J_N , with the sup norm. Let $\mathcal{C}(\tilde{J}_{-1})$ denote the Banach space of continuous functions in

$$\tilde{J}_{-1} := \{q \mid \Re(q) \leq -\omega, \Im(q) \in [0, \varepsilon]\}$$

We denote by $\mathcal{R}$ the class of functions which are real-analytic in α for all $\alpha \in \mathbb{R}^+$ and in $q \in \mathbb{H}_\ell$, continuous on $\overline{\mathbb{H}_\ell}$ and with possible square root branch points at $q \in 1 + \omega\mathbb{Z}$.

Remark 11. By the usual properties of the Laplace transform, g is analytic in the upper half plane. Since below we are interested in the properties of g in the lower half plane it is convenient to place the branch cuts in the upper half plane. Later, in Sect. ??, when we deform the contour of an inverse Laplace transform (in q it is horizontal, in the upper half plane), the points on the curve are moved vertically down, yielding a collection of vertical Hankel contours⁴ around the branch points $1 + n\omega$ and residues. For this particular purpose, placing the cuts in the upper or lower half plane can be seen to be equivalent.

Lemma 12. (i) For $|\alpha| < \alpha_A$, the operator $\mathcal{N}$ defined in (29) is contractive in the ball $\|\rho\| < A$ in $\mathcal{C}(J_N)$; the contractivity factor is $\frac{1}{4}\alpha^2(1 + o(1))$ as $\alpha \rightarrow 0$. Thus (29) has a unique fixed point $\rho \in \mathcal{C}(J_N)$. Also, ρ is analytic in α for $|\alpha| < \alpha_A$ and satisfies $|\rho(q)| \leq \frac{1}{2}\alpha(1 + o(1))$ as $\alpha \rightarrow 0$.

(ii) The operator $\mathcal{M}$ is contractive in a ball $\|\Omega\| < A$ in $\mathcal{C}(\tilde{J}_{-1})$.

Let us state first a more general result, valid for all $\alpha \in \mathbb{R}^+$ (where now the dependence of ρ on α is made explicit):

Lemma 13. (i) For any fixed α_0 and large enough $q_0 > 0$ the operator $\mathcal{N}$ in (29) is contractive in the ball

$$B = \{\rho \mid \sup_{q \geq q_0, |\alpha| \leq \alpha_0} |\rho(q, \alpha)| \leq \alpha_0 q_0^{-1/2}\}$$

and thus it has a unique solution, which is analytic in (q, α) . Similar estimates hold for (30).

⁴ A Hankel contour is a path surrounding a singular point, originating and ending at infinity, [16].

(ii) For large $q > 0$,

$$\rho(|q|; \alpha) = -\frac{1}{2}\alpha|q|^{-1/2}(1 + o(1)), \quad \Omega(-q) = -\frac{1}{2}\alpha q^{-1/2}(1 + o(1)) \quad (q \rightarrow +\infty) \quad (31)$$

(iii) For large q in the lower half-plane,

$$|\rho(q; \alpha)| = \frac{1}{2}\alpha|\Im q|^{-1/2}(1 + o(1)), \quad |\Omega(q)| = \frac{1}{2}\alpha|\Im q|^{-1/2}(1 + o(1)) \quad (\Im q \rightarrow -\infty) \quad (32)$$

Proof of Lemma 12. (i) We note that the minimum of $|\sqrt{q-1}-i|$ in J_N is $\sqrt{N\omega} + O(\varepsilon)$ implying $\|h\| < \frac{1}{2}$ for ε small enough.

We first show that $\mathcal{N}$ leaves the ball $\|\rho\| \leq A$ invariant. A straightforward estimate shows that if $|\alpha| \leq \alpha_0$ then $\mathcal{N}$ is well defined on the ball and

$$\|\mathcal{N}(\rho)\| < A$$

The contractivity factor is obtained by taking the sup of the norm of the Fréchet derivative of $\mathcal{N}$ with respect to ρ :

$$\left\| \frac{\partial \mathcal{N}}{\partial \rho} \right\| = \left\| \frac{\alpha^2 T^{-1} h T h}{(1 - \alpha T(h\rho))^2} \right\| \quad (33)$$

where

$$(Tg)(q) := g(q + \omega) \quad (34)$$

Under the assumptions in the Lemma, we have

$$\left\| \frac{\partial \mathcal{N}}{\partial \rho} \right\| \leq \frac{\alpha^2 \|h\|^2}{(1 - \alpha \|h\| A)^2} < 1, \quad \text{and} \quad \left\| \frac{\partial \mathcal{N}}{\partial \rho} \right\| < \frac{1}{4}\alpha^2(1 + o(1)) \quad \text{as } \alpha \rightarrow 0 \quad (35)$$

The same analysis goes through in the space of functions $\rho(q, \alpha)$ which are of class $\mathcal{R}$ for $q \in J_N$ and analytic in α for $|\alpha| \leq \alpha_A$, in the joint sup norm, in α and q , proving joint analyticity in α, q , except for the mentioned square root branch points. To show that $q = 1 + n\omega$ are square root branch points, we return to the $\sigma + n\omega$ representation of (21). For simplicity of presentation assume $\omega > 1$. We repeat the arguments above, now in the space of functions of the form $A_1\sqrt{q-1} + A_2$ where A_1, A_2 are analytic near $q = 1$, in the norm $\|A_1\| + \|A_2\|$.

Moreover, since the only singularities of $h(q)$ are a pole of order one at $q = 0$ and a square root branch point at $q = 1$, a similar analysis shows that ρ is of class $\mathcal{R}$.

