Probabilistic Cellular Automata:
Some Statistical Mechanical Considerations

Joel L. Lebowitz [1,2], Christian Maes [1,3], and

Eugene R. Speer

Department of Mathematics

Rutgers University

New Brunswick, NJ 08903

Abstract

We sketch some recent developments in the statistical mechanical analysis of Probabilistic Cellular Automata with emphasis on rigorous results.

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1. Supported in part by NSF Grant DMR86–12369

2. Also Department of Physics

3. Present Address: Dipartimento di Matematica, II Universita degli Studi di Roma, Via Fontanile di Carcaricola, 00133 Roma, Italy: Aangesteld Navorser N.F.W.O., Instituut voor Theoretische Fysica, K.U. Leuven, Belgium

1. Introduction

Spin systems evolving in continuous or discrete time under the action of stochastic dynamics are used to model phenomena as diverse as the structure of alloys and the functioning of neural networks. While in some cases the dynamics are secondary, designed to produce a specific stationary measure whose properties one is interested in studying, there are other cases in which the only available information is the dynamical rule. Prime examples of the former are computer simulations, via Glauber dynamics, of equilibrium Gibbs measures with a specified interaction potential. Examples of the latter include various types of majority rule dynamics used as models for pattern recognition and for error–tolerant computations. The present note discusses ways in which techniques found useful in equilibrium statistical mechanics can be applied to a particular class of models of the latter type. These are cellular automata with noise: systems in which the spins are updated stochastically at integer times, simultaneously at all sites of some regular lattice. These models were first investigated in detail in the Soviet literature of the late sixties and early seventies [1,2]. They are now generally referred to as Stochastic or Probabilistic Cellular Automata (PCA) [3], and may be considered to include deterministic automata (CA) as special limits.

There are various ways in which one may try to extend the methods of equilibrium statistical mechanics, which have been very successful in elucidating the "interesting" behavior of Gibbs measures, e.g. coexistence of phases and critical phenomena, to more general processes. In particular it has been recognized for a long time [1,2] that there is an intimate relation between stationary measures of reversible PCA (i.e. those with dynamics satisfying detailed balance) and Gibbs states. We shall use as our starting point here however the relation between PCA in d dimensions and Equilibrium Statistical Models (ESM) in $(d+1)$ dimensions, the extra dimension being the discrete time. This connection has been exploited by Domany [4] and others [5, 6] to obtain information about the equilibrium properties of some $(d+1)$–dimensional spin systems from knowledge of specially constructed d–dimensional PCA's. There has also been, more recently, some general study of this connection from a mathematically rigorous point of view [7, 8]. The present note is closely related to the work in [9, 10] and uses the same notation. It consists of three parts: Section 2 presents background material and the basic PCA–ESM connection. Section 3 gives some illustrative examples including some new results. Section 4 considers PCA which satisfy some form of detailed balance including ones which are updated in two or more time steps. These PCA are generalizations of the Domany models where the updating rules satisfy detailed balance conditions with respect to specified Gibbs measures. We refer the reader to [9] and to references given there for additional background and results.

2. PCA Formalism

We consider PCA which describe the stochastic discrete time evolution of Ising spin variables on some regular lattice $\mathcal{L}$ which we generally take to be $\mathbb{Z}^d$. We denote the value of the spin at site $i \in \mathbb{Z}^d$ at time $n \in \mathbb{Z}$ by $\sigma_{n,i} = \pm 1$, and write $\underline{\sigma}_n = \{\sigma_{n,i}\}$ for the configuration at time n; we will occasionally let $\underline{\eta}$ denote a generic configuration on $\mathbb{Z}^d$. The PCA evolves by simultaneous independent updating of spins. That is, the spin configuration $\underline{\sigma}_{n-1}$ determines the probabilities $p(\sigma_{n,i}|\underline{\sigma}_{n-1})$ of the spin values at each site i at time n, and the conditional probability distribution of the corresponding $\underline{\sigma}_n$, $P(\underline{\sigma}_n|\underline{\sigma}_{n-1})$, is a product measure,

$$P(\underline{\sigma}_n|\underline{\sigma}_{n-1}) = \prod_i p(\sigma_{n,i}|\underline{\sigma}_{n-1}) \qquad (2.1)$$

Without loss of generality we write

$$p(\sigma_{n,i}|\underline{\sigma}_{n-1}) = \tfrac{1}{2}(1 + \sigma_{n,i}h_i(\underline{\sigma}_{n-1})) \qquad (2.2)$$

with $|h_i(\underline{\sigma}_{n-1})| \leq 1$. We assume that the rules are translation invariant so that $h_i(\underline{\eta})$ is obtained from $h_o(\underline{\eta})$ via a translation by the lattice vector $i \in \mathbb{Z}^d$. We also assume that $h_o(\underline{\eta})$ depends only on the configurations of spins in a finite set U near the origin.

