



# Approach to Hyperuniformity in the One-Dimensional Facilitated Exclusion Process

S. Goldstein<sup>1</sup>, J. L. Lebowitz<sup>1,2</sup>, E. R. Speer<sup>1</sup> 

<sup>1</sup> Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

<sup>2</sup> Department of Physics, Rutgers University, New Brunswick, NJ 08903, USA

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**Abstract:** For the one-dimensional Facilitated Exclusion Process with initial state a product measure of density  $\rho = 1/2 - \delta$ ,  $\delta \geq 0$ , there exists an infinite-time limiting state  $\nu_\rho$  in which all particles are isolated and hence cannot move. We study the variance  $V(L)$ , under  $\nu_\rho$ , of the number of particles in an interval of  $L$  sites. Under  $\nu_{1/2}$  either all odd or all even sites are occupied, so that  $V(L) = 0$  for  $L$  even and  $V(L) = 1/4$  for  $L$  odd: the state is *hyperuniform* (Torquato in Phys Rep 745:1–95, 2018), since  $V(L)$  grows more slowly than  $L$ . We prove that for densities approaching  $1/2$  from below there exist three regimes in  $L$ , in which the variance grows at different rates: for  $L \gg \delta^{-2}$ ,  $V(L) \simeq \rho(1 - \rho)L$ , just as in the initial state; for  $A(\delta) \ll L \ll \delta^{-2}$ , with  $A(\delta) = \delta^{-2/3}$  for  $L$  odd and  $A(\delta) = 1$  for  $L$  even,  $V(L) \simeq CL^{3/2}$  with  $C = 2\sqrt{2/\pi}/3$ ; and for  $L \ll \delta^{-2/3}$  with  $L$  odd,  $V(L) \simeq 1/4$ . The analysis is based on a careful study of a renewal process with a long tail. Our study is motivated by simulation results showing similar behavior in higher dimensions; we discuss this background briefly.

## 1. Introduction

In the Facilitated Exclusion Process (FEP) on  $\mathbb{Z}^d$ , also known as the Conserved Lattice Gas process, each site of the lattice can be occupied by at most one particle, so that a configuration  $\eta$  is an element of the configuration space  $X = \{0, 1\}^{\mathbb{Z}^d}$ . At Poisson-distributed times a particle chooses one of its nearest-neighbor sites at random and attempts to jump to it, succeeding only if the target site is unoccupied and the original site has at least one occupied (*facilitating*) neighbor. (Variations of this dynamics, with simultaneous updating or with some other rule for choosing the target site, have also been considered [13].) Note that when  $d = 1$  a chosen particle can jump in at most one direction. We will always assume that the system is started in a Bernoulli initial state  $\mu_0^{(\rho)}$  of density  $\rho < 1$ , that is, a product measure in which the  $\eta_i$  are independent and take value 1 with probability  $\rho$ . The evolved state at time  $t$  will then be denoted  $\mu_t^{(\rho)}$ ; it is clearly translation invariant (TI).

The evolution of this system, or of minor variations of it, has been investigated for  $d = 1$  [1–5, 7, 8, 10–13, 17, 22], primarily theoretically, and for  $d \geq 2$  [13, 14, 16, 19], primarily via simulation in a cubical box with periodic boundary conditions. These investigations strongly suggest the existence of a TI limiting state

$$v_\rho := \lim_{t \rightarrow \infty} \mu_t^{(\rho)}. \quad (1)$$

This existence was proved for  $d = 1$  and  $\rho \leq 1/2$  in [22] (and the limiting state was first described there). Moreover, there appears to be a *critical density*  $\rho_c$  such that if  $\rho \leq \rho_c$  then  $v_\rho$  is a *frozen* state in which all particles are isolated and hence unable to move, while if  $\rho > \rho_c$  then  $v_\rho$  is an *active* stationary state in which there is a nonzero density of particles with an occupied neighboring site. Necessarily  $\rho_c \leq 1/2$ , since for  $\rho > 1/2$  it is geometrically impossible for all the particles to be isolated, and indeed equality holds for  $d = 1$ . But for  $d \geq 2$ , simulations suggest values of  $\rho_c$  which are much smaller, for example,  $\rho_c \approx 0.3308$  [13] for  $d = 2$ .

Our main interest here will be in *fluctuations* in the measure  $v_\rho$ , that is, in the variance  $V_\rho(L) := \text{Var}_{v_\rho}(N(L))$  of the number  $N(L)$  of particles in a cubical box of side  $L$ . In general, if such a variance computed from a TI measure  $\mu$  grows as  $L^d$  when  $L \nearrow \infty$ , we say that  $\mu$  has *normal fluctuations*. Hexner and Levine [14] observe that, in 2d and 3d,  $v_{\rho_c}$  is not of this character but rather is *hyperuniform* [21]:  $V_{\rho_c}(L)$  grows more slowly than  $L^d$ , specifically,  $V_{\rho_c}(L) \sim L^{\lambda_1}$ , with  $\lambda_1 \approx 1.57$  in 2d and  $\lambda_1 \approx 2.76$  in 3d [14]. (We will relate two quantities by “ $\sim$ ” or, respectively, by “ $\asymp$ ”, to express the fact that asymptotically their ratio is bounded away from both 0 and  $\infty$  or, respectively, is equal to 1. The symbol “ $\asymp$ ”, used extensively in this paper to express and prove our results, will be more precisely defined in Definitions 1.)

Hexner and Levine also discuss the behavior of  $V_\rho(L)$  as  $\rho \nearrow \rho_c$ . Further consideration of this behavior has led us [13] to the following conjecture, which we state for  $d \geq 2$ ; the  $d = 1$  version is Theorem 1 below. We introduce the notation  $\delta = \rho_c - \rho > 0$ .

**Fluctuation Conjecture.** For the FEP with  $d \geq 2$  a critical density  $\rho_c$  as described above exists, and  $v_{\rho_c}$  is hyperuniform. For  $\rho$  less than but close to  $\rho_c$ , three regimes in  $L$  may be identified. For *small*  $L$  (but still with  $L \gg 1$ ) the variances grow approximately as in the hyperuniform state at  $\rho_c$ :  $V_\rho(L) \simeq C_1 L^{\lambda_1}$ . At some (approximately defined) scale  $L_1(\delta)$  the variances enter the regime of *intermediate*  $L$ , in which they grow as  $V_\rho(L) \simeq C_2(\delta) L^{\lambda_2}$  with  $\lambda_2 > d > \lambda_1$  and  $C_2(\delta) > 0$ . Then above an (approximate) scale  $L_2 = L_2(\delta)$  the growth is as  $V_\rho(L) \simeq \rho(1 - \rho)L^d$ , that is, exactly as in the initial Bernoulli measure  $\mu_0^{(\rho)}$ . Finally, as  $\rho \nearrow \rho_c$ ,  $L_1(\delta)$  and  $L_2(\delta)$  increase as  $L_i \sim \delta^{-\gamma_i}$  for some exponents  $\gamma_1, \gamma_2$  satisfying  $\gamma_2 > \gamma_1 > 0$ .

In the remainder of the paper we restrict our consideration to the  $d = 1$  model. In Sect. 2 we state in Theorem 1 our main result, the one-dimensional version of the conjecture, and in Sect. 3 we describe the 1d limiting measure  $v_\rho$ . The proof of Theorem 1 is given in Sect. 4.

## 2. Statement of the Result

The key to the rigorous establishment of a version of the Fluctuation Conjecture in dimension  $d = 1$  is that there the existence and exact value of the critical density are known— $\rho_c = 1/2$ —and that we also have a complete description of the limiting

measure  $\nu_\rho$  for  $\rho \leq 1/2$  [1, 10, 11, 22]. (This measure was first identified and discussed in [22], a reference which has just come to our attention. We regret that we did not properly credit this work in earlier papers.) We will discuss  $\nu_\rho$  for  $\rho < 1/2$  in Sect. 3; for the moment let us note that the measure at the critical density is particularly simple:

$$\nu_{1/2} = \frac{1}{2}(\delta_{\eta^*} + \delta_{\eta^\dagger}), \quad (2)$$

where  $\eta^*$  and  $\eta^\dagger$  are the two configurations in  $X = \{0, 1\}^{\mathbb{Z}}$  in which holes and particles strictly alternate.