Straightforward estimates in (29) show that for small α we have

$$\|\rho\| < \frac{1}{2}\alpha(1 + o(1))$$

(ii) We note that $\max_{\bar{J}_{-1}} |h(q)| = |h(-\omega)|$ hence $\|h\| < \frac{1}{2}$. The rest of the proof is as for (i). $\square$

Proof of Lemma 13. We note that for any A and α , we have, for large enough q

$$\|\mathcal{N}(\rho)\| \leq \frac{1}{2}\alpha q^{-1/2}(1 + o(1)) \quad \text{and} \quad \left\| \frac{\partial \mathcal{N}}{\partial \rho} \right\| \leq \frac{1}{4}\alpha^2 q^{-1}(1 + o(1)) \quad (36)$$

$\square$

We use Pringsheim's notation for continued fractions $A + B/(C + D/(E + \dots)) = A + \frac{B}{C} + \frac{D}{E} + \dots$

Corollary 14. *Iteration of (29) yields a continued fraction*

$$\rho(q) = -\frac{\alpha h(q - \omega)}{1} + \frac{\alpha^2 h(q + \omega)h(q)}{1} + \frac{\alpha^2 h(q + 2\omega)h(q + \omega)}{1} + \dots \quad (37)$$

which is convergent for $\Re(q) \geq N\omega$, $\Im q \in [0, \varepsilon]$ and $\alpha < 2/3$. For small α , the rate of convergence is $4^{-n}\alpha^{2n}$.

Similarly, iteration of (30) yields a convergent continued fraction

$$\Omega(q) = \frac{\alpha h(q)}{1} + \frac{\alpha^2 h(q - \omega)h(q - 2\omega)}{1} + \frac{\alpha^2 h(q - 2\omega)h(q - 3\omega)}{1} + \dots \quad (38)$$

Proof. Convergence of the continued fraction, by definition, means that the n -th truncate of the continued fraction, that is $\mathcal{N}^{\text{on}}(0)$, converges to the fixed point $\rho = \mathcal{N}(\rho)$. Since zero is in the domain of contractivity of $\mathcal{N}$, convergence follows directly from Lemma 12. The norm of the Fréchet derivative of $\mathcal{N}^{\text{on}}$ is $(\frac{1}{4}\alpha^2)^n(1+o(1))$ implying the last statement. Convergence of (38) is similar. $\square$

Corollary 15. *As $n \rightarrow -\infty$, there is a solution of (22) with $f = 0$ which is $O(2^{-|n|}\alpha^{|n|}|n|^{-|n|/2})$; a second, linearly independent solution, has the property $1/g_n = O(2^{-|n|}\alpha^{|n|}|n|^{-|n|/2})$, that is, such a solution grows factorially. A similar statement holds as $n \rightarrow \infty$.*

Proof. The first part follows from the fact that $g_{n-1}/g_n = \Omega_n$. For the second part, one looks as usual for a second solution in the form $h_n = g_n u_n$ and notes that u_n satisfies a first order recurrence relation that can be solved in closed form in terms of g_n . $\square$

3.4. Proof of Theorem 4(i): location of the singularities for small α . By Proposition 9 the resolvent can only be singular if $\Im \sigma < 0$, which we will assume henceforth. By Remark 7 we can then work with the simpler operator K_0 . We place branch cuts in the upper half plane, see Remark 11.

Theorem 16. *There is a $\delta > 0$ such that for all complex α with $|\alpha| < \delta$ the following hold.*

- (i) *There exists a unique $\sigma = q_0(\alpha)$ in the strip $\Re \sigma \in [0, \omega)$ so that $\text{Ker}(I - K(\sigma, \alpha)) \neq \{0\}$. More precisely, $q_0(\alpha) = \alpha^2 s_0(1 + \alpha f(\alpha))$ where*

$$s_0 = -\frac{\sqrt{1 + \omega} + i\sqrt{\omega - 1}}{2\omega} \quad (39)$$

for some f analytic at zero.

- (ii) *For $\omega \in (\frac{1}{m}, \frac{1}{m-1})$, $m - 1 \in \mathbb{N}$ we have⁵*

$$q_0(\alpha) = \alpha^2 s_0(1 + o_a) + i\alpha^{2m} \xi_0(1 + o_a) \quad (40)$$

where s_0 is real, given by (39), and ξ_0 is real, given by

$$\xi_0 = -\frac{1}{m\omega} \frac{\sqrt{m\omega - 1}}{\prod_{1 \leq k \leq m-1} (1 - \sqrt{1 - k\omega})^2} \frac{1}{2^{2m+1}} \quad (41)$$

⁵ Note that for $\omega < 1$ we have $s_0 = -(2\omega)^{-1}(\sqrt{1 + \omega} - \sqrt{1 - \omega}) \in \mathbb{R}$.

Remark. Since $p = -iq$, the poles in the p -plane are at $-iq_0(\alpha) + i\omega\mathbb{Z}$, hence Theorem 16 completes the proof of Theorem 4(i).

For the proof of Theorem 16 we first show, in Lemma 17, that any singularities of $(I - K(\sigma, \alpha))^{-1}$ are $O(\alpha^2)$ distance to $\omega\mathbb{Z}$, if α is small enough. Then location of the poles is found by series expansions in α .

Lemma 17. *There is a $C(\omega) > 0$ such that for $|\alpha|$ small enough equation (25) has a unique solution for any σ in the strip $\Re\sigma \in [0, \omega)$ with $\text{dist}(\sigma, \{0, \omega\}) \geq C(\omega)\alpha^2$. In particular, for such σ , $\text{Ker}(I - K(\sigma, \alpha)) = \{0\}$.*

Proof. As mentioned above, we can additionally assume that $\Im\sigma < 0$, hence invertibility of $I - K(\sigma, \alpha)$ is equivalent to $\text{Ker}(I - K(\sigma, \alpha)) = \{0\}$, which is equivalent to $\text{Ker}(I - K_0(\sigma, \alpha)) = \{0\}$.

Let $K_0(\sigma, \alpha) = \alpha K_1(\sigma)$ (where now K_1 does not depend on α). If u is such that $(I - K_0(\sigma, \alpha))u = 0$, then $u = K_0u = \alpha K_1u = \alpha^2 K_1^2 u$. But straightforward estimates show that $\|\alpha^2 K_1^2\| < 1/2$, if C is large enough and $\text{dist}(\sigma, \{0, \omega\}) > C\alpha^2$. This implies that $\alpha^2 K_1^2$ is contractive and thus $u = 0$. $\square$

Proposition 18. *The functions $\rho(q, \alpha)$, $\Omega(q, \alpha)$ are meromorphic in (q, α) for q in the open lower half plane and $\alpha \in \mathbb{C}$.*

Proof. Let α be fixed and $\rho = \rho(q; \alpha)$ be the fixed point provided by Lemma 13, analytic in (q, α) for $q > q_0$. Using the recurrence relation (29) ρ can be continued to a meromorphic function for all q with smaller real part (the coefficients in (29) are meromorphic except for square root branch points on the real line).