Given $\underline{\sigma}_{n-1}$ the spins at time n are independent with averages given by

$$< \prod_{i \in A} \sigma_{n,i} \mid \underline{\sigma}_{n-1} > = \prod_{i \in A} h_i(\underline{\sigma}_{n-1}). \tag{2.3}$$

When $|h_i(\underline{\eta})| = 1$ for all i and η we have a deterministic evolution, i.e. a CA. An interesting way of characterizing a general PCA is by specifying a finite collection of deterministic rules M_α together with their probabilities q_α. M_α is a local function of the configuration taking only the values $+1$ and -1, and $\sum_\alpha q_\alpha = 1, q_\alpha \geq 0$. The evolution may be pictured as follows: at each site i at time n a choice of deterministic rule is made, independent of other sites or times and of the input $\underline{\sigma}_{n-1}$, so that

$$\sigma_{n,i} = M_{\alpha,i}(\underline{\sigma}_{n-1}) \text{ with probability } q_\alpha \tag{2.4}$$

where $M_{\alpha,i}$ is the translate of M_α with lattice vector i. This procedure gives

$$h_i(\underline{\eta}) \equiv \sum_\alpha q_\alpha M_{\alpha,i}(\underline{\eta}). \tag{2.5}$$

Note that in general many different choices of the deterministic rules M_α may give rise to the same transition probabilities (2.5). The case where there is only one M, i.e. in which $\sigma_{n,i}$ follows $M_i(\underline{\eta})$ with probability $(1-\epsilon)$ and does the opposite $(-M_i)$ with probability ϵ, is of special interest. We can then write $h_i(\underline{\eta})$ as

$$h_i(\underline{\eta}) = (1-2\epsilon)M_i(\underline{\eta}), \quad 0 \leq \epsilon \leq 1 \tag{2.6}$$

which can be interpreted to mean that we follow the deterministic rule M with probability $(1-2\epsilon)$ and with probabilities ϵ choose $\sigma_{n,i} = \pm 1$ independent of $\underline{\eta}$.

One of the main problems for these systems is to characterize their time invariant probability measures or, more generally, the stationary space–time process generated by the Markov transition rates (2.1). To do this within the framework of statistical physics it is natural to consider $\{\underline{\sigma}_n\}_{n \in \mathbb{Z}}$ as defining a spin configuration $\underline{\sigma}$ on the space–time lattice $\mathbb{Z}^{d+1}$; we will write $x = (n,i)$ for a typical site in this lattice and let $\mathbb{Z}^d_n$ denote the d–dimensional layer corresponding to time n. By the usual convention of cellular automata, we will visualize the time axis in $\mathbb{Z}^{d+1}$ as vertical and oriented so that the past is at the top and the future at the bottom. If $\rho(\underline{\eta})$ is a measure on the state space of the PCA and we "start" the time evolution with measure $\rho(\underline{\eta})$ on the layer $\mathbb{Z}^d_{-N}$, then the Markov transition rates (1.1) define a measure μ_ρ^{-N} on the set of "future" configurations $\{\underline{\sigma}_n\}_{n \geq -N}$. Suppose now that we chose a $\rho = \nu$ which is stationary for the time evolution, i.e.

$$\sum_{\underline{\eta}'} P(\underline{\eta}|\underline{\eta}')\, \nu(\underline{\eta}') = \nu(\underline{\eta}), \tag{2.7}$$

then the measure μ_ν^{-N} will "reproduce" ν in the time direction in the sense that its projection on any $\mathbb{Z}^d_n$, for $n > -N$, will be just $\nu(\underline{\sigma}_n)$. We can then take the limit $N \to \infty$ to produce a measure μ_ν on the set of space–time configurations $\underline{\sigma} = \{\sigma_x\}$, $x \in \mathbb{Z}^{d+1}$. The measure μ_ν is translation invariant in the time direction and its projection on any $\mathbb{Z}^d_n$ is just ν. Similar conclusions hold when ν is periodic.

The observant reader may have noticed that we have been very cavalier about transition rates $P(\underline{\eta}|\underline{\eta}')$ and measures $\nu(\underline{\eta})$ on the infinite set of spin configurations. It is most convenient to consider this as a limit of a PCA defined on a box $\Gamma \subset \mathbb{Z}^d$ with periodic or other specified boundary conditions. This will become clearer in the next section when we consider the corresponding ESM. In case of doubt simply think of the PCA as defined on configurations in a finite periodic box.

