

RESEARCH ARTICLE | FEBRUARY 01 2012

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J. Math. Phys. 53, 023301 (2012)

<https://doi.org/10.1063/1.3679069>



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Proof of rounding by quenched disorder of first order transitions in low-dimensional quantum systems

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(Received 19 April 2011; accepted 4 January 2012; published online 1 February 2012)

We prove that for quantum lattice systems in $d \leq 2$ dimensions the addition of quenched disorder rounds any first order phase transition in the corresponding conjugate order parameter, both at positive temperatures and at $T = 0$. For systems with continuous symmetry the statement extends up to $d \leq 4$ dimensions. This establishes for quantum systems the existence of the Imry–Ma phenomenon which for classical systems was proven by Aizenman and Wehr. The extension of the proof to quantum systems is achieved by carrying out the analysis at the level of thermodynamic quantities rather than equilibrium states. © 2012 American Institute of Physics. [doi:[10.1063/1.3679069](https://doi.org/10.1063/1.3679069)]

I. INTRODUCTION

Quenched disorder is known to have a pronounced effect on phase transitions in low dimensional systems. As Imry and Ma pointed out in 1975,¹ disorder can prevent the appearance of discontinuities, and of long range order, associated with first order transitions. Nonetheless, for a number of years there was considerable uncertainty about the mechanisms involved in such a “rounding effect,” and consequently also of the circumstances in which this effect appears.² Rigorous results in a series of works in the 1980s^{3–8} established the conditions for rounding in classical systems. The situation in quantum systems remained uncertain, with recent suggestions⁹ that for such the effect might not have the same general character as in the context of classical statistical mechanics.

In this work we present a modified version of the argument of Ref. 8 which allows us (as previously announced in Ref. 10) to extend the classical results to quantum systems. It was not obvious that this should be possible, since the statistical mechanical description of a d dimensional quantum systems often has the appearance of a $(d + 1)$ dimensional classical system, and, in addition, some of the tools used in the classical case have no clear quantum generalizations. The latter difficulty is resolved here by focusing on thermodynamic quantities, bypassing some of the subtle issues related to equilibrium states which played a role in Ref. 8.

Before presenting the formalism in which the general result is expressed, we start with some specific examples to which the results apply.

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II. EXAMPLES OF THE ROUNDING EFFECT

A. Transverse field Ising model

The transverse-field Ising model (TFIM) in a random longitudinal field is defined by the Hamiltonian

$$\mathcal{H}^{\text{TFIM}} = -J \sum_{\langle x,y \rangle} \sigma_{3,x} \sigma_{3,y} - \lambda \sum \sigma_{1,x} - \sum (h + \epsilon \eta_x) \sigma_{3,x}, \quad (2.1)$$

where $\sigma_{j,x}$ are Pauli matrices describing the $j = 1, 2, 3$ components of a spin at site x , and the first summation is over pairs of neighboring sites $x, y \in \mathbb{Z}^d$. The symbol η_x represents a random field in the $j = 3$ direction, whose values at different sites are given by independent and identically distributed (i.i.d.) random variables, satisfying certain mild regularity conditions (to be specified in Sec. IV B below).

In $d \geq 2$ dimensions the nonrandom version of the system (with $\epsilon = 0$) exhibits spontaneous (or residual) magnetization at $h = 0$ if J/λ and $J\beta$ are large enough. In $d = 1$ dimension this occurs only at zero temperature, $T \equiv \beta^{-1} = 0$, i.e., in the ground state. More explicitly, existence of *spontaneous magnetization* means that

$$\overline{M}(\beta) := \frac{1}{2} [M_{0+}(\beta) - M_{0-}(\beta)] > 0, \quad (2.2)$$

where $M_{0\pm}(\beta)$ are the two limits of the Gibbs state expectation values in finite (rectangular) domains $\Gamma \subset \mathbb{Z}^d$,

$$M_{0\pm}(\beta) := \lim_{h \rightarrow 0 \pm} \lim_{\Gamma \nearrow \mathbb{Z}^d} \frac{1}{|\Gamma|} \left\langle \sum_{x \in \Gamma} \sigma_{3,x} \right\rangle_{\Gamma}^{h,\beta}. \quad (2.3)$$

Since the h limit is taken last, the boundary conditions on the finite systems do not affect the value of the limits in (2.3) (for the convenience of translation covariance our default choice is periodic boundary conditions). In view of the symmetry of the model, $M_{0+} = -M_{0-} = \overline{M}(\beta)$.