Similarly, Ω can be continued to a meromorphic function. $\square$

Proof of Theorem 16 (i). Let ρ , Ω be given by Proposition 18. The value(s) of $\sigma = \sigma(\alpha)$ for which $\text{Ker}[I - K(\sigma, \alpha)] \neq \{0\}$ are those for which

$$\rho(\sigma, \alpha) = \frac{1}{\Omega(\sigma, \alpha)} \quad (42)$$

as discussed in Sect. 3.3. By Lemma 17 any such σ has the form $\sigma = \alpha^2 s$ with $|s| < C(\omega)$. We now show that for small α there is exactly one solution of (42) in the strip $\Re\sigma \in [0, 1 - \lfloor \omega^{-1} \rfloor \omega)$. (Note that $0 < 1 - \lfloor \omega^{-1} \rfloor \omega < \omega$ using the assumption that $\omega^{-1} \notin \mathbb{Z}$ which ensures that the poles and the branch points do not coincide.)

Expanding ρ , respectively Ω , in a power series in α we have

$$\begin{aligned} \rho(\alpha^2 s, \alpha) &= \frac{i\alpha}{2} \frac{1}{\sqrt{\omega+1} - 1} \frac{1}{1 - \frac{1}{2s} \frac{1}{i\sqrt{\omega-1+1}}} (1 + O(\alpha^2)), \\ \text{and } \Omega(\alpha^2 s, \alpha)^{-1} &= -i\alpha s (1 + O(\alpha^2)) \end{aligned} \quad (43)$$

Let $F(r, s) := \rho(\alpha^2 s) - \Omega(\alpha^2 s)^{-1}$. It can be checked that F is analytic in $\{(s, \alpha) \mid |s| < C(\omega), |\alpha| < \delta\}$ for small enough δ . Equating the dominant terms in (43) we obtain that F has only one simple zero, of the form (39).

Finally, note that if equation (42) had a solution of the form $\sigma = \omega - \alpha^2 s_1$ then $\sigma = -\alpha^2 s_1$ would also be a solution, but this is ruled out by the uniqueness of $O(\alpha^2)$ solutions. $\square$

Proof of Theorem 16 (ii). As in the proof of Corollary 14, the n -th truncate of the continued fraction defining ρ , that is $\mathcal{N}^{\circ n}(0)$, converges to the fixed point $\rho = \mathcal{N}(\rho)$. Similarly, $\mathcal{M}^{\circ n}(0)$, converges to the fixed point $\Omega = \mathcal{M}(\Omega)$.

With the notation

$$x_k = \alpha^2 t_k, \quad t_k = h(\sigma + k\omega) h(\sigma + (k-1)\omega), \quad k \in \mathbb{Z}$$

we obtain from (37), (38)

$$\begin{aligned} \mathcal{N}^{\circ(m+1)}(0) &= -\frac{\alpha h(\sigma - \omega)}{1} + \frac{x_1}{1} + \frac{\alpha^2 t_2}{1} \dots + \frac{\alpha^2 t_m}{1}, \\ \mathcal{M}^{\circ(m+1)}(0) &= \frac{\alpha h(\sigma)}{1} + \frac{\alpha^2 t_{-1}}{1} + \frac{\alpha^2 t_{-2}}{1} \dots + \frac{\alpha^2 t_{-m}}{1} \end{aligned} \quad (44)$$

Note that, for small α , $x_1 = O(1)$ and all other $x_k = O(\alpha^2)$ and, inductively, $\mathcal{N}^{\circ(k+1)}(0) = \mathcal{N}^{\circ(k)}(0) + O(\alpha^{2k-1})$ and $\mathcal{M}^{\circ(k+1)}(0) = \mathcal{M}^{\circ(k)}(0) + O(\alpha^{2k+1})$. Therefore

$$\rho(\sigma) = \mathcal{N}^{\circ(m+1)}(0) + O(\alpha^{2m+1}), \quad \Omega(\sigma) = \mathcal{M}^{\circ(m+1)}(0) + O(\alpha^{2m+3})$$

Noting that

$$h(\sigma + k\omega) = \frac{1}{2i} \frac{1}{\sqrt{1 - \sigma - k\omega} - 1} \quad \text{for } k < m$$

it follows that all t_k with $k < m$ have a power series expansion in α with real coefficients for $\omega \in (\frac{1}{m}, \frac{1}{m-1})$. Then so does the denominator in the right of (44), as well as the denominator on the left, truncated to $m-1$ terms.

We now look for a solution to

$$1 = \rho(\sigma)\Omega(\sigma) = \mathcal{N}^{\circ(m+1)}(0)\mathcal{M}^{\circ(m+1)}(0)(1 + O(\alpha^{2m+1})) \quad (45)$$

in the form $\sigma = \alpha^2 s + O(\alpha^{2m+2})$ where $s = s_0 + \alpha^2 s_1 + \dots + i\alpha^{2m-2}\xi$ with ξ and all s_k real.