The Fluctuation Conjecture concerns the asymptotic behavior of the quantity  $V_\rho(L)$  with  $L$  “small,” “intermediate,” or “large,” yet in each case also sufficiently large. To give a precise result in 1d we introduce some notation for the description of this behavior.

**Definition 1.** Assume that  $L$  is a positive integer and  $\delta$  a positive real number (in applications we will have  $\delta = \rho_c - \rho$ ), that  $A(\delta, L)$  and  $B(\delta, L)$  are real valued functions whose asymptotic behavior in  $L$  we wish to compare, and that  $L_1(\delta)$  and  $L_2(\delta)$  are positive functions (these play the role of setting the scales of the various regions). Then we write respectively

$$A(\delta, L) \simeq B(\delta, L) \quad \text{for} \quad \begin{cases} L \ll L_1(\delta), \\ L_1(\delta) \ll L \ll L_2(\delta), \\ L \gg L_2(\delta), \end{cases} \quad (3)$$

if for any  $\epsilon > 0$  there exists a  $\delta_0 > 0$ , together with a (small) number  $s > 0$  and/or a (large) number  $l > 0$ , such that for  $\delta < \delta_0$  we have, respectively,

$$1 - \epsilon < \frac{A(\delta, L)}{B(\delta, L)} < 1 + \epsilon \quad \begin{cases} \text{for } L < sL_1(\delta), \\ \text{for } lL_1(\delta) < L < sL_2(\delta), \\ \text{for } L > lL_2(\delta). \end{cases} \quad (4)$$

If  $A$  and  $B$  depend also on some additional parameter(s)  $\alpha$  we say that (3) holds *uniformly* for  $\alpha$  in some (possibly  $L$ - or  $\delta$ -dependent) set if  $s$  and/or  $l$ , as well as  $\delta_0$ , may be chosen so that (4) holds for all such  $\alpha$ .

With this notation established we may state our main result; we assume that  $0 < \rho < 1/2$ , that  $\delta = 1/2 - \rho$ , and that  $\nu_\rho$  is the measure (1) of the 1d FEP. In contrast to the  $d \geq 2$  behavior described in the Fluctuation Conjecture, for  $d = 1$  the variances  $V_\rho(L)$  behave differently for  $L$  odd and  $L$  even.

**Theorem 1.** Let  $N^{(\delta)}(L)$  be the number of particles on the sites  $1, 2, \dots, L$ , with distribution determined by  $\nu_\rho$ . Then:

(a) For  $L$  odd,

$$V_\rho(L) = \text{Var}(N^{(\delta)}(L)) \simeq \begin{cases} \frac{1}{4}, & \text{for } L \ll \delta^{-2/3}, \\ \frac{2}{3}\sqrt{\frac{2}{\pi}}\delta L^{3/2}, & \text{for } \delta^{-2/3} \ll L \ll \delta^{-2}, \\ \frac{1}{4}L, & \text{for } L \gg \delta^{-2}. \end{cases} \quad (5)$$

(b) For  $L$  even,

$$V_\rho(L) = \text{Var}(N^{(\delta)}(L)) \simeq \begin{cases} \frac{2}{3}\sqrt{\frac{2}{\pi}}\delta L^{3/2}, & \text{for } 1 \ll L \ll \delta^{-2}, \\ \frac{1}{4}L, & \text{for } L \gg \delta^{-2}. \end{cases} \quad (6)$$

We actually have a stronger result for the asymptotics in the large- $L$  region. The “right” estimate for  $\text{Var}(N^{(\delta)}(L))$  there is  $\rho(1 - \rho)L$ , as stated in the Fluctuation Conjecture and as we discuss further in Remark 1 below, and this is true for all, not just small,  $\delta$ :

**Corollary 2.** (to the proof of Theorem 1) *For any  $\epsilon > 0$  there exists an  $l > 0$  such that for any  $\delta \in (0, 1/2)$ ,*

$$1 - \epsilon < \frac{\text{Var}(N^{(\delta)}(L))}{\rho(1 - \rho)L} < 1 + \epsilon \quad \text{for } L > l\delta^{-2}. \quad (7)$$

In fact, it follows from (82) below that (7) holds with  $l = (2\epsilon)^{-1}$ .

A comparison of Theorem 1(a) with the Fluctuation Conjecture shows that for  $L$  odd the behavior of  $V_\rho(L)$  in one dimension corresponds directly to the conjectured behavior in higher dimension (but without the condition  $L \gg 1$ ). In particular it follows from (2) that

$$V_{1/2}(L) = \begin{cases} \frac{1}{4}, & \text{if } L \text{ is odd,} \\ 0, & \text{if } L \text{ is even,} \end{cases} \quad (8)$$

which implies that  $v_{1/2}$  is hyperuniform and also explains the  $V_\rho(L) \simeq 1/4$  behavior in Theorem 1 for small odd  $L$ . The variables introduced in the conjecture become  $\lambda_1 = 0$ ,  $C_1 = 1/4$ ,  $\lambda_2 = 3/2$ ,  $C_2(\delta) = \sqrt{8/\pi} \delta/3$ ,  $\gamma_1 = 2/3$ , and  $\gamma_2 = 2$ . On the other hand, for even  $L$  the “small” growth region is absent in one dimension: for small and moderate values of  $L$  the variances grow as  $C_2(\delta)L^{3/2}$ . This odd/even distinction may be regarded as a legacy of (8) when  $\delta$  is perturbed away from 0.

*Remark 1.* Corollary 2 certainly implies that  $\lim_{L \rightarrow \infty} V_\rho(L)/L = \rho(1 - \rho)$  for all  $\rho < 1/2$ , and this part of the result, although not the scale  $\delta^{-2}$  at which the limit is achieved, may be obtained by an elementary argument [11]. For with probability 1 each particle will move only a finite distance during the evolution, so that for  $L$  sufficiently large  $N(L)$  will, to high relative accuracy, be the same at the end of the evolution as it was at the beginning, and  $\text{Var}(N(L))$  will be the same as for the original Bernoulli measure.

*Remark 2.* There are several one-dimensional models with exclusion and facilitation, closely related to the FEP, for which also  $\rho_c = 1/2$  and for which  $\lim_{t \rightarrow \infty} \mu_t^{(\rho)}$  is for  $\rho \leq 1/2$  the measure  $\nu_\rho$  that we are considering here, and hence for which the fluctuations  $V_\rho(L)$  satisfy Theorem 1. In particular, this is true of the totally asymmetric, discrete-time (parallel) updating in which all particles attempt to jump at the same time, and only to the right [11]. It is also true of an asymmetric version of the continuous-time model of Sect. 1 in which particles attempt to jump to the left or right at different rates [1].

### 3. The Limiting Measure $\nu_\rho$ for $\rho < 1/2$

A key ingredient for understanding the behavior described in Theorem 1, especially the behavior in the intermediate regime, is the renewal structure of the stationary state  $\nu_\rho$  defined in (1),  $\rho < 1/2$ ; we now describe this structure [11,22]. Since this state is frozen, the occurrence of adjacent 1's has probability zero, so that the state is supported on configurations of the form

$$\dots 1 0 1 0 1 0 1 0 \hat{0} 1 0 1 0 \dots 1 0 \hat{0} \hat{0} \hat{0} 1 0 1 0 1 0 \dots 1 0 \hat{0} \dots \quad (9)$$

$$= \dots 0 (1 0)^{X_{-1}} 0 (1 0)^{X_0} 0 (1 0)^{X_1} 0 (1 0)^{X_2} 0 \dots \quad (10)$$

We focus particular attention on the 00's in (9); the *second* 0 of each such pair, marked as  $\hat{0}$  in (9) and corresponding to a 0 outside the parentheses in (10), will be called a *renewal event*. (Note that adjacent renewal events correspond to a zero value for the corresponding  $X_i$ .) If we let  $\hat{\nu}_\rho$  be the measure  $\nu_\rho$  conditioned on the occurrence of a renewal event at the origin, then under  $\hat{\nu}_\rho$  the  $X_i$ 's in (10) are independent random variables that are identically distributed, with the distribution described in (11).