From PCA to ESM

It is a simple observation that if the transition probabilities $p_i(\sigma_{n,i}|\underline{\sigma}_{n-1})$ are all strictly positive, i.e., if $|h_o(\underline{\eta})| < 1$ for all $\underline{\eta}$, then μ_ν is a Gibbs measure for the Hamiltonian [4]

$$\mathscr{H}(\sigma) = \sum_{x \in \mathbb{Z}^{d+1}} H(\sigma_x, \underline{\sigma}). \tag{2.8}$$

Here the single site energy for $x = (n,i)$ is defined by

$$\exp[-H(\sigma_x, \underline{\sigma})] \equiv p_i(\sigma_{n,i}|\underline{\sigma}_{n-1}), \tag{2.9}$$

and the reciprocal temperature β which usually multiplies the energy in the exponent of (2.9) has been absorbed into H. The normalization condition of the PCA gives

$$\sum_{\sigma_x=\pm 1} \exp[-H(\sigma_x, \underline{\sigma}_{n-1})] = 1. \tag{2.10}$$

In fact we can always write

$$\exp[-H(\sigma_0, \underline{\eta})] = \exp[-\sigma_0 Q_0(\underline{\eta})]/2 \cosh Q_0(\underline{\eta}), \tag{2.11}$$

with

$$h_0(\underline{\eta}) = -\tanh Q_0(\underline{\eta}).$$

We can therefore think of $h_i(\underline{\eta})$ as the tanh of the "field" acting on the spin at site i when the configuration in the layer "above" it is $\underline{\eta}$. When $|h_0(\underline{\eta})| = 1$ for some configurations $\underline{\eta}$, making the field infinite, we may still be able to consider non–trivial μ_ν – but there will now be certain "hard–core" constraints on the possible configuration $\underline{\sigma}_n$ given $\underline{\sigma}_{n-1}$.

An important consequence of (2.10) is that the finite volume free energy for the ESM is identically zero for certain classes of boundary conditions, i.e. for $d+1$ dimensional cubes or parallelopipeds with periodic boundary conditions in the space directions, arbitrary configurations (initial conditions) at the top, and free boundary conditions at the bottom. More generally given any region $\Lambda \subset \mathbb{Z}^{d+1}$ and a spin configuration $\underline{\sigma}_{\Lambda^c}$ on the complement of Λ we can define the finite volume Gibbs measure $\mu_p(\underline{\sigma}_\Lambda | \underline{\sigma}_{\Lambda^c})$ on the spins $\underline{\sigma}_\Lambda$ in Λ with "PCA boundary conditions" by

$$\mu_p(\underline{\sigma}_\Lambda | \underline{\sigma}_{\Lambda^c}) \equiv \exp[-\sum_{x\in\Lambda} H_x(\sigma_x, \underline{\sigma})]. \tag{2.12}$$

<u>N.B.</u> The right side of (2.12) is properly normalized since the trace over the spins in Λ, is by (2.10) identically one, independent of $\underline{\sigma}_{\Lambda^c}$. Note also that if we weight the $\underline{\sigma}_{\Lambda^c}$ in (2.12) according to a probability distribution corresponding to an infinite volume Gibbs measure $\mu_{\mathcal{H}}$ for $\mathcal{H}$ given in (2.8) the resulting measure on $\underline{\sigma}_\Lambda$ is <u>not</u> the same as that obtained by projecting $\mu_{\mathcal{H}}$ on Λ. The difference arises from the fact that many of the interactions <u>across</u> the boundary of Λ are omitted from the exponent in (2.12). Nevertheless, as is well known, in the limit $\Lambda \nearrow \mathbb{Z}^{d+1}$ <u>all</u> boundary conditions lead to some

Gibbs measure $\mu_{\mathscr{H}}$.

Now if the set of Gibbs measures for $\mathscr{H}$ is unique, i.e. there is only one $\mu_{\mathscr{H}}$ then clearly it is the same as μ_ν. If however there is more than one $\mu_{\mathscr{H}}$ as occurs when the ESM has more than one phase then the question naturally arises as to which boundary conditions give Gibbs measures which coincide with a space–time stationary PCA measure μ_ν for some ν. More generally what is the correspondence between the set of Gibbs measures $\{\mu_{\mathscr{H}}\}$ and PCA measures $\{\mu_\nu\}$? As remarked above $\{\mu_\nu\} \subset \{\mu_{\mathscr{H}}\}$. A partial answer to this question was given in [8] where it was shown that those measures in each class which are translation invariant or periodic, in <u>all</u> $d+1$ directions, <u>coincide</u>. What happens for other measures is an open question.

We remark here also that for <u>any</u> boundary condition, the finite volume free energy of the system with interaction $\mathscr{H}$ is at most of the order of the size of the boundary region. Hence the infinite volume free energy density, which is independent of boundary conditions, is identically zero. In particular, it is analytic in the parameters of the PCA, even when there is a phase transition in the sense of ESM. The same analyticity will hold for the dependence of the free energy on the interaction coefficients entering the Hamiltonian (2.8). In certain cases this analyticity may be shown to hold separately in the entropy and energy densities, even when there is a phase transition as the parameters change; see [9].