The general result which is proven below (Theorem 4.1) implies:

In $d \leq 2$, the system described above, with the Hamiltonian (2.1), at $\epsilon \neq 0$ has $\overline{M}(\beta) = 0$ at all $\beta \leq \infty$. In particular, even if there is more than one ground state, they all yield the same bulk-average value for the mean magnetization.

This can also be restated in thermodynamic terms: Letting $\mathcal{F}(h, \beta)$ denote the thermodynamic limit of the specific free energy, which is defined below in (3.6), the different magnetizations can be expressed as directional derivatives¹¹

$$M_{0\pm}(\beta) = \left. \frac{\partial \mathcal{F}}{\partial h \pm} \right|_{h=0}. \quad (2.4)$$

Non-vanishing spontaneous magnetization corresponds therefore to a discontinuity of a first order derivative of the free energy, here at $h = 0$, and thus to a first order phase transition in the terminology of Ehrenfest.¹²

Note that the order parameter for the phase transition is the expectation of the volume average of the local operator $\sigma_{3,x}$ which in the Hamiltonian (2.1) appears coupled to the random field. Such conjugacy is the essential condition for the general result presented here. The theorem does not imply that the rounding of the magnetization in the “ z -direction” would occur when random terms are added only to the transverse field λ . Indeed, as has long been known,¹³ when only such disorder is present, spontaneous magnetization does persist in two dimensions. Such transversal disorder can have other subtle effects on the ferromagnetic-paramagnetic transition,¹⁴ but these are beyond the scope of the present work.

B. Isotropic Heisenberg model

The isotropic, nearest-neighbor Heisenberg ferromagnet in a random magnetic field is described by the Hamiltonian

$$\mathcal{H}^{\text{Heis}} = -J \sum_{\langle x, y \rangle} \vec{\sigma}_x \cdot \vec{\sigma}_y - \sum_x (\vec{h} + \epsilon \vec{\eta}_x) \cdot \vec{\sigma}_x \quad (2.5)$$

where J is a positive real number, $\vec{\sigma}_x$ is a vector spin operator associated with the site $x \in \mathbb{Z}^d$, and $\vec{\eta}_x$ is a magnetic field which varies randomly from site to site. We now assume that these random field variables are not only independently and identically distributed, but also that their distribution is rotation invariant. We also assume that $\vec{\eta}_x$ is nonzero with probability 1.

In the case of a uniform magnetic field ($\epsilon = 0$) it is universally believed that for $d \geq 3$ this system exhibits spontaneous magnetization at $\vec{h} = \vec{0}$ below some J -dependent critical temperature (although a rigorous proof is lacking). Letting $\langle \cdot \rangle_{\Gamma}^{\vec{h}, \beta}$ now denote the Gibbs state of this system on a finite domain $\Gamma \subset \mathbb{Z}^d$ with periodic boundary conditions at inverse temperature β , spontaneous magnetization can be expressed, similarly to the previous cases, by the statement that

$$\vec{M}_0(\beta) := \lim_{h \searrow 0} \lim_{\Gamma \nearrow \mathbb{Z}^d} \frac{1}{|\Gamma|} \left\langle \sum_{x \in \Gamma} \vec{\sigma}_x \right\rangle_{\Gamma}^{h\hat{e}, \beta} \neq 0, \quad (2.6)$$

where $\hat{e}$ is any fixed unit vector (indicating the direction along which $\vec{h}$ is taken to $\vec{0}$). In terms of the thermodynamic limit of the specific free energy the same quantity can be expressed as¹¹

$$\vec{M}_0(\beta) = \lim_{h \searrow 0} \frac{d}{dh} \mathcal{F}(h\hat{e}, \beta). \quad (2.7)$$

Note that as in the previous example the randomness is conjugate to the quantity whose density is the magnetization considered here.