The measure  $\nu_\rho$  may now be described as the unique ergodic TI measure such that conditioning on the occurrence of a renewal event at the origin yields the measure  $\hat{\nu}_\rho$  as just described. Explicitly, if  $\hat{\nu}_\rho^{(k)}$ ,  $k \geq 1$ , denotes the restriction of the measure  $\hat{\nu}_\rho$  to configurations in which the first renewal event to the right of the origin lies at site  $k$ , and  $\hat{\nu}_\rho^{(k,i)}$ ,  $0 \leq i < k$ , its translation by  $i$  sites to the left, then  $\nu_\rho = Z^{-1} \sum_{k \geq 1} \sum_{i=1}^{k-1} \hat{\nu}_\rho^{(k,i)}$ , with  $Z$  a normalizing factor.

**Remark 3.** The density of the renewal events, i.e., the probability under  $\nu_\rho$  of finding such an event at, say, site 1 (or equivalently of finding adjacent 0's at sites 0 and 1), is  $1 - 2\rho = 2\delta$ , since under  $\nu_\rho$  the probability of adjacent 1's is zero.

It is shown in [11,22] that the distribution of the  $X_i$ 's under  $\hat{\nu}_\rho$  is that of a random variable  $\hat{X}^{(\delta)}$  for which

$$\begin{aligned} P(\hat{X}^{(\delta)} = n) &= C_n \rho^n (1 - \rho)^{n+1} \\ &= \frac{1 + 2\delta}{2 \cdot 4^n} C_n (1 - 4\delta^2)^n, \quad n = 0, 1, 2, \dots, \end{aligned} \quad (11)$$

with  $C_n$  the  $n^{\text{th}}$  Catalan number [20]:

$$C_n := \frac{1}{n+1} \binom{2n}{n} = \frac{4^n}{n^{3/2} \sqrt{\pi}} \left( 1 + O\left(\frac{1}{n}\right) \right). \quad (12)$$

Here we have used Stirling's formula with error bounds. Thus for  $n \gg 1$ ,

$$P(\hat{X}^{(\delta)} = n) \simeq \frac{1 + 2\delta}{2n^{3/2} \sqrt{\pi}} (1 - 4\delta^2)^n \simeq \frac{1}{2n^{3/2} \sqrt{\pi}} e^{-4\delta^2 n}, \quad (13)$$

where the first approximation holds for all  $\delta$ ,  $0 \leq \delta < 1/2$ , and the second for  $\delta^2 n \leq 1$ . (The Catalan number  $C_n$  arises here as the number of random walks of length  $2n$ , with steps  $\pm 1$ , which begin and end at the origin and take only nonnegative values.)

It follows immediately from (2) and Remark 3 that

$$\nu_{1/2} = \lim_{\delta \rightarrow 0} \nu_\rho \quad (14)$$

in the sense of weak convergence, i.e., that  $\nu_{1/2}(A) = \lim_{\delta \rightarrow 0} \nu_\rho(A)$  for every  $A \subset X$  defined in terms of the configuration on a finite set of sites. In particular, the limiting measure contains no renewal events. On the other hand, the limit  $\hat{\nu}_{1/2} = \lim_{\delta \rightarrow 0} \hat{\nu}_\rho$  is not so trivial:  $\hat{\nu}_{1/2}$  is the probability distribution on configurations of the form (10) for which there is a renewal event at the origin and the i.i.d. random variables  $X_i$  have the distribution of  $\hat{X}^{(0)}$ :

$$P(\hat{X}^{(0)} = n) = \lim_{\delta \rightarrow 0} P(\hat{X}^{(\delta)} = n) = \frac{C_n}{2 \cdot 4^n} \simeq \frac{1}{2\sqrt{\pi} n^{3/2}}. \quad (15)$$

In particular, there exist  $c_1, c_2 > 0$  such that

$$c_1 n^{-3/2} < P(\widehat{X}^{(0)} = n) < c_2 n^{-3/2}, \quad n \geq 1. \quad (16)$$

Note that  $\widehat{X}^{(0)} = \lim_{\delta \rightarrow 0} \widehat{X}^{(\delta)}$  (limit in distribution); note also that although  $\hat{v}_\rho$  for  $\rho < 1/2$  was obtained from  $v_\rho$  by conditioning on a renewal event at the origin,  $\hat{v}_{1/2}$  cannot be so obtained from  $v_{1/2}$ .

$\widehat{X}^{(0)}$  has a  $3/2$  power-law tail, and this  $3/2$  is, as we shall show, the origin of the  $3/2$  in the  $L^{3/2}$  behavior of the variance in the intermediate regime. (If  $3/2$  were replaced by  $\gamma$ , with  $1 < \gamma \leq 2$ , we would have had  $L^\gamma$  behavior there [13]). Further, the fact that the exponential decay in (13) becomes significant when  $n$  is of order  $\delta^{-2}$  is the origin of the fact that the transition to the large  $L$  regime occurs for  $L$  of order  $\delta^{-2}$ .

**Notation:** Here, for the reader's convenience, we summarize our notation, reviewing some that was introduced earlier and also defining some new notation that will be used in the sequel. We write  $\rho = 1/2 - \delta$ , with  $0 \leq \delta < 1/2$ , define  $J_L = \{1, 2, \dots, L\}$ , and call the second of a pair of consecutive empty sites a renewal event.  $v_\rho$  denotes the infinite-time limit state (1) for the one-dimensional FEP at density  $\rho$ , and  $\hat{v}_\rho$  the state defined for  $\delta > 0$  by conditioning  $v_\rho$  on the occurrence of a renewal event at the origin, and for  $\delta = 0$  as  $\lim_{\delta \searrow 0} \hat{v}_\rho$ . We will use the following random variables:

- $N^{(\delta)}(L)$ , the number of particles in  $J_L$  under  $v_\rho$ ;
- $N_{\text{ren}}^{(\delta)}(L)$  and  $\widehat{N}_{\text{ren}}^{(\delta)}(L)$ , the number of renewal events in  $J_L$  under  $v_\rho$  and  $\hat{v}_\rho$ , respectively (and, by convention,  $\widehat{N}_{\text{ren}}^{(\delta)}(0) = 0$ );
- $\widehat{X}^{(\delta)}$ , a random variable with the distribution under  $\hat{v}_\rho$  of the  $X_i$  in (10);
- $\widehat{Y}^{(\delta)} = 2\widehat{X}^{(\delta)} + 1$ , the distance between renewal events under  $\hat{v}_\rho$ ;
- $F^{(\delta)}$ , the location of the first renewal event to the right of the origin under  $v_\rho$ .

We make a further remark and a general notational convention. First, the translation invariance of  $v_\rho$  implies that the number of renewal events in any interval of  $L$  sites, conditioned on the occurrence of a renewal event at the immediately preceding site, has the same distribution as  $\widehat{N}_{\text{ren}}^{(\delta)}(L)$ . Second, we will let  $P$  denote probabilities for random variables such as those above, whether defined using  $v_\rho$  or  $\hat{v}_\rho$ ; thus, for example,  $P(N_{\text{ren}}^{(\delta)}(L) = n) = v_\rho(N_{\text{ren}}^{(\delta)}(L) = n)$  while  $P(\widehat{N}_{\text{ren}}^{(\delta)}(L) = n) = \hat{v}_\rho(\widehat{N}_{\text{ren}}^{(\delta)}(L) = n)$ .