A simple calculation gives

$$t_m = \mathcal{P} + iq_m + O(\alpha^2), \quad \text{where } q_m = -\frac{\sqrt{m\omega - 1}}{4m\omega} \frac{1}{\sqrt{1 - (m-1)\omega} - 1}, \quad \mathcal{P} \in \mathbb{R} \quad (46)$$

which implies

$$\frac{\alpha^2 t_{m-1}}{1 + \alpha^2 t_m} = \mathcal{P} - i\alpha^4 t_{m-1} q_m + O(\alpha^6)$$

where here, and in the following, $\mathcal{P}$ denotes quantities that depend polynomially on α and have real coefficients. Then, inductively, we obtain that

$$\begin{aligned} \frac{1}{1} + \frac{\alpha^2 t_2}{1} + \dots + \frac{\alpha^2 t_m}{1} &= 1 + \alpha^2 \mathcal{P} + i(-1)^{m-1} \alpha^{2m-2} t_2 \dots t_{m-1} q_m + O(\alpha^{2m}) \\ &:= 1 + \alpha^2 \mathcal{P} + i\alpha^{2m-2} Q + O(\alpha^{2m}) \end{aligned} \quad (47)$$

Equation (45) becomes

$$\frac{-\alpha h(\alpha^2 s - \omega)}{1 + x_1(1 + \alpha^2 \mathcal{P} + i\alpha^{2m-2} \mathcal{Q} + O(\alpha^{2m}))} \frac{\alpha h(\alpha^2 s)}{1 + \alpha^2 \mathcal{P}} = 1 + O(\alpha^{2m+1}) \quad (48)$$

We have

$$-\alpha^2 h(\alpha^2 s) h(\alpha^2 s - \omega) = -\frac{1}{2s} \frac{1}{\sqrt{1+\omega}-1} + O(\alpha^2) \quad (49)$$

$$x_1 = \alpha^2 h(\alpha^2 s) h(\alpha^2 s + \omega) = \frac{1}{2s} \frac{1}{\sqrt{1-\omega}-1} + O(\alpha^2) \quad (50)$$

Using (49), (50) in (48) and equating the coefficient of α^0 we obtain (39).

Rewriting (48) using (49), (50) as

$$\frac{\frac{1}{\sqrt{1+\omega}-1} + s O(\alpha^2)}{2s + (\frac{1}{\sqrt{1-\omega}-1} + s O(\alpha^2))(1 + \alpha^2 \mathcal{P} + i\alpha^{2m-2} \mathcal{Q} + O(\alpha^{2m}))} \frac{1}{1 + \alpha^2 \mathcal{P}} = 1 + O(\alpha^{2m+1})$$

and equating the dominant imaginary terms, of order $O(\alpha^{2m-2})$, we obtain $2\xi + \frac{\mathcal{Q}}{\sqrt{1+\omega}-1} = O(\alpha^{2m})$. Using (47), (46) and noting that

$$t_k = -\frac{1}{4} \frac{1}{\sqrt{1-k\omega}-1} \frac{1}{\sqrt{1-(k-1)\omega}-1} + O(\alpha^2)$$

we obtain formula (41). $\square$

3.5. Proof of Theorem 1(ii), (iii): structure of the resolvent $(I - K(\sigma, \alpha))^{-1}$ for any $\alpha \neq 0$.

Theorem 19. *Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$. $(I - K(\sigma, \alpha))^{-1}$ is analytic for $\sigma \in \mathbb{H}_\ell$ except for one array of simple poles and it is continuous on $\mathbb{H}_\ell$.*

The poles are located at $\sigma = q_0(\alpha) + n\omega$ (for all $n \in \mathbb{Z}$) where $\Im q_0(\alpha) < 0$ and $q_0(\alpha)$ is real-analytic for $\alpha > 0$.

As a consequence $(I - K(\sigma, \alpha))^{-1}$ is analytic in a strip $\Re \sigma \in [0, \omega)$, $\Im \sigma < 0$ except for one simple pole.

Note that going back to the variable p , Theorem 19 implies the first statement of Theorem 1(ii) and (11).

The structure of the proof of Theorem 19 is as follows. The results were proved for small α in Sect. 3.4, see also Lemma 23 (i). We extend them to all $\alpha \neq 0$ using a general result on the constancy of number of zeros of analytic, periodic functions depending on a parameter contained in Lemma 20. To apply this Lemma, we construct a periodic operator isospectral to $I - K(\sigma, \alpha)$ (Lemma 21). This is especially convenient since working in the whole lower half plane would mean working with infinitely many poles while restricting σ to a strip introduces a number of unnecessary complications.

Note that continuity up to the real line follows from Proposition 9. Therefore it suffices to consider $\Im \sigma < 0$, and by Remark 7, we can work with the operator K_0 .

Lemma 20. *Let $A, Q, \omega > 0$ and define*

$$S = \{z \mid \Re z \in (0, \omega), \Im z < 0\}; \quad I_A = (0, A) \quad (51)$$

Let $H(z, \alpha)$ be a function which is real-analytic in $\alpha \in I_A$, analytic and exponentially bounded in $z \in S$. Assume further that H is periodic, $H(z+\omega, \alpha) = H(z, \alpha)$, continuous in $\bar{S} \times I_A$, and

$$H(z, \alpha) \neq 0 \quad \text{if} \quad (z, \alpha) \in \{z, \Im z \leq -Q \text{ or } \Im z = 0\} \times I_A \quad (52)$$

Let $Z(\alpha)$ be the number of zeros, counting multiplicity, of H in $\bar{S} \times I_A$.

Then the function Z is constant. If $Z = 1$, then defining q_0 by $H(q_0(\alpha), \alpha) = 0$, q_0 is real-analytic in $\alpha \in I_A$.

Proof. Multiplying H by $e^{-2\pi i N z / \omega}$ for some $N \in \mathbb{N}$ we can arrange that $H \rightarrow 0$ as $\Im z \rightarrow -i\infty$. By (52) and continuity, there is an $\varepsilon = \varepsilon(A)$ so that for all $\alpha < A$ all the zeros of $H(\cdot, \alpha)$ are in $\{\Im z \in (-Q, -\varepsilon)\}$.

Fix $\alpha \in (0, A)$ and choose a small $\beta \geq 0$ so that $H \neq 0$ if q is on ∂S_β , where $S_\beta = S + \beta - i\varepsilon$. By the argument principle, the number of zeros in S counting multiplicity is

$$Z(\alpha) = -\frac{1}{2\pi i} \int_{-i\varepsilon/2}^{\omega - i\varepsilon/2} \frac{\partial_s H(s, \alpha)}{H(s, \alpha)} ds \quad (53)$$

noting that by periodicity the contributions of the vertical sides of ∂S_β cancel out and the integral over $[\beta - i\varepsilon, \omega + \beta - i\varepsilon]$ equals the integral in (53).