Theorem 4.2, below, has the following implication for this system at $\epsilon \neq 0$:

In $d \leq 4$, the random-field Heisenberg ferromagnet described above has $\vec{M}_0(\beta) = 0$ (and hence $\mathcal{F}(\vec{h}, \beta)$ is differentiable in $\vec{h}$ at $\vec{h} = \vec{0}$) for any β including $\beta = \infty$.

As will be seen in Sec. IV C, the vanishing of the spontaneous magnetization implies also the absence of ferromagnetic long range order.

III. A GENERAL FORMULATION

A. The system and its Hamiltonian

We consider here systems on homogeneous d dimensional lattices, which for simplicity we take to be $\mathbb{Z}^d$. Associated with each lattice site $x \in \mathbb{Z}^d$ is a quantum system whose state space is isomorphic to a common finite dimensional Hilbert space $\mathcal{H}_0$. The local systems are coupled through a Hamiltonian which for a finite region $\Gamma \subset \mathbb{Z}^d$, with free boundary conditions, takes the form

$$H_{\Gamma,0}^{h,\epsilon,\eta} = \sum_{A \subset \Gamma} Q_A + \sum_{x \in \Gamma} (h + \epsilon \eta_x) \kappa_x. \quad (3.1)$$

Here Q_A is an operator which acts on the quantum degrees of freedom in A (i.e., it is described by an operator acting in the space $\otimes_{x \in A} \mathcal{H}_x$). The interaction is assumed to be translation invariant ($Q_{T_x A} = U_x^\dagger Q_A U_x$, with T_x denoting translations on the lattice and U_x the corresponding unitary operators). Unless stated otherwise it is assumed here that the interactions are of finite range (i.e., $Q_A = 0$ for all sets with $\text{diam } A > R$, at some finite R). However, the stated results apply also to a broader class of long range interactions with a sufficiently fast power law decay, which is described in the Appendix. In particular, Theorem 4.1 applies to two-body interactions satisfying $\|Q_{x,y}\| \leq C/|x - y|^{3d/2}$, and Theorem 4.2 applies if $\|Q_{x,y}\| \leq C/|x - y|^\alpha$ with some $\alpha > d - 2$.

Disorder is expressed in (3.1) through the random coefficients η_x . Their variance is fixed at 1, so that the strength of the disorder is controlled by the parameter ϵ . The terms η_x multiply are assumed to be of the form $\kappa_x \equiv U_x^\dagger \kappa_0 U_x$, with κ_0 an operator which acts on a finite cluster of sites, whose

size may be greater than 1. It should be noted that $\{\eta_x\}$ appear in (3.1) as additions to a uniform parameter h . This parameter plays an essential role for the results presented here. The symbol η without the subscript indicates the collection of random fields at all sites, and η_Γ denotes the random fields in some finite region $\Gamma \subset \mathbb{Z}^d$. In systems with several families of independent random terms our results hold for the observable associated with each family considered separately, but we will not discuss this point here (see Ref. 15).

Hamiltonians with other boundary conditions, $H_{\Gamma,B}^{h,\epsilon,\eta}$, are defined analogously, so that the terms within Γ are the same as for free boundary conditions, while at a finite distance from the boundary there can be additional terms which are uniformly bounded in norm and in the size of the clusters of the directly affected sites. Among the allowed options are periodic boundary conditions, denoted by $B = \text{per}$, which will be the default choice if no other is explicitly specified.