3. <u>Some Examples</u>

1. An example of a PCA with a particularly simple stationary ν is obtained by letting $\sigma_{i,n} = M_i(\underline{\sigma}_{n-1}) = \prod_{j \in U(i)} \sigma_{j,n-1}$ with probability $(1-\varepsilon)$ and $\sigma_{n,i} = -M_i$, with probability ε for some ε, $0 < \varepsilon \leq 1$, where $U(i)$ is the translate of the neighborhood of the origin U by i. This gives, as per (2.6) and (2.11),

$$h_i(\underline{\eta}) = (1-2\varepsilon) \prod_{j \in U(i)} \eta_j, \tag{3.1}$$

and

$$H(\sigma_{n,i},\underline{\sigma}) = -\alpha\,\sigma_{n,i}\Pi\sigma_{n-1,j} - \ell n[2\cosh\alpha] \tag{3.2}$$

with $(1-2\varepsilon) = \tanh\alpha$. The stationary measure ν for this PCA, corresponding to the projection of the Gibbs measure μ_ν on $\mathbb{Z}^d$, is somewhat surprisingly just the Bernoulli product measure with $<\sigma_i> = 0$; c.f. [1] for a special case of this. This can be seen by noting from (2.3) and (2.7) that in a periodic box $\Lambda \subset \mathbb{Z}^d$, the product measure ν for which $<\eta_j> = 0$ is time invariant. The uniqueness of this stationary measure is easily established, c.f. Section 4 in [9].

2. The Bennett–Grinstein version of Toom's model [11] is a PCA on the square lattice $\mathbb{Z}^2$ in which every spin becomes (up to errors) equal to the majority at the previous time of the spin itself and its northern and eastern nearest neighbor. Errors favoring up spins are made with a probability p, and errors favoring down spins with a probability q. Using (2.4) we then have for $i = (i_1,i_2)$,

$$h_i(\underline{\eta}) = (1-p-q)\operatorname{sgn}(\sum_{U(i)}\eta_i) + (p-q) \tag{3.3}$$

with $U(i) = (i, i+e_1, i+e_2)$, e_1 and e_2 being the unit vectors on $\mathbb{Z}^2$. The resulting ESM on $\mathbb{Z}^3$ has the per site energy

$$\begin{aligned} H(\sigma_{n,i},\underline{\sigma}) = -(\beta\sigma_{n,i} + J)(\sum_{j\in U(i)}\sigma_{n-1,j} - \prod_{j\in U(i)}\sigma_{n-1,j}) \\ - b\,\sigma_{n,i} - \gamma \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \beta = \tfrac{1}{8}\log\frac{(1-p)(1-q)}{pq}, \quad b = \tfrac{1}{4}\log\frac{p(1-q)}{q(1-p)} \\ J = \tfrac{1}{8}\log\frac{q(1-q)}{p(1-p)}, \quad \gamma = \tfrac{1}{4}\log[pq(1-p)(1-q)]. \end{aligned} \tag{3.5}$$

Toom [12] has proven that there is more than one invariant measure for this PCA whenever p and q are sufficiently small. A discussion of the phase diagram for this model obtained from computer simulations is contained in [11] and [9], from which Fig. 1 is taken.

We note here in particular that this system can exist in two phases even when p (or q) is zero. In this case one of the phases consists of all spins down (up). Such a state has, of course, zero entropy and since its free energy is zero so is its energy (understood as the limit $p \to 0$ for fixed small q). The other state has most spins up and as q is increased past some critical value $q_c \simeq .07$ it disappears abruptly – corresponding to a first order type phase transition. Bennett and Grinstein [11] believe that the transition is in fact first order all along the coexistence curve in Fig. 1 except for the symmetric case $p = q$ where it is second order with presumably the same exponents as the two dimensional Ising model. Nothing however is known about this rigorously.

It should be remarked also that the planar region in Fig. 1 where the system can exist in (at least) two phases is to be thought of as a two–dimensional region in the four dimensional parameter space, β, J, b and γ, of the Hamiltonian (3.4). Crossing the coexistence line in Fig. 1 the free energy remains analytic (since it is zero everywhere in the parameter region specified by (3.5)). There will however be "other" directions of parameter space in which the free energy of the Hamiltonian (3.4) will have a discontinuous slope – since the ESM has more than one phase.

Finally we show in Figs. 2 and 3 typical configurations of the stationary state of this system in a 256×256 square with + boundary conditions on top and – boundary conditions on the right side, a la (2.12), for $p = q = \varepsilon = .02$ and $.09$ (the boundary conditions on the left and bottom are irrelevant since they do not influence the interior). When there is a bias in favor of the +'s (–'s) the slope of the interface between the two phase increases (decreases). The interface is expected to broaden as the length, L, of the side of the square is increased as $L^{1/3}$ [13].

4. Stationary Measures and Detailed Balance

Using the normalization condition on the transition probability $P(\underline{\eta}|\underline{\eta}')$, Eq.

(2.7) for the stationary ν can be rewritten in the form

$$\sum_{\underline{\eta}'} [P(\underline{\eta}|\underline{\eta}')\nu(\underline{\eta}') - P(\underline{\eta}'|\underline{\eta})\nu(\underline{\eta})] = 0. \tag{4.1}$$

The transition P is said to be reversible or to satisfy detailed balance with respect to ν iff the summand in (4.1) vanishes for all $\underline{\eta}'$, i.e. when

$$P(\underline{\eta}|\underline{\eta}')\,\nu(\underline{\eta}') = P(\underline{\eta}'|\underline{\eta})\,\nu(\underline{\eta}) \tag{4.2a}$$

or

$$P(\underline{\eta}|\underline{\eta}') = S(\underline{\eta},\underline{\eta}')/\nu(\underline{\eta}') \tag{4.2b}$$

with S symmetric,

$$S(\underline{\eta},\underline{\eta}') = S(\underline{\eta}',\underline{\eta}). \tag{4.2c}$$

The condition of detailed balance implies that a motion picture made of the PCA in the stationary state ν, will look the same when run forwards or backwards, i.e. the stationary space–time process is reversible.