The right side of (53) is manifestly real-analytic in α and integer-valued, thus constant. Real-analyticity when $Z = 1$ is an immediate consequence of the implicit function theorem. $\square$

Let T denote the forward shift in ℓ^2 (cf. (34)). A straightforward calculation shows that for any σ

$$K_0(\sigma + \omega, \alpha) = T K_0(\sigma, \alpha) T^{-1} \quad (54)$$

Lemma 21. *The operator*

$$K_2(\sigma, \alpha) := T^{-\sigma/\omega} K_0(\sigma, \alpha) T^{\sigma/\omega}$$

is periodic in σ .

$I - K_2(\sigma, \alpha)$ is a periodic operator isospectral to $I - K_0(\sigma, \alpha)$ and $\|K_2\| = O(e^{2\pi|\Im \sigma|/\omega})$ for large σ .

Proof. Note that T is a unitary operator, and thus $T = e^{iA}$ for some self-adjoint bounded operator A . In $L^2(\mathbb{T})$, the Fourier transform space, T is multiplication by $e^{-i\varphi}$ and A is multiplication by $-\varphi$. This reduces the analysis to a compact analytic manifold. $\square$

Lemma 22. *For small $|\alpha|$ there is a unique pole of $(I - K_0(\sigma, \alpha))^{-1}$ in the strip $\Re q \in [0, \omega)$. The pole is simple and analytic in α .*

Proof. Since the position of the unique pole is analytic for small α , and is manifestly simple (see (25)) when $\alpha = 0$, this follows from the argument principle. $\square$

Lemma 23. (i) For small $\alpha > 0$ the resolvent $(I - K_0(\cdot, \alpha))^{-1}$ has only one array of poles, located in the lower half plane at $\sigma = \alpha^2 s_0(1 + \alpha g(\alpha)) + n\omega$, $n \in \mathbb{Z}$ with g analytic.

(ii) For any α, q , $\dim \text{Ker}(I - K_0(\sigma, \alpha)) \in \{0, 1\}$ and $\dim \text{Ker}(I - K_0(\sigma, \alpha)^*) \in \{0, 1\}$, where $*$ denotes the adjoint.

proof. (i) is an immediate corollary of Lemma 22 and Theorem 16.

(ii) follows from Corollary 15. Indeed, if $\mathbf{g} \in \text{Ker}(I - K_0)$ then $\mathbf{g}$ is an ℓ^2 solution of the linear recurrence (27) which cannot have a two dimensional space of solutions decaying at both $n \rightarrow \pm\infty$ by Corollary 15. Similar arguments apply to $\text{Ker}(I - K_0^*)$, since $K_0^* \mathbf{y} = \alpha \bar{h}(T^{-1} - T)\mathbf{y}$ and $K_0^* \mathbf{y} = \mathbf{y}$ yields a second order difference equation similar to that of K_0 , having two solutions $O((n!)^{1/2}(-2\sqrt{\omega}/\alpha)^n)$, respectively $O((n!)^{-1/2}(\alpha/2\sqrt{\omega})^n)$ for $n \rightarrow \infty$ and two similar solutions (one decreasing to 0 and another one increasing) as $n \rightarrow -\infty$. $\square$

Proposition 24. For any $A > 0$ there is $Q, \varepsilon > 0$ such that the Neumann series

$$(I - K(\sigma, \alpha))^{-1} = \sum_{m=0}^{\infty} K(\sigma, \alpha)^m \quad (55)$$

converges to an operator valued function, analytic in $\{(\sigma, \alpha) | \Re \alpha \in [0, A), |\Im \alpha| < \varepsilon, \Im \sigma < -Q\}$.

A similar statement holds for $K_0(\sigma, \alpha)$.

Proof. This simply follows from the fact that the shift operators have norm one, $h(q) = O((\Im q)^{-1/2})$, and are analytic in this regime. $\square$

Proof of Theorem 19. By Lemma 21 it suffices to prove these results for $(I - K_2(\sigma, \alpha))^{-1}$.

Let F_k be finite rank operators converging to K_2 as $k \rightarrow \infty$ and P_k the projectors on the range of F_k . Let $A > 0$ and choose, cf. Proposition 24, $Q = Q(A)$ so that $I - K_2$ is invertible if $\Im \sigma < -Q$, and $\alpha \in (0, A]$. Let $\varepsilon > 0$ be small enough and $k = k_A$ large enough so that for $\alpha \in [0, A + \varepsilon]$ and q in B where

$$B := \{z | \Re z \in [-\varepsilon, \omega + \varepsilon], \Im z \in [-Q - \varepsilon, 0]\}$$

we have $\|K_2 - F_k\| < \varepsilon$. The easily checked identity (cf. in [20] p. 202)

$$(I - K_2)^{-1} = (I - F)^{-1}(I - (K_2 - F_k)) \quad \text{where } F := F_k(I - (K_2 - F_k))^{-1} \quad (56)$$

implies $I - K_2$ is invertible iff $I - F$ is invertible. Now F is finite rank and by the usual Fredholm alternative $I - F$ is not invertible iff $x = Fx$ has a nonzero solution. Since $F = P_k F$, if $x = Fx$ we have $x = P_k x$. Thus the condition for $(I - K_2)^{-1}$ to have a pole in S is $h_A := \det M_A = 0$ where M_A is the matrix of $P_k(I - F)P_k$.

To end the proof, we note that A is arbitrary, h_A satisfies the conditions of Lemma 20, by Proposition 9 and analyticity of the matrix elements of M_A . This also completes the proof of the theorem. $\square$

3.6. Proof of Theorem 1(iii). The first part is an immediate consequence of Lemma 23. The second statement follows from (42) and the Proposition 18.

3.7. Proof of Theorem 4(ii): calculation of the residues.

3.7.1. General expression for residues and proof of (12). The argument is based on a Laurent expansion of the resolvent and general properties of compact operators.

In this section we consider only σ with $\Im\sigma < 0$; therefore we can work with the operator K_0 , see Remark 7.

Note that $\text{Ran}(I - K_0(\sigma, \alpha))$ and $\text{Ran}(I - K_0(\sigma, \alpha)^*)$ are closed, since $K_0(\sigma, \alpha)$, and therefore $K_0(\sigma, \alpha)^*$ are compact [12].