Note that from Remark 3,

$$E(N_{\text{ren}}^{(\delta)}(L)) = 2\delta L. \quad (17)$$

Note also that while, from (14),  $N_{\text{ren}}^{(0)}(L) = \lim_{\delta \rightarrow 0} N_{\text{ren}}^{(\delta)}(L)$  (limit in distribution) is the zero random variable, since in  $v_{1/2}$  there are no renewal events,  $\widehat{N}_{\text{ren}}^{(0)}(L) = \lim_{\delta \rightarrow 0} \widehat{N}_{\text{ren}}^{(\delta)}(L)$  is non-trivial. We stress that the key to the  $\delta \searrow 0$  asymptotics described in Theorem 1 lies not in  $v_{1/2}$  but in  $\hat{v}_{1/2}$ .

#### 4. Proof of Theorem 1

The proof of Theorem 1 is broken into nine steps, as follows:

**Step 1:** Express  $N^{(\delta)}(L)$  in terms of  $N_{\text{ren}}^{(\delta)}(L)$ .

**Step 2:** Express the distribution and second moment of  $N_{\text{ren}}^{(\delta)}(L)$  in terms of those of  $\widehat{N}_{\text{ren}}^{(\delta)}(L)$ .

- Step 3:** Approximate the expressions found in Step 2 by replacing  $\widehat{N}_{\text{ren}}^{(\delta)}(L)$  by  $\widehat{N}_{\text{ren}}^{(0)}(L)$ .
- Step 4:** The expressions obtained in Step 3 involve the quantity  $p_\delta$  defined in (26). In Step 4 we express this quantity in terms of  $\widehat{Y}^{(\delta)}$ , then replace the occurrences of  $\widehat{Y}^{(\delta)}$  by  $\widehat{Y}^{(0)}$ .
- Step 5:** Obtain the large- $L$  asymptotics of the distribution of  $\widehat{N}_{\text{ren}}^{(0)}(L)$  and of its second moment, and insert these into the expressions found in Step 4.
- Step 6:** Use the asymptotics of the distribution of  $\widehat{Y}^{(0)}$  to further approximate the expressions found in Step 5.
- Step 7:** Obtain from the expressions found in Step 6 the asymptotics of  $\text{Var}(N_{\text{ren}}^{(\delta)}(L))$ .
- Step 8:** Use Step 1 to obtain the results of Theorem 1 for  $L \ll \delta^{-2}$  from the expression found in Step 7 for  $\text{Var}(N_{\text{ren}}^{(\delta)}(L))$ .
- Step 9:** Use some facts about the truncated two-point correlation function for  $v_\rho$  to obtain the results of Theorem 1 for  $L \gg \delta^{-2}$ .

We now consider these steps in order.

**Step 1:** In this step we again use the notation introduced in (9), so that the values which may be taken by a configuration  $\eta_i$  are  $\hat{0}$ , 0, and 1, where  $\hat{0}$  denotes a renewal event, 0 an empty site preceded by a 1, and 1 an occupied site. Now we observe that  $L - N_{\text{ren}}^{(\delta)}(L)$  is odd if and only if the pair  $(\eta_1, \eta_L)$  has value (0, 0), (0,  $\hat{0}$ ), ( $\hat{0}$ , 1), or (1, 1); moreover,

$$N^{(\delta)}(L) = \frac{1}{2} [L - (N_{\text{ren}}^{(\delta)}(L) + \sigma^{(\delta)}(L))], \quad (18)$$

where

$$\sigma^{(\delta)}(L) = \begin{cases} 0, & \text{if } L - N_{\text{ren}}^{(\delta)}(L) \text{ is even,} \\ 1, & \text{if } L - N_{\text{ren}}^{(\delta)}(L) \text{ is odd and } (\eta_1, \eta_L) \text{ is } (0, 0) \text{ or } (0, \hat{0}), \\ -1, & \text{if } L - N_{\text{ren}}^{(\delta)}(L) \text{ is odd and } (\eta_1, \eta_L) \text{ is } (\hat{0}, 1) \text{ or } (1, 1). \end{cases} \quad (19)$$

One checks this by induction on  $N_{\text{ren}}^{(\delta)}(L)$ ; the case  $N_{\text{ren}}^{(\delta)}(L) = 0$  is easy. For the induction step one passes from a configuration  $\eta$  to another  $\eta'$  by removing a  $\hat{0}$  from some site  $i$  with  $1 \leq i \leq L$  and setting  $\eta'_j = \eta_j$  if  $j < i$  and  $\eta'_j = \eta_{j+1}$  if  $j \geq i$ ; one then applies the induction assumption to  $\eta'$  on  $J_{L-1}$ , noting that then  $L$  and  $N_{\text{ren}}^{(\delta)}(L)$  both decrease by 1, and observing that  $(\eta'_1, \eta'_{L-1}) \neq (\eta_1, \eta_L)$  only if  $i = 1$  and  $\eta'_1 = 1$  or  $i = L$  and  $\eta'_{L-1} = 0$ , and that in each of these cases  $\sigma$  is unchanged and (18) remains valid.

From (18) we have that

$$\text{Var}(N^{(\delta)}(L)) = \frac{1}{4} \text{Var}(N_{\text{ren}}^{(\delta)}(L) + \sigma^{(\delta)}(L)). \quad (20)$$

To simplify this expression further we note that, writing  $E$  for expectation, we have

$$E(\sigma^{(\delta)}(L) \mid N_{\text{ren}}^{(\delta)}(L) = n) = 0, \quad \text{for any } n \geq 0, \quad (21)$$

as we will argue shortly. But then

$$E(\sigma^{(\delta)}(L)) = 0 \quad (22)$$

and

$$E(N_{\text{ren}}^{(\delta)}(L)\sigma^{(\delta)}(L)) = \text{Cov}(N_{\text{ren}}^{(\delta)}(L), \sigma^{(\delta)}(L)) = 0, \quad (23)$$

so that

$$\text{Var}(N_{\text{ren}}^{(\delta)}(L) + \sigma^{(\delta)}(L)) = \text{Var}(N_{\text{ren}}^{(\delta)}(L)) + \text{Var}(\sigma^{(\delta)}(L)). \quad (24)$$

Thus from (20) and (22),

$$\begin{aligned} \text{Var}(N^{(\delta)}(L)) &= \frac{1}{4} (\text{Var}(N_{\text{ren}}^{(\delta)}(L)) + \text{Var}(\sigma^{(\delta)}(L))) \\ &= \frac{1}{4} (\text{Var}(N_{\text{ren}}^{(\delta)}(L)) + P(\sigma^{(\delta)}(L) \neq 0)). \end{aligned} \quad (25)$$

To verify (21) we first note that, from (1) and the reflection invariance of the Bernoulli measure and of the dynamics,  $v_\rho$  is invariant under reflection about any (integer or half-integer) point. (21) then follows from the observation that reflection about  $L/2$ , the midpoint of the interval  $\{0, 1, \dots, L\}$  (not of  $J_L$ ), leaves  $N_{\text{ren}}^{(\delta)}(L)$  unchanged and, when  $L - N_{\text{ren}}^{(\delta)}(L)$  is odd, changes the sign of  $\sigma^{(\delta)}(L)$ , as one sees by checking separately for the four possible values of  $(\eta_1, \eta_L)$  which can then occur.