It was noted by Stavskaja [2] that the detailed balance condition (4.2) implies that

$$\begin{aligned}\nu(\underline{\eta}) &= \nu(-1)\, P(\underline{\eta}|-1)/P(-1|\underline{\eta}) \\ &= \nu(-1)\, \Pi\{[1 + \eta_i h_i(-1)]/[1 - h_i(\underline{\eta})]\}\end{aligned} \tag{4.3}$$

where we have written -1 for the reference configuration $\eta_i = -1$, $\forall$i, and have used (2.2). Stavskaja also introduces a more general detailed balance condition than (4.2) which is sometime useful. For any η' let $S_{\underline{\eta}'}$ be a transformation of the set of $\{\eta_i\}$ which can include shifts, inversions, etc. We say that the PCA satisfies "Stavaskaja balance" whenever

$$P(S_{\underline{\eta}'}\cdot\underline{\eta}|\underline{\eta}')\,\nu(\underline{\eta}') = P(\underline{\eta}'|\underline{\eta})\,\nu(\underline{\eta}). \tag{4.4}$$

Letting $S_{-1}\cdot\underline{\eta} = \underline{\eta}^*$ we then have in analogy with (4.3) that (4.4) implies

$$\nu(\underline{\eta}) = \nu(-1)\, \prod_i\{[1 + \eta_i^* h_i(-1)]/[1 - h_i(\underline{\eta})]\}. \tag{4.5}$$

We refer the reader to [1] and [2] for more information on the S–condition.

Returning now to (4.2) the question naturally arises as to when a given P satisfies (4.2). It turns out that a very simple criterion can be given using (2.9) and (2.11): (4.2) will be satisfied iff $h_i(\underline{\eta})$ has a form of the type frequently used in neural networks

$$h_i(\underline{\eta}) = \tanh\,[\lambda_i + \sum_j J_{ij}\eta_j], \tag{4.6a}$$

with symmetric inputs,

$$J_{ij} = J_{ji}. \tag{4.6b}$$

The corresponding stationary ν is then given by

$$\nu = C \exp \sum_i \{\lambda_i \eta_i + \log 2 \cosh[\lambda_i + \sum_j J_{ij}\eta_j]\} \tag{4.7}$$

with C a normalization constant. The second term in the exponent in (4.7) can be expanded into a polynomial of the form

$$\log\{2 \cosh[\lambda_0 + \sum_j J_{0j}\eta_j]\} = \sum_A J_A^{(d)} \eta_A. \tag{4.8}$$

The sum in (4.8) goes over all subsets $A \subset U$, with $\eta_A = \prod_{k \in A} \eta_k$: U being the neighborhood of the origin for which $J_{0j} \neq 0$. When the dynamics are translation invariant, $\lambda_i = \lambda$ and $J_{ij} = J(i-j)$, the stationary ν is a Gibbs measure on $\mathbb{Z}^d$ for the Hamiltonian

$$H_d(\underline{\eta}) = -\sum_{i \in \mathbb{Z}^d} \{\lambda \eta_i + \sum_A J_A^{(d)} \eta_{A+i}\} = \sum_i H_i^{(d)}(\underline{\eta}). \tag{4.9}$$

Phase transitions for this d–dimensional equilibrium system will thus correspond to non ergodicity of the PCA and to phase transitions in the $(d+1)$–dimensional ESM system with Hamiltonian per site $(n,i) \in \mathbb{Z}^{d+1}$ of the form

$$H_{n,i}(\underline{\sigma}) = -2\lambda\sigma_{n,i} - \sum_j \sigma_{n,i} J(i-j)\sigma_{n-1,j} - H_i^{(d)}(\underline{\sigma}_{n-1}). \tag{4.10}$$

It should be noted that the special form of the Hamiltonian H_d in (4.9) precludes, in general, the construction of a PCA dynamics which will satisfy detailed balance for an a priori given Gibbs measure $\tilde{\nu}$ corresponding to a Hamiltonian $\tilde{H}_d$. This is in contrast to continuous time or sequential updatings where a suitable choice of stochastic dynamics can always be made. This difficulty can be overcome, following Domany [4], when the lattice $\mathcal{L}$ can be divided into two parts, $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$, e.g. for

$\mathscr{L} = \mathbb{Z}^d$ the even and odd sublattices. Let $\underline{\eta} = (\underline{\eta}^{(1)},\underline{\eta}^{(2)})$, and assume that the conditional measures on $\mathscr{L}_1$ and $\mathscr{L}_2$, obtained from the measure $\tilde{\nu}(\underline{\eta}) = \tilde{\nu}(\underline{\eta}^{(1)},\underline{\eta}^{(2)})$ in which we are interested, are product measures, i.e.