Let $q_0 = q_0(\alpha)$ be as in Theorem 19. Then $\text{Ker}(I - K_0(q_0, \alpha)) \neq \{0\}$, therefore it is one dimensional by Lemma 23. Let $\mathbf{y}_0$ be a unit vector generating this kernel.

Since 0 belongs to the spectrum of $L_0 := I - K_0(q_0, \alpha)$ then it also belongs to the spectrum of $L_0^* = I - K_0(q_0, \alpha)^*$. This means that 1 is an eigenvalue of the compact operator $K_0(q_0, \alpha)^*$, hence it is in the point spectrum. Let $\mathbf{y}_0^*$ be a unit vector generating $\text{Ker } L_0^*$.

By Theorem 19 equation (25) has a solution $\mathbf{g}$ which has a pole of order one when $z := q - q_0 = 0$. Thus $\mathbf{g} = \mathbf{g}_{-1}z^{-1} + \mathbf{g}_0 + z\tilde{\mathbf{g}}$ where $\tilde{\mathbf{g}}$ is analytic at $z = 0$.

Since $K_0(\sigma, \alpha)$ is analytic in σ at $\sigma = q_0$ we can write $I - K_0(\sigma, \alpha) = L_0 + zS(z, \alpha)$ with S analytic. With this notation equation (22) becomes

$$L_0 \left(z^{-1}\mathbf{g}_{-1} + \mathbf{g}_0 + z\tilde{\mathbf{g}} \right) + zS(z, \alpha) \left(z^{-1}\mathbf{g}_{-1} + \mathbf{g}_0 + z\tilde{\mathbf{g}} \right) = \mathbf{f} \quad (57)$$

Since $\mathbf{f}$ is analytic, this implies that $\mathbf{g}_{-1} = \lambda\mathbf{y}_0$ for some scalar λ .

Decompose $\mathbf{f} = \langle \mathbf{f}, \mathbf{y}_0^* \rangle \mathbf{y}_0^* + \mathbf{w}$ where $\mathbf{w} \in (\text{Ker } L_0^*)^\perp = \text{Ran } L_0$. With the notations $S_0 := S(0, \alpha) = \frac{\partial L_0}{\partial q} \big|_{z=0}$ and $\mathbf{f}_0 = \mathbf{f}(q_0, \alpha) + O(z)$, Eq. (57) becomes

$$L_0\mathbf{g}_0 + \lambda S_0\mathbf{y}_0 + O(z) = \langle \mathbf{f}, \mathbf{y}_0^* \rangle \mathbf{y}_0^* + \mathbf{w} \quad (58)$$

Noting that L_0 is invertible from $(\text{Ker } L_0)^\perp$ to $\text{Ran } L_0$ Eq. (58) is solvable only if

$$\langle \mathbf{f}_0, \mathbf{y}_0^* \rangle \mathbf{y}_0^* - \lambda S_0\mathbf{y}_0 \in \text{Ran } L_0 = (Sp \mathbf{y}_0^*)^\perp$$

The argument above works for arbitrary $\mathbf{f}_0 \in \ell^2$, $\langle S_0\mathbf{y}_0, \mathbf{y}_0^* \rangle \neq 0$ and $\lambda = \langle \mathbf{f}_0, \mathbf{y}_0^* \rangle / \langle S_0\mathbf{y}_0, \mathbf{y}_0^* \rangle$, thus

$$\mathbf{g} = \frac{1}{\sigma - q_0} \frac{\langle \mathbf{f}_0, \mathbf{y}_0^* \rangle}{\langle S_0\mathbf{y}_0, \mathbf{y}_0^* \rangle} \mathbf{y}_0 + O(1) \quad (\sigma \rightarrow q_0) \quad (59)$$

Proposition 25. *The residues $R = (R_n)_{n \in \mathbb{Z}}$ defined in Theorem 1(ii) are multiples of $\mathbf{y}_0$, given by (59), therefore they satisfy $(I - K_0)R = 0$ and consequently $R_n = O(2^{-n}\alpha^n n^{-n/2})$, implying (12).*

Proof. To see this we consider disks $\mathbb{D}_n$ around the poles of Φ small enough to contain no pole of the nonhomogeneous part of the equation and write the equation for $\frac{1}{2\pi i} \oint_{\partial\mathbb{D}_n} \Phi$, which is just the homogeneous part of the recurrence. The solution is thus a multiple of the eigenvector $\mathbf{y}_0$ (noting that $R_n = -i\text{Res}(g, q_n)$) and Corollary 15 completes the proof. $\square$

3.8. Concrete calculations: proof of Theorem 4(ii). This follows directly from the following

Lemma 26. *For $\omega \in (\frac{1}{m}, \frac{1}{m-1})$ the 0th component of the residue of $(I - K)^{-1}\mathbf{f}$ is*

$$R_0 = \frac{i m \alpha^m p_0}{2^m \prod_{k < m} (1 - \sqrt{1 - k\omega})} (1 + o_a) \quad (60)$$

(with o_a defined in the beginning of Sect. 2.2. The other components are of higher order in α .)

Proof. Straightforward calculations based on the continued fraction (37), (38), and their matching condition (42), (42) yield power series in small α for $\mathbf{y}_0$, $\mathbf{y}_0^*$ for (59). Note that the residues do not depend on the normalization of these vectors.

We prove Theorem 4(ii) only for $\omega > 1$ [this corresponds to $m = 1$ in (60)]. For the other cases the proof is similar, retaining a sufficient number of components of the vectors $\mathbf{y}_0$, $\mathbf{y}_0^*$.

Choosing the component -1 of $\mathbf{y}_0$, $y_{0,-1} = 1$, we obtain that $y_{0,0} = O(\alpha)$, and $y_{0,1} = -y_{0,-1} + O(\alpha^2)$. Inductively, $y_{0,\pm n} = O(\alpha^{|n|-1})$. Similarly, we chose $y_{0,0}^* = 1$.