**Step 2:** With  $F^{(\delta)}$  as defined towards the end of Sect. 3, let

$$p_\delta(l) = P(F^{(\delta)} = l). \quad (26)$$

The critical observation for Step 2 is that the number of renewal events in  $J_L$ , if there are any, is one more than the number of such events to the right of  $F^{(\delta)}$ . Thus for  $n \geq 1$ ,

$$\begin{aligned} P(N_{\text{ren}}^{(\delta)}(L) = n) &= \sum_{l=1}^L p_\delta(l) P(N_{\text{ren}}^{(\delta)}(L) = n \mid F^{(\delta)} = l) \\ &= \sum_{l=1}^L p_\delta(l) P(\widehat{N}_{\text{ren}}^{(\delta)}(L-l) = n-1), \end{aligned} \quad (27)$$

and so

$$E(N_{\text{ren}}^{(\delta)}(L)^2) = \sum_{l=1}^L p_\delta(l) E((\widehat{N}_{\text{ren}}^{(\delta)}(L-l) + 1)^2). \quad (28)$$

**Step 3:** As indicated earlier, the next step is to control the approximation arising from the replacement of  $\widehat{N}_{\text{ren}}^{(\delta)}$  by  $\widehat{N}_{\text{ren}}^{(0)}$  in (27) and (28). Specifically, we will show that for  $n \geq 1$ ,

$$P(N_{\text{ren}}^{(\delta)}(L) = n) \simeq \sum_{l=1}^L p_\delta(l) P(\widehat{N}_{\text{ren}}^{(0)}(L-l) = n-1) \quad \text{for } L \ll \delta^{-2}, \quad (29)$$

uniformly in  $n \leq k\sqrt{L}$  for  $k$  any fixed positive integer (see Definitions 1), and also that

$$E(N_{\text{ren}}^{(\delta)}(L)^2) \simeq \sum_{l=1}^L p_\delta(l) E((\widehat{N}_{\text{ren}}^{(0)}(L-l) + 1)^2) \quad \text{for } L \ll \delta^{-2}. \quad (30)$$

Note that the right hand sides of (29) and (30) both mix quantities defined for  $\delta > 0$  with those defined for  $\delta = 0$  ( $p_\delta$  and  $\widehat{N}_{\text{ren}}^{(0)}$  respectively). These equations are more delicate than they may appear because they demand that we control the errors in these



approximations by requiring merely that, for small  $\delta$ , the quantity  $L\delta^2$  be sufficiently small regardless of the size of  $L$  itself.

Let us fix  $\tilde{L} < L$  ( $\tilde{L}$  plays the role of  $L - l$  in (29) and (30)) and let  $\eta^{(\tilde{L})}$  and  $\hat{\eta}^{(\tilde{L},\delta)}$  denote respectively a fixed and a random configuration on  $J_{\tilde{L}}$ , such that (i)  $\eta^{(\tilde{L})}$  contains  $m = n - 1$  renewal events, with the convention that if  $\eta_1^{(\tilde{L})} = 0$  then this is a renewal event, and (ii)  $\hat{\eta}^{(\tilde{L},\delta)}$  is the restriction to  $J_{\tilde{L}}$  of a configuration distributed according to  $\hat{\nu}_\rho$ . Then for  $\delta \geq 0$ ,

$$P(\widehat{N}_{\text{ren}}^{(\delta)}(\tilde{L}) = n - 1) = \sum_{\eta^{(\tilde{L})}} P(\hat{\eta}^{(\tilde{L},\delta)} = \eta^{(\tilde{L})}). \quad (31)$$

(29) will now follow from (31) once we show that for all  $\eta^{(\tilde{L})}$ , uniformly in  $m \leq k\sqrt{\tilde{L}}$ ,

$$P(\hat{\eta}^{(\tilde{L},\delta)} = \eta^{(\tilde{L})}) \simeq P(\hat{\eta}^{(\tilde{L},0)} = \eta^{(\tilde{L})}) \quad \text{for } L \ll \delta^{-2}. \quad (32)$$

To verify (32) we let  $2x_i + 1$ ,  $i = 1, \dots, m$ , be the distances between the renewal events in  $\eta^{(\tilde{L})}$ , with  $2x_1 + 1$  the distance from the origin to the first renewal event. Further, we define

$$q_\delta(L) = P(\widehat{Y}^{(\delta)} > L) = P\left(\widehat{X}^{(\delta)} > \left\lfloor \frac{L-1}{2} \right\rfloor\right) \quad (33)$$

with  $\widehat{Y}^{(\delta)} = 2\widehat{X}^{(\delta)} + 1$  as defined at the end of Sect. 3, and

$$r_\delta(L) = \frac{q_\delta(L)}{q_0(L)}. \quad (34)$$

Now from (11) and (15),

$$P(\widehat{X}^{(\delta)} = l) = (1 + 2\delta)(1 - 4\delta^2)^l P(\widehat{X}^{(0)} = l). \quad (35)$$

Then, with  $L'$  the distance from the last renewal event in  $\eta^{(\tilde{L})}$  to the right boundary  $\tilde{L}$  of  $J_{\tilde{L}}$ , we have

$$\begin{aligned} P(\hat{\eta}^{(\tilde{L},\delta)} = \eta^{(\tilde{L})}) &= \prod_{i=1}^m P(\widehat{X}^{(\delta)} = x_i) q_\delta(L') \\ &= (1 + 2\delta)^m (1 - 4\delta^2)^{\sum_{i=1}^m x_i} r_\delta(L') P(\hat{\eta}^{(\tilde{L},0)} = \eta^{(\tilde{L})}). \end{aligned} \quad (36)$$

We complete the argument for (32) by showing that each of the first three factors on the right hand side of (36) is asymptotic to 1 for  $L \ll \delta^{-2}$ , uniformly in  $n \leq k\sqrt{\tilde{L}}$  for any fixed positive  $k$ . First, under this condition,

$$1 \leq (1 + 2\delta)^m \leq (1 + 2\delta)^{k\sqrt{\tilde{L}}} \leq e^{2k\delta\sqrt{\tilde{L}}}. \quad (37)$$

Next, from  $1 - x \geq e^{-2x}$  for  $0 \leq x \leq 1/2$  we have for any  $M \geq 0$ ,

$$1 \geq (1 - 4\delta^2)^M \geq e^{-8M\delta^2} \geq 1 - 8M\delta^2 \quad \text{for } 0 \leq \delta \leq \frac{1}{2\sqrt{2}}, \quad (38)$$

and with  $M = \sum_{i=1}^m x_i \leq L/2$  this gives the desired asymptotics of the second factor. We will finally show that

$$r_\delta(L) \simeq 1 \quad \text{for } L \ll \delta^{-2}; \quad (39)$$

this, with  $L$  replaced by  $L'$ , gives the needed control of the right hand side of (36).

We first observe that (16) implies that there exists a  $c > 0$ , independent of  $L$ , such that if for  $\epsilon > 0$  we set  $K = c\epsilon^{-2}L$  then the tail of the series for  $q_0(L)$  beyond  $K$  is relatively small, i.e.,  $q_0(K) < \epsilon q_0(L)$ , which implies

$$q_0(L) < \frac{1}{1-\epsilon} \sum_{L < l \leq K} P(\widehat{Y}^{(0)} = l). \quad (40)$$

Further, for  $L\delta^2 < \epsilon^3/(8c)$  it follows from (38) that

$$\sum_{L < l \leq K} P(\widehat{Y}^{(\delta)} = l) > (1-\epsilon) \sum_{L < l \leq K} P(\widehat{Y}^{(0)} = l). \quad (41)$$

Then from (11), (40), and (41),

$$1 + 2\delta \geq \frac{q_\delta(L)}{q_0(L)} > (1-\epsilon) \frac{\sum_{L < l \leq K} P(\widehat{Y}^{(\delta)} = l)}{\sum_{L < l \leq K} P(\widehat{Y}^{(0)} = l)} > (1-\epsilon)^2. \quad (42)$$

This completes the verification of (39). (32) then follows from (36)–(39).

We remark that in fact the first inequality in (42) can be strengthened:

$$\frac{q_\delta(L)}{q_0(L)} \leq 1 \quad \text{for all } L \geq 1. \quad (43)$$

For from (11) the difference  $q_0(L) - q_\delta(L)$  is increasing for  $L \leq L^*$  and decreasing for  $L > L^*$ , where  $L^* = -\log(1+2\delta)/\log(1-4\delta^2)$ , and vanishes at  $L = 0$  and as  $L \rightarrow \infty$ .