As usual, the finite volume partition function is denoted

$$Z_{\Gamma,B}(h, \beta, \epsilon; \eta) := \text{Tr} \exp(-\beta H_{\Gamma,B}^{h,\epsilon,\eta}), \quad (3.2)$$

and the corresponding Gibbs state is

$$\langle \dots \rangle_{\Gamma,B}^{h,\beta,\epsilon}(\eta) := \frac{\text{Tr} \left(\dots e^{-\beta H_{\Gamma,B}^{h,\epsilon,\eta}} \right)}{Z_{\Gamma,B}(h, \beta, \epsilon; \eta)}. \quad (3.3)$$

When the values of the superscripts is deemed obvious from the context, they will sometimes be omitted.

B. The quenched free energy and its infinite volume limit

The free energy for a finite system is denoted here by

$$F_{\Gamma,B}(h, \beta, \epsilon; \eta) := \frac{-1}{\beta} \log Z_{\Gamma,B}(h, \beta, \epsilon; \eta), \quad (3.4)$$

We will mostly be concerned with square domains $\Gamma_K = [-K, K]^d \cap \mathbb{Z}^d$ with the periodic boundary conditions, and so we introduce the abbreviation

$$F_K^h(\eta) \equiv F_{\Gamma_K, \text{per}}(h, \beta, \epsilon; \eta). \quad (3.5)$$

The free energy density, per unit volume, is

$$\mathcal{F}_{\Gamma,B}(h, \beta, \epsilon; \eta) := F_{\Gamma,B}(h, \beta, \epsilon; \eta)/|\Gamma|. \quad (3.6)$$

where $|\Gamma|$ is the number of sites in Γ . As the notation suggests, for finite systems the free energy depends on the choice of boundary conditions and on the disorder variables. In the thermodynamic limit, however, the dependence of its density on the boundary conditions disappears and so does the dependence of its typical value on the disorder variables. The following generally known result (we refer here to such statements as propositions) applies to all systems of the type described above.

Proposition 3.1: If the random fields are of finite variance and form a translation invariant and ergodic process, then for any $\beta \in [0, \infty]$ there is a full measure set $\mathcal{N}$ of field configurations for which the infinite volume limit

$$\mathcal{F}(h, \beta, \epsilon) := \lim_{L \rightarrow \infty} \mathcal{F}_{\Gamma_L, B}(h, \beta, \epsilon; \eta) \quad (3.7)$$

exists for all h and its value is independent of η and the boundary conditions B .

Since much of the early discussion of this result^{16,17} was limited to the classical case let us add that the argument used in that context extends directly also to quantum statistical mechanics. In essence: the free energy density can be approximated up to an arbitrarily small correction, of order $O(L_0^{-1})$ (see Inequality (6.1)) by that of a system obtained by partitioning the space into large blocks (of length L_0) among which the couplings were removed. For the approximants the density converges by either the law of large numbers, in case of independent $\{\eta_x\}$, or by the ergodic theorem in the more general case stated above. That implies Proposition 3.1.

The following remarks are of relevance:

(i) Since convexity is automatically inherited by the limit, $\mathcal{F}(h, \beta, \epsilon)$ is concave as a function of h .

(ii) The case $\beta = \infty$ corresponds to the ground state energy: for almost every η the limit

$$\mathcal{F}(h, \infty, \epsilon) := \lim_{\beta \rightarrow \infty} \mathcal{F}(h, \beta, \epsilon) \quad (3.8)$$

exists and is equal to the limit of the energy densities of the finite volume ground states of the random Hamiltonian (whose value is independent of the boundary conditions). In other words, for the free energy the limits $\beta \rightarrow \infty$ and $L \rightarrow \infty$ are interchangeable.

(iii) The uniqueness of the free energy density does not extend to uniqueness of the Gibbs states, or ground states in case $\beta = \infty$. The question which our results address is whether the different Gibbs states (or ground states) of the given Hamiltonian can differ in their mean magnetization, that is in the volume averages of $\langle \kappa_x \rangle$.