$$\tilde{\nu}(\underline{\eta}^{(\alpha')}\,|\,\underline{\eta}^{(\alpha)}) \equiv \tilde{\nu}(\underline{\eta}^{(1)},\underline{\eta}^{(2)})/\tilde{\nu}(\underline{\eta}^{(\alpha)}) = \prod_{i\in\mathscr{L}_{\alpha'}} g_i^{(\alpha)}(\eta_i^{(\alpha')};\underline{\eta}^{(\alpha)}),$$

$$\text{with } \alpha'=\alpha+1 \bmod 2, \tag{4.11}$$

where $\tilde{\nu}_\alpha(\underline{\eta}^{(\alpha)})$ is the trace over $\underline{\eta}^{(\alpha')}$ of $\tilde{\nu}(\underline{\eta})$. (An example of this is the standard equilibrium Ising model with nearest neighbor interactions.) We can then do the updatings in two steps, first $\mathscr{L}_1$ and then $\mathscr{L}_2$, and choose the transition rates of our PCA so that detailed balance is satisfied for $\tilde{\nu}$ in the sense of reversibility of the stationary space–time process.

To do this we define the PCA subsystem updatings $P^{(\alpha)}(\underline{\sigma}^{(\alpha)}_{n+1}\,|\,\underline{\sigma}^{(1)}_n, \underline{\sigma}^{(2)}_n)$ as in (2.1). The full updating, which will in general no longer be of the PCA form (2.1), is then given by applying first $P^{(1)}$ then $P^{(2)}$,

$$\hat{P}(\underline{\sigma}_{n+1}\,|\,\underline{\sigma}_n) \equiv P^{(2)}(\sigma^{(2)}_{n+1}\,|\,\underline{\sigma}^{(1)}_{n+1}, \underline{\sigma}^{(2)}_n)\,P^{(1)}(\underline{\sigma}^{(1)}_{n+1}\,|\,\underline{\sigma}^{(1)}_n, \underline{\sigma}^{(2)}_n). \tag{4.12}$$

Its "time reversed adjoint" is

$$\hat{P}^{+}(\sigma_{n+1}\,|\,\underline{\sigma}_n) \equiv P^{(1)}(\underline{\sigma}^{(1)}_{n+1}\,|\,\underline{\sigma}^{(1)}_n, \underline{\sigma}^{(2)}_{n+1})\,P^{(2)}(\underline{\sigma}^{(2)}_{n+1}\,|\,\underline{\sigma}^{(1)}_n, \underline{\sigma}^{(2)}_n). \tag{4.13}$$

The detailed balance condition, which ensures that the stationary stochastic process looks statistically the same when run forwards or backwards in time, is then

$$\hat{P}(\underline{\eta}\,|\,\underline{\eta}')\;\nu(\underline{\eta}') = \hat{P}^{+}(\underline{\eta}'\,|\,\underline{\eta})\;\nu(\underline{\eta}). \tag{4.14}$$

This condition of detailed balance will clearly be satisfied for measures $\tilde{\nu}$ of the form (4.11) whenever

$$P^{(\alpha)}(\underline{\eta}^{(\alpha)}\,|\,\underline{\eta}) = \tilde{\nu}(\underline{\eta}^{(\alpha)}\,|\,\underline{\eta}^{(\alpha')}). \tag{4.15}$$

The dynamics used by Domany and others [4–6] for Ising models on different lattices are of this type with

$$P^{(\alpha)}(\underline{\eta}^{(\alpha)}\,|\,\underline{\eta}') = \prod_{i\in\mathscr{L}_\alpha} \frac{\exp\{-\eta_i[\Sigma\, J_\alpha(i-j)\eta'_j + b_\alpha]\}}{2\cosh[\,\Sigma\, J_\alpha(i-j)\eta'_j + b_\alpha]},\quad j\in\mathscr{L}_{\alpha'}. \tag{4.16}$$

Here b_α is the external magnetic field on sublattice $\mathscr{L}_\alpha$ and $J_\alpha(k)$ is the pair interaction between a spin on $\mathscr{L}_\alpha$ and a spin on $\mathscr{L}_{\alpha'}$ separated by the vector k.