We now turn to (30). For any positive  $k$  we write  $\widehat{N}_{\text{ren}}^{(\delta)}(L) = \widehat{N}_{\leq k\sqrt{L}}^{(\delta)}(L) + \widehat{N}_{> k\sqrt{L}}^{(\delta)}(L)$ , where

$$\widehat{N}_{\leq x}^{(\delta)}(L) := \widehat{N}_{\text{ren}}^{(\delta)}(L) I_{\{\widehat{N}_{\text{ren}}^{(\delta)}(L) \leq x\}} \quad (44)$$

and

$$\widehat{N}_{> x}^{(\delta)}(L) := \widehat{N}_{\text{ren}}^{(\delta)}(L) I_{\{\widehat{N}_{\text{ren}}^{(\delta)}(L) > x\}}, \quad (45)$$

with  $I_{\{\cdot\}}$  denoting the indicator function of the set  $\{\cdot\}$ . Then from (32),

$$E(\widehat{N}_{\leq k\sqrt{L}}^{(\delta)}(L)^2) \simeq E(\widehat{N}_{\leq k\sqrt{L}}^{(0)}(L)^2) \quad \text{for } L \ll \delta^{-2}. \quad (46)$$

(30) will follow easily once we strengthen (46) to

$$E(\widehat{N}_{\text{ren}}^{(\delta)}(L)^2) \simeq E(\widehat{N}_{\text{ren}}^{(0)}(L)^2) \quad \text{for } L \ll \delta^{-2}. \quad (47)$$

There are two crucial facts for obtaining (47) from (46), via the approximations expressed in (54) below. The first, to be proved shortly, is that for any  $\epsilon > 0$  there exists a  $k > 0$  such that

$$E(\widehat{N}_{> k\sqrt{L}}^{(\delta)}(L)^2) \leq \epsilon L \quad \text{for } L \ll \delta^{-2} \quad (48)$$

(where  $L \ll \delta^{-2}$  holds for all  $L$  if  $\delta = 0$ ). The second, to be proved in Step 5, is that  $\lim_{L \rightarrow \infty} E(\widehat{N}_{\text{ren}}^{(0)}(L)^2)/L = 1$ , so that for some constant  $C > 0$ ,

$$E(\widehat{N}_{\text{ren}}^{(0)}(L)^2) \geq CL \quad \text{for all } L \geq 1. \quad (49)$$

(47) follows from (46), (48) (for both  $\delta = 0$  and  $\delta > 0$ ) and (49).

To see this, fix  $\epsilon > 0$  and take  $k$  so that (48) holds. Then from (48) for  $\delta = 0$  and (49) we have (for all  $L$ )

$$\left| \frac{E(\widehat{N}_{\leq k\sqrt{L}}^{(0)}(L)^2)}{E(\widehat{N}_{\text{ren}}^{(0)}(L)^2)} - 1 \right| = \frac{E(\widehat{N}_{> k\sqrt{L}}^{(0)}(L)^2)}{E(\widehat{N}_{\text{ren}}^{(0)}(L)^2)} < \frac{\epsilon}{C}. \quad (50)$$

Now by (46) we may take  $L\delta^2$  so small that

$$\left| \frac{E(\widehat{N}_{\leq k\sqrt{L}}^{(\delta)}(L)^2)}{E(\widehat{N}_{\leq k\sqrt{L}}^{(0)}(L)^2)} - 1 \right| < \epsilon. \quad (51)$$

From this,

$$\begin{aligned} E(\widehat{N}_{\leq k\sqrt{L}}^{(\delta)}(L)^2) &> E(\widehat{N}_{\leq k\sqrt{L}}^{(0)}(L)^2)(1 - \epsilon) \\ &= [E(\widehat{N}_{\text{ren}}^{(0)}(L)^2) - E(\widehat{N}_{> k\sqrt{L}}^{(0)}(L)^2)](1 - \epsilon) \\ &> (C - \epsilon)(1 - \epsilon)L, \end{aligned} \quad (52)$$

and then we have from (48) (possibly with a further restriction on  $L\delta^2$ ) that

$$\left| \frac{E(\widehat{N}_{\leq k\sqrt{L}}^{(\delta)}(L)^2)}{E(\widehat{N}_{\text{ren}}^{(\delta)}(L)^2)} - 1 \right| \leq \frac{E(\widehat{N}_{> k\sqrt{L}}^{(\delta)}(L)^2)}{E(\widehat{N}_{\leq k\sqrt{L}}^{(\delta)}(L)^2)} < \frac{\epsilon}{(C - \epsilon)(1 - \epsilon)}. \quad (53)$$

Since  $\epsilon$  here is arbitrary, (50) and (53) imply respectively that

$$E(\widehat{N}_{\text{ren}}^{(0)}(L)^2) \simeq E(\widehat{N}_{\leq k\sqrt{L}}^{(0)}(L)^2) \quad \text{and} \quad E(\widehat{N}_{\text{ren}}^{(\delta)}(L)^2) \simeq E(\widehat{N}_{\leq k\sqrt{L}}^{(\delta)}(L)^2) \quad (54)$$

for  $L \ll \delta^{-2}$ , and with (46), (47) follows.

To conclude Step 3 we must establish (48). To do so we first note that for any integer  $n \geq 1$ ,

$$P(\widehat{N}_{\text{ren}}^{(\delta)}(L) \geq n) \leq P(\widehat{Y}^{(\delta)} \leq L)^n = (1 - q_\delta(L))^n, \quad (55)$$

since the distances between the successive renewal events in  $J_L$  (including the distance of the first such event from the origin), which are independent, must each be no greater than  $L$ .

In the remainder of this section we write  $q = q_\delta(L)$ . For any integer-valued random variable  $N$ , and any integer  $n_c \geq 1$ , we have (as a consequence of summation by parts) that

$$E(N^2 I_{\{N \geq n_c\}}) = n_c^2 P(N \geq n_c) + \sum_{n=n_c+1}^{\infty} (2n-1) P(N \geq n), \quad (56)$$

provided that  $n^2 P(N \geq n) \rightarrow 0$  as  $n \rightarrow \infty$ . Thus from (55),

$$\begin{aligned} E(\widehat{N}_{>n_c}^{(\delta)}(L)^2) &\leq n_c^2(1-q)^{n_c} + \sum_{n=n_c+1}^{\infty} (2n-1)(1-q)^n \\ &\leq \left(n_c^2 + \frac{2n_c}{q} + \frac{2}{q^2}\right) (1-q)^{n_c} \\ &\leq 2 \left(n_c + \frac{1}{q}\right)^2 e^{-qn_c}. \end{aligned} \quad (57)$$

If  $\delta > 0$  then it follows from (13) and (33) that for  $L\delta^2$  sufficiently small there is an  $A > 0$  such that

$$q \geq \frac{A}{\sqrt{L}}. \quad (58)$$

Moreover, (15) implies the same conclusion, for any  $L$ , when  $\delta = 0$ . If now for any  $k > 0$  we set  $n_c = \lfloor k\sqrt{L} \rfloor$  then (57) and (58) yield

$$E(\widehat{N}_{>k\sqrt{L}}^{(\delta)}(L)^2) = E(\widehat{N}_{>n_c}^{(\delta)}(L)^2) \leq \left(k + \frac{1}{A}\right)^2 e^A e^{-Ak} L, \quad (59)$$

and (48) will hold for sufficiently large  $k$ .