(iv) The following condition will allow us to weaken other assumptions. We will say that our system satisfies the *weak Fortuin-Kasteleyn-Ginibre (FKG) condition with respect to κ* if

$$\langle \kappa_x \rangle_{\Gamma}^h(\eta) \geq \langle \kappa_x \rangle_{\Gamma}^{h'}(\eta'), \quad (3.9)$$

whenever $x \in \Gamma$, $h \geq h'$, and $\eta_y \geq \eta'_y$ for all $y \in \Gamma$. This condition is known to hold in the transverse field Ising model described in Sec. II A.¹³

C. Quenched disorder with continuous symmetry

For systems with a continuous symmetry, the Hamiltonian (with the free boundary conditions) is of the form

$$H_{\Gamma,0}^{\vec{h},\epsilon,\vec{\eta}} = \sum_{A \subset \Gamma} Q_A + \sum_{x \in \Gamma} (\vec{h} + \epsilon \vec{\eta}_x) \cdot \vec{\kappa}_x, \quad (3.10)$$

where the Q interaction is not only translation invariant but also rotation invariant in the following sense. We assume that the group of rotations $SO(N)$ has a representation by local unitary operators whereby for each $R \in SO(N)$ there is a corresponding $\hat{R}_x$ acting in $\mathcal{H}_x$. The interaction Q then is rotation invariant if and only if

$$Q_A = \left(\prod_{x \in A} \hat{R}_x^{-1} \right) Q_A \left(\prod_{x \in A} \hat{R}_x \right) \quad (3.11)$$

for all finite $A \subset \mathbb{Z}^d$ and all $R \in SO(N)$. At each site $x \in \mathbb{Z}^d$ instead of a single operator κ_x there is now a collection of operators $\vec{\kappa}_x = (\kappa_{x,1}, \dots, \kappa_{x,N})$ which under the above action of $SO(N)$ transform as a vector. That is: for each $R \in SO(N)$ and $\vec{v} \in \mathbb{R}^N$

$$\hat{R}_x^{-1}(\vec{v} \cdot \vec{\kappa}_x) \hat{R}_x = (R\vec{v}) \cdot \vec{\kappa}_x. \quad (3.12)$$

Hamiltonians with other boundary conditions, Gibbs states, etc., are defined in analogy with the previous section. (In the classical case,⁸ it was also necessary to assume that the interactions transformed smoothly under nonuniform rotations, however, for a finite-dimensional local state space this is automatically true.)

Likewise, the random fields are now given by random vectors $\vec{\eta}_x$ in $\mathbb{R}^N$. Clearly, rotation invariance is lost in the presence of such terms in the Hamiltonian. However, we will assume that the symmetry is retained in the distributional sense, i.e., that for any rotation R the random variables $\vec{\eta}_x$ and $R\vec{\eta}_x$ have the same distribution.

IV. STATEMENT OF THE MAIN RESULTS

A. Two perspectives on 1st order phase transitions

As was done in the context of the examples of Sec. II, the general statements will be presented in two equivalent ways: in their thermodynamic formulations, which is the level at which the results

are derived here, and then in the language of statistical mechanics, i.e., properties of Gibbs states. Since the results concern (the absence of) first order phase transitions let us recall the latter's dual manifestations.

1. In thermodynamic terms, a 1st order phase transition is associated with the discontinuity of the first derivative of the free energy with respect to one its parameters. By default this parameter will be denoted here by h .
2. In the terms of statistical mechanics a 1st order phase transition is expressed in the non-uniqueness, among the infinite volume Gibbs equilibrium states, of the bulk density of some extensive quantity. Our discussion concerns the case when this is the quantity whose coupling parameter in H is the field h which is randomized by the disorder.

One occasionally finds that the equivalence of these two statements is not fully trusted in the present context (of disordered systems).¹⁸ We shall therefore briefly recall below, in Sec. IV C, some pertinent known results.

B. Thermodynamic formulation

Our first result concerning systems of the type described in Sec. III A, with η_x given by i.i.d. random variables, is

Theorem 4.1: *In dimensions $d \leq 2$, assuming the variables η_x are i.i.d. with absolutely continuous distribution and with more than two finite moments, the quenched free energy density $\mathcal{F}$ is differentiable in h at all values of h , $\epsilon \neq 0$, and $\beta \leq \infty$.*

Furthermore, for $\beta < \infty$, the assumption of absolute continuity can be relaxed, requiring instead that the distribution of η has no isolated point masses, or alternatively that the system satisfies the weak FKG property with respect to κ .