The right side of (4.16) can be generalized by replacing the terms in the square bracket there by the more general form, see (2.11),

$$[\ ] \to Q_i^{(\alpha)}(\underline{\eta}'^{(\alpha')}) + f_i(\underline{\eta}')\eta_i\eta_i' \tag{4.17}$$

where $f_i(\underline{\eta}') = 0$ whenever $Q_i^{(\alpha)}(\underline{\eta}'^{(\alpha')}) \neq 0$ and is constant, independent of $\underline{\eta}'$, when $Q_i^{(\alpha)}(\underline{\eta}'^{(\alpha')}) = 0$, e.g. in the case of (4.16) when $b_\alpha = 0$, $J(i-j) = J$ for nearest neighbor sites, half of which are up. The resulting $\hat{P}$ of (4.12) will then still satisfy detailed balance with respect to the Gibbs measure $\tilde{\nu}$,

$$\tilde{\nu}(\underline{\eta}) = Z^{-1}\exp[-H_d(\underline{\eta})] \tag{4.18}$$

with

$$H_d(\underline{\eta}) = \sum_{i\in\mathscr{L}_1} \eta_i Q_i^{(1)}(\underline{\eta}^{(2)}) + \sum_{i\in\mathscr{L}_2} \eta_i Q_i^{(2)}(\underline{\eta}^{(1)}). \tag{4.19}$$

The generalization (4.17) of the Domany rules permits us to write the stationary measure of a one dimensional PCA which is updated alternately, on the even and odd sites, according to a majority rule of itself and its two neighbors with some noise, as the Gibbs state of a one dimensional Ising model with nearest neighbor interactions. It follows then that this PCA will be ergodic and the corresponding two dimensional ESM will have no phase transitions. The corresponding result for simultaneous updatings of all sites has only been proven recently by Gray [14].

We note here that $\mathscr{L}_1$ and $\mathscr{L}_2$ can be very different sets, like sites and bonds, with bond variables being on and off according to whether $\eta_i = +1$ or -1. The updating rules can also be quite different and the PCA can then serve as a model of a neural net with two different types of elements. Further generalizations to more than two subsets and to cases where the σ_i can take on more than two values are of course also possible.

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Figure Captions

Fig. 1

In this figure the phase plane is parameterized by the noise $p + q$ and the bias $(p - q)/p + q$. Several mean field approximations to the model, whose transition curves are also shown, are discussed in [9], [10].

Figs. 2 and 3

Typical configurations of the Toom PCA with $+$ b.c. on top and $-$ b.c. on the right side at two different noise levels ε, $\varepsilon = .02$ and $\varepsilon = .09$.

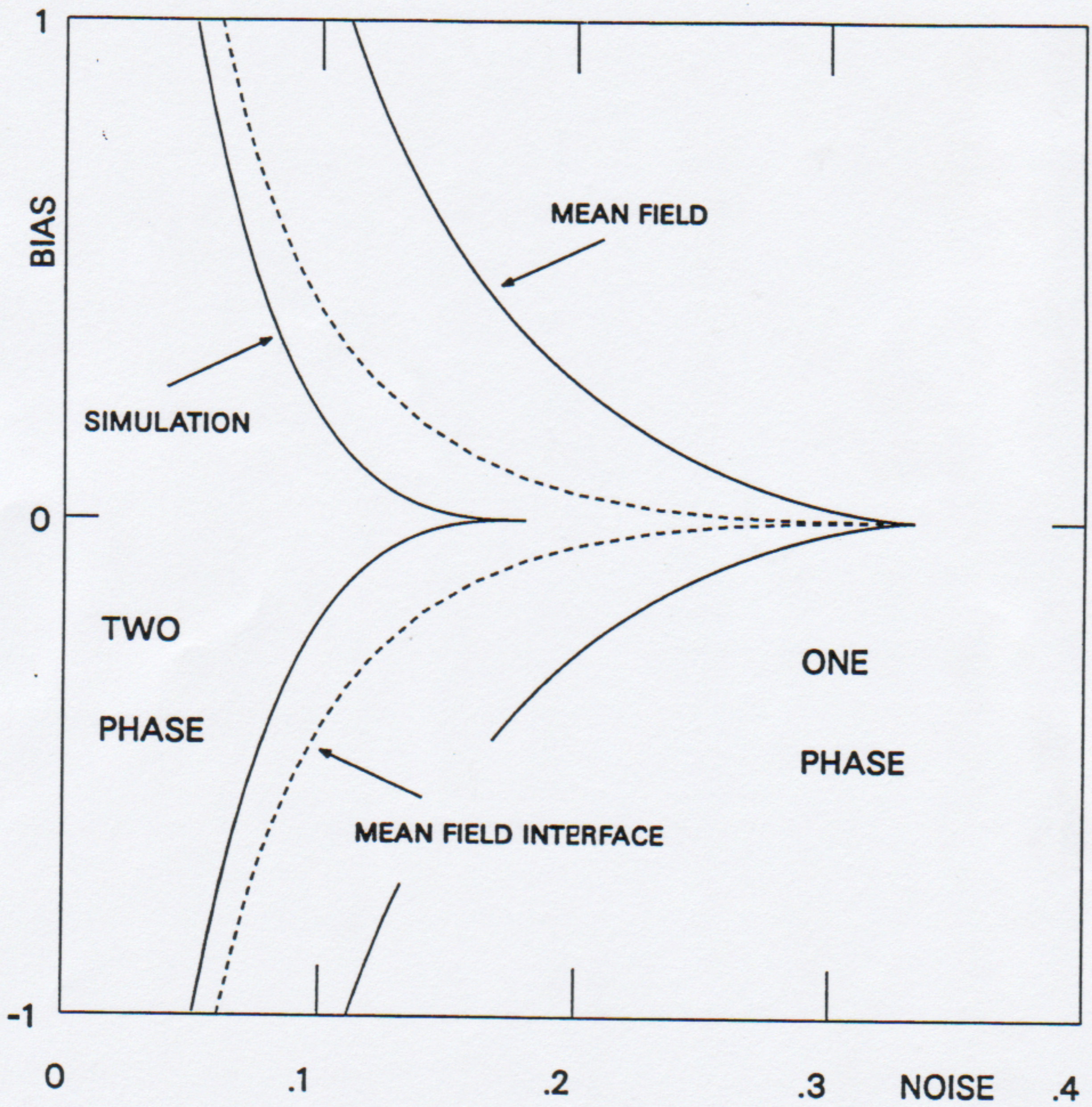
1
0
-1
BIAS
MEAN FIELD
SIMULATION
TWO
PHASE
ONE
PHASE
MEAN FIELD INTERFACE
0
.1
.2
.3
NOISE
.4