More concretely, we let P be the orthogonal projector in ℓ^2 on the components $-1, 0, 1$ and, to simplify the notation, we omit the factor $(1 + o_a)$ in the formulas below. Then

$$P\mathbf{y}_0 = \begin{bmatrix} -1 \\ \frac{\alpha(i\sqrt{\omega+1}-\sqrt{\omega-1})}{2\omega} \\ 1 \end{bmatrix}, \quad P\mathbf{y}_0^* = \begin{bmatrix} \frac{\alpha(\sqrt{\omega-1}-i)}{2\omega} \\ 1 \\ \frac{-i\alpha(\sqrt{\omega+1}+1)}{2\omega} \end{bmatrix}; \quad P\mathbf{f}_0 = \begin{bmatrix} \frac{-i\sqrt{\omega-1}+\sqrt{\omega+1}}{2\alpha} \\ \frac{\alpha}{\omega} \\ \frac{i\sqrt{\omega-1}-\sqrt{\omega+1}}{2\alpha} \end{bmatrix} \quad (61)$$

and

$$PS_0P = \begin{bmatrix} 0 & \frac{2i(i\sqrt{\omega-1}\sqrt{\omega+1}-1)}{\alpha^3} & 0 \\ \frac{(2i\sqrt{\omega-1}+\omega-2)\alpha}{4\omega^2\sqrt{\omega-1}} & 0 & \frac{-i\alpha(2\sqrt{\omega+1}+\omega+2)}{4\omega^2\sqrt{\omega+1}} \\ 0 & \frac{-2i(i\sqrt{\omega-1}\sqrt{\omega+1}-1)}{\alpha^3} & 0 \end{bmatrix} \quad (62)$$

Using (59) we get $\lambda = \alpha/2(1 + o_a)$. The contribution from the pole close to p_0 to the inverse Laplace transform of Φ is $\frac{\alpha}{2}e^{p_0 t}$. The rest is a straightforward calculation based on (7). $\square$

3.9. Proof of the transseries representation.

3.9.1. Proof of Theorem 2. We now go back from the discretized quantity $\mathbf{g}$ to the continuous one, g , using (21). Then by (8) and (19) we have

$$\varphi(t) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} g(ip)e^{pt} dp \quad (63)$$

We can choose $c = 0$ since g is L^1 . $\mathbf{g}$ solves (23) and, by Proposition 9 and Theorem 19, it is analytic in σ except for an array of poles and of branch points, therefore the same holds for g . We proceed as described in Remark 11 and deform the contour of the inverse Laplace transform in (63) (the Fourier transform of g) into a sum of Hankel contours. In the process we collect the residues $R_n e^{p_n t}$.

More precisely, we obtain from (63)

$$\begin{aligned}
 \varphi(t) &= \frac{1}{2\pi} \int_{ic-\infty}^{ic+\infty} g(q) e^{-iqt} dq = i \sum_n \text{Res}[g(q) e^{-iqt}, q_n] \\
 &\quad + \frac{1}{2\pi i} \sum_n e^{-i\beta_n t} \int_0^\infty [g(-i\tau + \beta_n + 0) - g(-i\tau + \beta_n - 0)] e^{-\tau t} d\tau \\
 &= i \sum_n R_n e^{P_n t} + \frac{1}{2\pi i} \sum_n e^{-i\beta_n t} \int_0^\infty \Delta g(-i\tau + \beta_n) e^{-\tau t} d\tau
 \end{aligned} \tag{64}$$

We only need to check the convergence of the sum of the residues, and of the integrals, which is rather straightforward but for completeness we outline below.

The sum of the residues converges factorially fast, by (12). We claim that $\sup_{n \in \mathbb{Z}, \tau > 0} |(n^2 + 1)g(n\omega \pm 0 - i\tau)| < \infty$ ensuring the convergence of the sum of the branch-cut contributions.

Fix some $n_0 > 0$ (a similar argument applies for $n_0 < 0$) so that $\sup_{n \in \mathbb{Z}, \tau > 0} |h(n\omega \pm 0 - i\tau)| < 1/2$. Note that $g(n_0\omega \pm 0 - i\tau)$ are real-analytic in τ and vanish in the limit $\tau \rightarrow \infty$. Thus $\sup_{\tau > 0} |g(n_0\omega \pm 0 - i\tau)| < C$ for some C . By analytic continuation, $g(n\omega \pm 0 - i\tau)$ are given by $(I - K_0)^{-1}f$ and in particular are in ℓ^2 for all $\tau \geq 0$, and satisfy the recurrence (22). With P the projection on $\ell^2(n_0 + \mathbb{N})$ (the ℓ^2 sequences indexed starting with n_0), Pg satisfies the equation

$$Pg = PKPg + Pf + E \tag{65}$$

where E is the vector whose only nonzero component is $E_{n_0} = -\alpha h_{n_0} g_{n_0}$. We consider (65) in the space $\ell_2^\infty := \{x : \|x\| = \sup_{n > n_0} n^2 |x_n| < \infty\}$. Since $\|PK\| < 1/2$, $\|f\| < C_1$, $\|E\| < C$ for some C_1 , $(I - PK)^{-1}(f + E)$ is the unique ℓ_2^∞ solution of (65). Since it is obviously in ℓ^2 , it coincides with g .

Finally note that from (6) we obtain that $\theta(t) = -2i \int_t^\infty \varphi(s) ds$ (since $\theta(t) \rightarrow 0$ as $t \rightarrow \infty$). The series for θ and Θ are obtained by integration of (64) completing the proof.

3.10. Proof of Theorem 4 (iii) and (iv). Formula (19) is obtained from (14) and the Neumann series for $(I - K_0)^{-1}$, noting that $\Phi = (I - K_0)^{-1}f$.

For small α , the branch cut contributions are most easily found from the Neumann series $\mathbf{g} = (I - \alpha \tilde{K}_0(\sigma, \omega))^{-1}\mathbf{f} = \mathbf{f} + \alpha \tilde{K}_0\mathbf{f} + O(\alpha^2)$ noting that g_0 is analytic and g_n for $n \neq \pm 1$ has square root branch points in the order α^3 of the expansion (recall that $\mathbf{f}$ is a multiple of α). Since $g_{\pm 1} = \mp h_0 f_0 + \text{analytic} + O(\alpha^3)$ we obtain, for small $|z|$,

$$g_{\pm 1}(1+z) = \pm \frac{\alpha^2 \omega}{2} \frac{1}{\sqrt{z} - i} \frac{\omega}{(1+z)^2 - \omega^2} + O(\alpha^3) + f(z) \tag{66}$$

where $f(z)$ is a function analytic at 0.