Let us recall that a probability distribution of a variable q is said to be *absolutely continuous* if it is of the form $\rho(q)dq$, with a density function $\rho(q)$. In particular, the corresponding measure on $\mathbb{R}$ has neither “delta function” terms (point masses), nor any component which is supported on a Cantor type fractal set. Point masses are isolated if they are separated from the continuous part of the distribution. The statement that the random variables η_x have more than two finite moments means that

$$\text{Av } |\eta_x|^{2+\delta} < \infty \quad (4.1)$$

for some $\delta > 0$, where Av denotes the average over the random fields.

As in the classical case, in the presence of continuous symmetry, as understood above, the “rounding effect” extends to higher dimensions:

Theorem 4.2: *In dimensions $d \leq 4$, any isotropic system of the type described in Sec. III C, with the random terms being independent with an identical, rotation invariant distribution with more than two finite moments and $\text{Prob}(\vec{\eta} = \vec{0}) = 0$, the quenched free energy density $\mathcal{F}$ is differentiable in $\vec{h}$ at $\vec{h} = \vec{0}$, for any $\epsilon \neq 0$ and $\beta \leq \infty$.*

For $\beta < \infty$, the conclusion still holds if there is a nonzero probability of a random term being zero.

We note that an apparently weaker condition on the distribution of $\vec{\eta}$ compared to Theorem 4.1 is adequate because what will ultimately be important is the distribution of the component in an arbitrary direction. With the assumption of an isotropic distribution for the vector, the components can easily be seen to satisfy the stronger conditions used in Theorem 4.1 or Proposition 7.2.

C. Statistical mechanical implications (no long range order)

In this section, we establish the relationship between phase transitions (understood in terms of non-differentiability of free energy) and long range order, culminating in corollaries of Theorems 4.1 and 4.2 which reframe the results in statistical mechanics terms.

We say that a system exhibits long range order with order parameter κ (for some set of parameters and disorder variables) if the mean value of the bulk averages $\bar{\kappa}_\Lambda = |\Lambda|^{-1} \sum_{x \in \Lambda} \kappa_x$, for $\Lambda \rightarrow \mathbb{Z}^d$, depends on the boundary conditions B with which the infinite volume state is constructed. That is: if there are two sequences of cubic domains Λ and Γ increasing to $\mathbb{Z}^d$, and two sets of boundary conditions which yield different values for

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \lim_{\Gamma \nearrow \mathbb{Z}^d} \langle \bar{\kappa}_\Lambda \rangle_{\Gamma, B}. \quad (4.2)$$

When the sensitivity to the boundary conditions is found for the full bulk averages, i.e., if different values occur for

$$\lim_{\Gamma \nearrow \mathbb{Z}^d} \langle \bar{\kappa}_\Gamma \rangle_{\Gamma, B}, \quad (4.3)$$

we say that there is long-long range order.

Remark: If for a given system there are multiple limits of the expectation values in (4.2), then the system has multiple infinite volume Kubo-Martin-Schwinger (KMS) states.¹⁹ Convex combinations of the different states (which will also form KMS states) will exhibit non-decaying correlations of κ : $\langle \kappa_x \kappa_y \rangle - \langle \kappa_x \rangle \langle \kappa_y \rangle \not\rightarrow 0$ as $|x - y| \rightarrow \infty$. The latter condition provides another aspect of (“short”-) long range order, which is often used as its definition.