**Step 4:** Equation (30), the starting point for our future investigations, involves  $p_\delta(l) = P(F^{(\delta)} = l)$ , the probability under  $\nu_\rho$  that the first renewal event to the right of the origin occurs at site  $l > 0$ . This happens precisely when there is a renewal event at some site  $-l' \leq 0$ , an event with probability  $2\delta$  (see Remark 3), and the next renewal event to its right is at  $l$ , so that

$$p_\delta(l) = 2\delta \sum_{l' \geq 0} P(\widehat{Y}^{(\delta)} = l' + l) = 2\delta P(\widehat{Y}^{(\delta)} \geq l). \quad (60)$$

(Note that  $p_\delta(l) = 2\delta q_\delta(l-1)$ .) Now (39) yields

$$p_\delta(l) \simeq 2\delta P(\widehat{Y}^{(0)} \geq l) \quad \text{for } l \ll \delta^{-2}, \quad (61)$$

and thus from (30) we have that

$$E(N_{\text{ren}}^{(\delta)}(L)^2) \simeq 2\delta \sum_{l=1}^L P(\widehat{Y}^{(0)} \geq l) E((\widehat{N}_{\text{ren}}^{(0)}(L-l) + 1)^2) \quad \text{for } L \ll \delta^{-2}. \quad (62)$$

**Step 5:** In this step we obtain the large- $L$  asymptotics of  $E(\widehat{N}_{\text{ren}}^{(0)}(L)^2)$ :

$$E(\widehat{N}_{\text{ren}}^{(0)}(L)^2) \simeq L, \quad \text{for } L \gg 1. \quad (63)$$

This formula may be obtained from [9], but we give a self-contained proof, arguing from the detailed form of the distribution of the renewal random variable  $\widehat{Y}^{(0)} = 2\widehat{X}^{(0)} + 1$ . Recall (see (11)) that  $P(\widehat{X}^{(0)} = n) = C_n 2^{-(2n+1)}$ ; the Catalan number  $C_n$  counts the number of paths between time 0 and time  $2n$  of a random walk which starts and ends at the origin while never taking any positive value. Thus  $\widehat{Y}^{(0)}$  has the same distribution as

the time of first arrival at site 1 of a simple symmetric random walk  $W_l$ ,  $l = 0, 1, \dots$ , which starts at the origin. As a consequence,  $\widehat{N}_{\text{ren}}^{(0)}(L)$  has the same distribution as the maximum value  $M(L)$  of  $W_l$  over the interval  $[0, L]$ . From [6], Section III.7, Theorem 1 we then have

$$P(\widehat{N}_{\text{ren}}^{(0)}(L) = n) = \begin{cases} P(W_L = n), & \text{if } L - n \text{ is even,} \\ P(W_L = n + 1), & \text{if } L - n \text{ is odd.} \end{cases} \quad (64)$$

An easy calculation from (64) gives, for  $L$  odd,

$$E(\widehat{N}_{\text{ren}}^{(0)}(L)^2) = E(W_L^2) - E(|W_L|) + \frac{1}{2}(1 - P(W_L = 0)), \quad (65)$$

and this yields (63), since  $E(W_L^2) = L$  and  $E(|W_L|) \leq \sqrt{L}$ , by the Cauchy-Schwarz inequality.

*Remark 4.* (a) From (64) one can show easily that  $\widehat{N}_{\text{ren}}^{(0)}(L)/\sqrt{L}$  converges in distribution, as  $L \rightarrow \infty$ , to  $|Z|$ , with  $Z$  a standard normal random variable.

(b) In [11, 22] a random walk representation of particle configurations (there called a height function or height process) was used to obtain the distribution (11). The Catalan numbers play the same role in this derivation that they do above.

**Step 6:** Equation (62) provides the leading order small- $\delta$  approximation to  $E(N_{\text{ren}}^{(\delta)}(L)^2)$ , valid for  $L \ll \delta^{-2}$ . In this step we use (63) and (66) below to approximate this moment to leading order in  $L$ , as well.

First, note that from (13) we have at once that

$$P(\widehat{Y}^{(0)} \geq l) = P(\widehat{X}^{(0)} \geq \frac{l-1}{2}) \simeq \sqrt{\frac{2}{\pi l}} \quad \text{for } l \gg 1. \quad (66)$$

Substituting (63) and (66) into (62) yields, at least formally,

$$E(N_{\text{ren}}^{(\delta)}(L)^2) \simeq 2\sqrt{\frac{2}{\pi}}\delta \sum_{l=1}^L \frac{L-l}{\sqrt{l}}, \quad \text{for } 1 \ll L \ll \delta^{-2}. \quad (67)$$

We will justify (67) shortly, but for the moment only note that the restriction  $L \gg 1$ , not present in (62), arises from (63) and (66). From (67) we have that for  $1 \ll L \ll \delta^{-2}$ ,

$$\begin{aligned} E(N_{\text{ren}}^{(\delta)}(L)^2) &\simeq 2\sqrt{\frac{2}{\pi}}\delta \int_1^L \frac{L-x}{\sqrt{x}} dx \\ &= 2\sqrt{\frac{2}{\pi}}\delta L^{3/2} \int_{1/L}^1 \frac{1-y}{\sqrt{y}} dy \simeq \frac{8}{3}\sqrt{\frac{2}{\pi}}\delta L^{3/2}. \end{aligned} \quad (68)$$

This is our final approximation for  $E(N_{\text{ren}}^{(\delta)}(L)^2)$ .

We now return to (67). We are justified (when  $l$  is not too large) in replacing  $E((\widehat{N}_{\text{ren}}^{(0)}(L-l)+1)^2)$  by  $E((\widehat{N}_{\text{ren}}^{(0)}(L-l))^2)$  in passing from (62) to (67) since, from (63) and the Cauchy-Schwarz inequality,  $E(\widehat{N}_{\text{ren}}^{(0)}(L)) \ll E(\widehat{N}_{\text{ren}}^{(0)}(L)^2)$  for  $L \gg 1$ . Moreover, the substitutions in (62) suggested by (63) and (66) are valid provided that  $l$

and  $L - l$  are sufficiently large. More precisely, using (63) and (66), we can conclude that there exists an integer  $l_*$ , which does not depend on  $L$ , such that

$$2\sqrt{\frac{2}{\pi}}\delta \sum_{l=l_*}^{L-l_*} \frac{L-l}{\sqrt{l}} \simeq 2\delta \sum_{l=l_*}^{L-l_*} P(\hat{Y}^{(0)} \geq l) E((\hat{N}_{\text{ren}}^{(0)}(L-l) + 1)^2). \quad (69)$$

But note that the sums in (62) and (67) over  $1 \leq l < l_*$  and  $L - l_* < l \leq L$  are  $O(L)$  and  $o(1)$ , respectively, as  $L \rightarrow \infty$ , while the sums over  $l_* \leq l \leq L - l_*$  are of order  $L^{3/2}$  (see (68)). This completes the justification.

**Step 7:** From (17) and (68),

$$\frac{E(N_{\text{ren}}^{(\delta)}(L))^2}{E(N_{\text{ren}}^{(\delta)}(L)^2)} \simeq \frac{3}{2} \sqrt{\frac{\pi}{2}} \delta L^{1/2} \ll 1 \quad \text{for } 1 \ll L \ll \delta^{-2}. \quad (70)$$

Thus, again from (68),

$$\text{Var}(N_{\text{ren}}^{(\delta)}(L)) \simeq E(N_{\text{ren}}^{(\delta)}(L)^2) \simeq \frac{8}{3} \sqrt{\frac{2}{\pi}} \delta L^{3/2}, \quad \text{for } 1 \ll L \ll \delta^{-2}. \quad (71)$$

**Step 8:** We can now combine the results of Step 1 and Step 7 to obtain the parts of Theorem 1 which concern  $L \ll \delta^{-2}$ . For from (25) and (71) we have that

$$\text{Var}(N^{(\delta)}(L)) \simeq \frac{2}{3} \sqrt{\frac{2}{\pi}} \delta L^{3/2} + \frac{1}{4} P(\sigma^{(\delta)}(L) \neq 0) \quad \text{for } 1 \ll L \ll \delta^{-2}, \quad (72)$$

with  $\sigma^{(\delta)}(L)$  defined in (19).