FIGURE 1

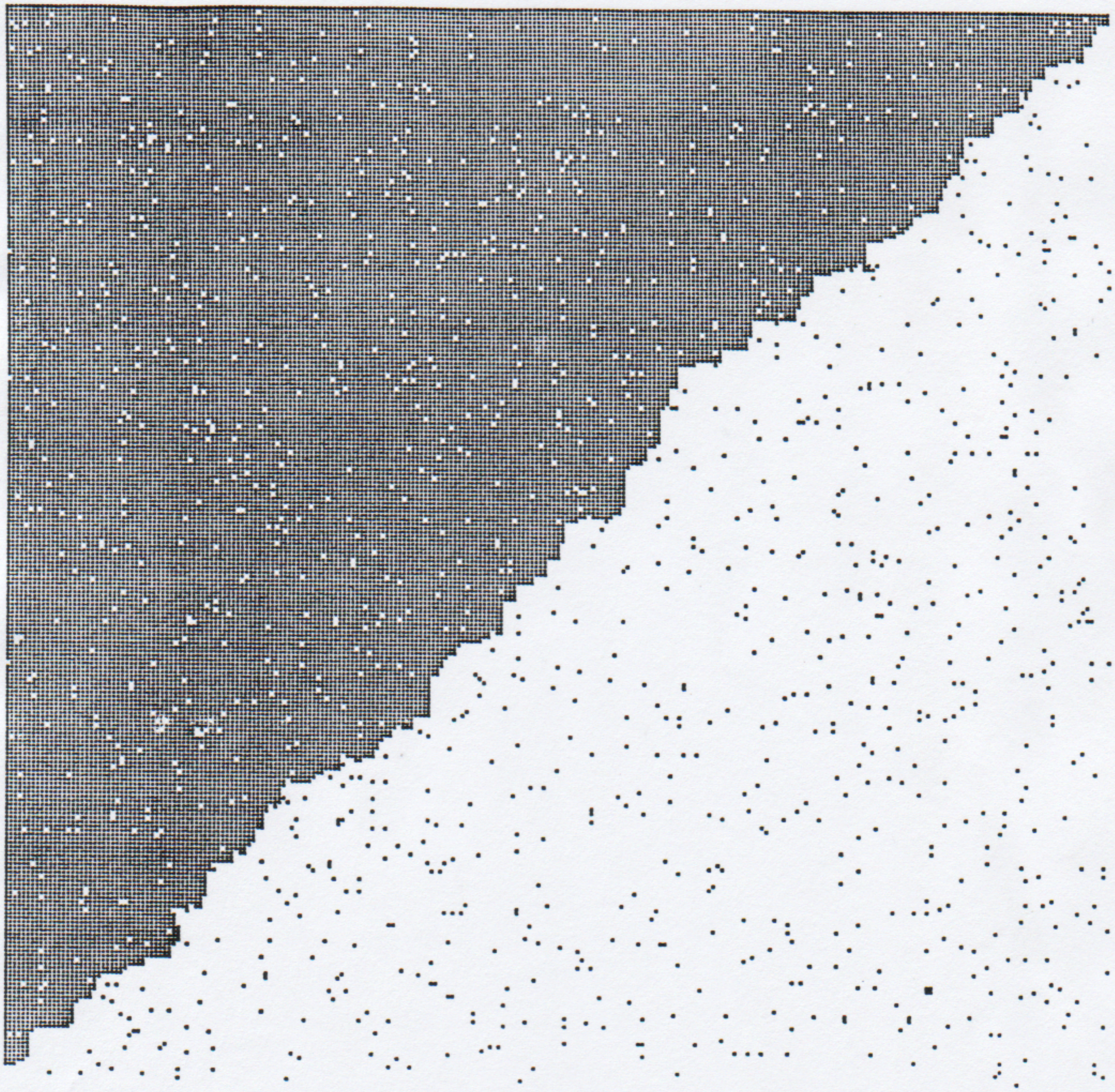

FIGURE 2

FIGURE 3