The rest follows from Theorem 2 using (64) and Theorem 4(ii).

4. Appendix

We sketch the argument in the proof of Proposition 9 with the notation of this paper.

Proof. Consider a solution $\mathbf{g} \in \ell^2$ of (27), the homogeneous part of (25). Note that $h_0 = h_0(\sigma)$ has a pole at $\sigma = 0$, hence, from (24), $b_1(0) = 0$. Then taking $n = 1$ in the homogeneous part of (25) we see that $g_0(0) = 0$.

Let $g_n h_n = u_n$; clearly $\mathbf{u} \in \ell^2$. Taking now $n = 0$ in the recurrence, we see that $u_1(0) = u_{-1}(0)$. Reversing now the steps that led to (25), we see that the homogeneous equation is equivalent to

$$h_n^{-1} u_n = \alpha u_{n+1} - \alpha u_{n-1} \quad (67)$$

where the left side is zero if $n = 0$, $\sigma = 0$. Taking scalar product with $\mathbf{u}$ in (67) we get

$$\sum_{n \in \mathbb{Z}} h_n^{-1} |u_n|^2 = \alpha \sum_{n \in \mathbb{Z}} u_{n+1} \bar{u}_n - \alpha \sum_{n \in \mathbb{Z}} u_{n-1} \bar{u}_n = 2i\alpha \Im \sum_{n \in \mathbb{Z}} u_{n+1} \bar{u}_n$$

Note that $h_n \in i\mathbb{R}$ for $n < 1$ and $\Re h_n > 0$ if $n \geq 1$ and thus, unless all u_n for $n > 0$ vanish, the left side above has a positive real part. Then, (67) implies $u_n = 0$ for all $n \in \mathbb{Z}$. $\square$

Acknowledgements. OC was partially supported by the NSF-DMS Grant 1515755 and JLL by the AFOSR Grant FA9550-16-1-0037. We thank David Huse for very useful discussions. JLL thanks the Systems Biology division of the Institute for Advanced Study for hospitality during part of this work.

References

1. Bandrauk, A.D., Fillion-Gourdeau, F., Lorin, E.: Atoms and molecules in intense laser fields: gauge invariance of theory and models. *J. Phys. B* **46**, 153001 (2013)
2. Bauer, D.: Theory of Intense Laser-Matter Interaction. Lecture Notes. University of Heidelberg, Heidelberg (2006)
3. Bauer, J.H.: Keldysh theory re-examined. *J. Phys. B* **49**, 145601 (2016)
4. Cohen-Tannoudji, C., Dupont-Roc, J., Grynberg, G.: Photons and Atoms. Introduction to Quantum Electrodynamics. Wiley, Hoboken (1997)
5. Cohen-Tannoudji, C., Diu, B., Laloe, F.: Quantum Mechanics, Volume 2. Wiley, Hoboken (1991)
6. Costin, O., Lebowitz, J.L., Rokhlenko, A.: Exact results for the ionization of a model quantum system. *J. Phys. A Math. Gen.* **33**, 1–9 (2000)
7. Costin, O., Costin, R., Jauslin, H., Lebowitz, J.L.: Ionization by an oscillating field: Where are the photons? (**in preparation**)
8. Costin, O., Costin, R.D., Lebowitz, J.L., Rokhlenko, A.: Evolution of a model quantum system under time periodic forcing: conditions for complete ionization *Comm. Math. Phys.* **221**(1), 1–26 (2001)
9. Costin, O., Lebowitz, J.L., Stucchio, C.: Ionization in a 1-dimensional dipole model. *Rev. Math. Phys.* **20**(7), 835–872 (2008)
10. Cycon, H.L., Froese, R.G., Kirsch, W., Simon, B.: Schrödinger operators. Springer, Berlin (1987)
11. Elberfeld, W., Kleber, M.: Tunneling from an ultrathin quantum well in a strong electrostatic field: a comparison of different methods. *Z. Phys. B Condens. Matter* **73**, 23–32 (1988)
12. Gohberg, I., Goldberg, S., Kaashoek, M.A.: Basic Classes of Linear Operators. Springer, Basel (2003)
13. Immink, G.K.: Asymptotics of Analytic Difference Equations. Springer, Berlin (1984)
14. Karnakov, B.M. et al.: Current progress in developing the nonlinear ionization theory of atoms and ions. *Phys. Uspekhi* **58**(1), 3 (2015)
15. Karnakov, B.M.: Nonperturbative generalization of the Fermi golden rule. *JETP Lett.* **101**, 825 (2015)
16. Krantz, S.G.: Handbook of Complex Variables, p. 159. Birkhäuser, Boston (1999)
17. Keldysh, L.V.: Ionization in the field of a strong electromagnetic wave. *Sov. Phys. JETP* (1964) (Engl. transl.), *Zh. Eksp. Teor. Fiz.* **47** (1945)
18. Popruzhenko, S.V.: Keldysh theory of strong field ionization: history, applications, difficulties and perspectives. *J. Phys. B Atomic Mol. Opt. Phys.* **47**(20), 204001 (2014)
19. Guérin, S., Dupont, J.-M., Monti, F., Jauslin, H.R.: On the relation between cavity-dressed states, Floquet states, RWA and semiclassical models. *J. Phys. A* **30**, 7193 (1997)
20. Reed, M., Simon, B.: Methods of Modern Mathematical Physics I. Academic Press, New York (1980)

21. Zhang, P., Lau, Y.Y.: Ultrafast strong-field photoelectron emission from biased metal surfaces: exact solution to time-dependent Schrödinger equation. *Sci. Rep.* **6**, 19894 (2016)
22. Zinn-Justin, J., Jentschura, U.D.: Multi-instantons and exact results I: conjectures, WKB expansions, and instanton interactions. *Ann. Phys.* **313**(1), 197–267 (2004)

Communicated by W. Schlag