If  $L$  is even then, from (17),

$$\begin{aligned} P(\sigma^{(\delta)}(L) \neq 0) &= P(N_{\text{ren}}^{(\delta)}(L) \text{ is odd}) \\ &\leq P(N_{\text{ren}}^{(\delta)}(L) > 0) \leq E(N_{\text{ren}}^{(\delta)}(L)) = 2\delta L. \end{aligned} \quad (73)$$

Since  $L \ll L^{3/2}$  for  $L \gg 1$ , the  $L \ll \delta^{-2}$  part of Theorem 1(b) follows.

On the other hand, if  $L$  is odd then, again from (17),

$$\begin{aligned} P(\sigma^{(\delta)}(L) \neq 0) &= P(N_{\text{ren}}^{(\delta)}(L) \text{ is even}) \\ &\geq P(N_{\text{ren}}^{(\delta)}(L) = 0) \geq 1 - 2\delta L \simeq 1 \quad \text{for } L \ll \delta^{-1}. \end{aligned} \quad (74)$$

Since  $\delta L^{3/2} \gg 1$  for  $L \gg \delta^{-2/3}$  and  $\delta L^{3/2} \ll 1$  for  $L \ll \delta^{-2/3}$ , we obtain from (74) and (72) the conclusions of Theorem 1(a) for  $1 \ll L \ll \delta^{-2}$ . To remove the restriction that  $L \gg 1$  we note that

$$\text{Var}(N_{\text{ren}}^{(\delta)}(L)) \leq E(N_{\text{ren}}^{(\delta)}(L)^2) \leq E(N_{\text{ren}}^{(\delta)}(L')^2) \quad (75)$$

for  $L \leq L'$ . Choosing  $L'$  such that also  $1 \ll L' \ll \delta^{-2/3}$ , we see using (25), (68), and (74) that  $\text{Var}(N^{(\delta)}(L)) \simeq \frac{1}{4}$  for  $1 \leq L \ll \delta^{-2/3}$ .

*Remark 5.* Concerning the estimate  $P(N_{\text{ren}}^{(\delta)}(L) > 0) \leq 2\delta L$  used in (73) and (74), note that it follows from (26), (61), and (66) that in fact for  $L \gg 1$ ,

$$P(N_{\text{ren}}^{(\delta)}(L) > 0) = P(F^{(\delta)} \leq L) \simeq 2\delta \int_0^L \sqrt{\frac{2}{\pi l}} dl \simeq 4\sqrt{\frac{2}{\pi}} \delta \sqrt{L}. \quad (76)$$

**Step 9:** We now turn to the  $L \gg \delta^{-2}$  part of Theorem 1 and to the related Corollary 2 (recall also Remark 1). We write

$$\begin{aligned} \text{Var}(N^{(\delta)}(L)) &= \sum_{1 \leq k \leq L} \text{Var}_{v_\rho}(\eta_k) + 2 \sum_{1 \leq k < i \leq L} \text{Cov}_{v_\rho}(\eta_k, \eta_i) \\ &= \rho(1 - \rho)L + 2 \sum_{k=1}^{L-1} \sum_{j=1}^k g_\rho^T(j), \end{aligned} \quad (77)$$

with  $g_\rho^T$  the truncated two-point correlation function for the TI state  $v_\rho$ :

$$g_\rho^T(k) = E(\eta_j \eta_{j+k}) - \rho^2. \quad (78)$$

It is shown in [11] that for all  $n \geq 0$ ,

$$g_\rho^T(2n+1) + g_\rho^T(2n+2) = 0, \quad (79)$$

so that (77) becomes

$$\text{Var}(N^{(\delta)}(L)) = \rho(1 - \rho)L + 2 \sum_{\substack{k=1 \\ k \text{ odd}}}^{L-1} g_\rho^T(k). \quad (80)$$

We will show shortly that for  $j \geq 0$ ,

$$|g_\rho^T(2j+1)| \leq \rho^2(1 - 4\delta^2)^j. \quad (81)$$

Then from (80),

$$\left| \frac{\text{Var}(N^{(\delta)}(L))}{\rho(1 - \rho)L} - 1 \right| \leq \frac{2\rho}{(1 - \rho)L} \sum_{j=0}^{\infty} (1 - 4\delta^2)^j < \frac{1}{2\delta^2 L}, \quad (82)$$

and this verifies the result stated in Corollary 2. The  $L \gg \delta^{-2}$  cases of Theorem 1 then follow by taking the  $\delta_0$  of Definition 1 sufficiently small.

Consider now (81). The generating function of the  $g_\rho^T$  is computed in [11]:

$$G_\rho(z) := \sum_{k=1}^{\infty} g_\rho^T(k) z^k = \frac{z(\sqrt{1 - z^2(1 - 4\delta^2)} - 2\delta)^2}{4(z - 1)(z + 1)^2}. \quad (83)$$

The numerator in (83) has double zeros (on its first sheet) at  $z = \pm 1$ , so that  $G_\rho$  is analytic at these points with singularities at  $z = \pm z_*$ , where  $z_* := (1 - 4\delta^2)^{-1/2}$ . Expressing  $g_\rho^T(k)$  via Cauchy's formula as an integral over a small circle around the origin, distorting this contour to obtain the sum of integrals of the discontinuity of  $G_\rho$  across cuts on the real axis from  $z_*$  to  $\infty$  and from  $-z_*$  to  $-\infty$ , and making the change of variable  $z \rightarrow -z$  in the second of these integrals, we obtain a representation of  $g_\rho^T(k)$  which for  $k$  odd is

$$g_\rho^T(k) = -\frac{2\delta}{\pi z_*} \int_{z_*}^{\infty} \frac{\sqrt{z^2 - z_*^2}}{z^k (z^2 - 1)^2} dz, \quad k \text{ odd}. \quad (84)$$

This implies that for  $k$  odd,  $|g_\rho^T(k)| \leq z_*^{-(k-1)} |g_\rho^T(1)|$ , and since  $g_\rho^T(1) = -\rho^2$ , (81) follows.

This completes the proof of Theorem 1.

## 5. Concluding Remarks

We note that on  $\mathbb{Z}$  the approach to hyperuniformity as  $\rho \searrow \rho_c$  is different from that for  $\rho \nearrow \rho_c$ ; in the former case the unique stationary measure for the FEP on  $\mathbb{Z}$  is known and [13, 15]

$$\lim_{L \rightarrow \infty} \frac{1}{L} V_\rho(L) = \rho(1 - \rho)(2\rho - 1) \quad \text{for } \rho > 1/2. \quad (85)$$

Thus from Theorem 1 and (8),  $\lim_{L \rightarrow \infty} L^{-1} V_\rho(L)$  is continuous in  $\rho$  from above, but not from below, at  $\rho = 1/2$ . We expect similar behavior on  $\mathbb{Z}^d$ .

The case of the FEP on a *ladder*, a system consisting of two (infinite) rows of sites, was studied numerically in [17, 18]. Here again, as on  $\mathbb{Z}^d$  with  $d \geq 2$ ,  $\rho_c < 1/2$  (for the continuous-time symmetric FEP on the ladder,  $\rho_c \approx 0.4755$  or, for the slightly different dynamics of [16],  $\rho_c \approx 0.4874$  [18]). The results of [18] also suggest that the Fluctuation Conjecture holds for this model (although the scaling behavior of the  $L_i(\delta)$  is not discussed). Rigorously, one may observe that for  $\rho < \rho_c$  the portions of the system to the left and right of an empty square are independent under the  $t \rightarrow \infty$  limiting measure  $\nu_\rho$ , implying in particular that the locations of the empty squares (when these have a nonzero density) form a renewal process and that the portions of the system between them are jointly independent. Further, we see that the critical density must satisfy  $\rho_c \geq 1/4$ , since at smaller densities there would always be a finite density of empty squares and the stationary state would be frozen [18]. We have no such lower bound for  $\rho_c$  on  $\mathbb{Z}^d$ ,  $d \geq 2$ .

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## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

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