The basis of the connection to the thermodynamic quantities discussed in Sec. III B is the relation

$$\frac{\partial \mathcal{F}_{\Gamma, B}^h(\eta)}{\partial h} = \frac{1}{|\Gamma|} \sum_{x \in \Gamma} \langle \kappa_x \rangle_{\Gamma, B}^h(\eta). \quad (4.4)$$

A useful implication of convexity is that if a sequence of differentiable convex functions, such as $\mathcal{F}$ of Theorem 3.1, converges pointwise (in h) then also their derivatives converge to the derivative of the limiting function, wherever that function is differentiable. Without assuming differentiability of the limiting function one may still conclude that all the derivatives’ accumulation points lie in the interval spanned by the left and right derivative of the limiting function. (An elementary proof of that can be obtained by considering the relations among quotients of the form $[\mathcal{F}(h_2) - \mathcal{F}(h_1)]/[h_2 - h_1]$ for suitably chosen collections of intervals $[h_1, h_2]$.) This has the following relevant implication.

Proposition 4.3: Under the assumptions of Proposition 3.1, for any set of the parameters (β, h, ϵ) at which $\mathcal{F}$ is differentiable in h

$$\lim_{L \rightarrow \infty} \frac{1}{|\Gamma_L|} \sum_{x \in \Gamma_L} \langle \kappa_x \rangle_{\Gamma_L, B}^h(\eta) = \frac{\partial \mathcal{F}}{\partial h} \quad (4.5)$$

for almost every realization of the disorder η , and any choice of the boundary conditions B . Furthermore, also

$$\lim_{L \rightarrow \infty} \lim_{K \rightarrow \infty} \frac{1}{|\Gamma_L|} \sum_{x \in \Gamma_L} \langle \kappa_x \rangle_{\Gamma_K, B}^h(\eta) = \frac{\partial \mathcal{F}}{\partial h}. \quad (4.6)$$

Without assuming differentiability, one may still conclude that

$$\text{LIM}_{L, K \rightarrow \infty; K \geq L} \frac{1}{|\Gamma_L|} \sum_{x \in \Gamma_L} \langle \kappa_x \rangle_{\Gamma_K, B}^h(\eta) \in \left[\frac{\partial \mathcal{F}}{\partial h-}, \frac{\partial \mathcal{F}}{\partial h+} \right], \quad (4.7)$$

with LIM denoting the collection of accumulation points, for different boundary conditions, and possibly different sequences of volumes, and $\frac{\partial}{\partial h \pm}$ denoting one-sided derivatives with respect to h .

Of particular relevance for us is the conclusion that when $\mathcal{F}$ is differentiable there is only one possible value for the limit (4.2), which by Proposition 3.1 is independent of the boundary conditions. For classical systems more can be said: if $\mathcal{F}$ is differentiable not only do the mean values of the observable $\bar{\kappa}_\Gamma$ converge, but the distribution of this quantity with respect to the Gibbs state collapses onto a point. Such a stronger statement concerning quantum fluctuations is not known to be true. Nevertheless, Eq. (4.5) holds also for the quantum systems.

Proof: The relations (4.5) and (4.7) follow directly by the above-stated property of convex functions (i.e., the implication for the derivatives of convex functions' pointwise convergence).

To prove (4.6) and the corresponding extension of (4.7), let $F_{\Gamma,B}^{h,\delta,\Lambda}$ denote the free energy with the fixed field within Λ changed by δ , so that

$$\frac{1}{|\Gamma_L|} \sum_{x \in \Gamma_L} \langle \kappa_x \rangle_{\Gamma_K,B}^h(\eta) = \frac{1}{|\Gamma_L|} \left. \frac{\partial F_{\Gamma_K,B}^{h,\delta,\Lambda}}{\partial \delta} \right|_{\delta=0}. \quad (4.8)$$

A standard estimate, which is presented in Proposition 6.1 below, implies that

$$F_{\Gamma_K,B}^{h,\delta,\Gamma_L} - F_{\Gamma_K,B}^h = F_{\Gamma_L,0}^{h+\delta} - F_{\Gamma_L,0}^h + O(\|V_{\Gamma_L}^\eta\|) \quad (4.9)$$

uniformly in K , where V_L^η denotes the terms in the Hamiltonian crossing the boundary of Γ_L , i.e.,

$$V_L^\eta := \sum_{A: A \cap \Gamma \neq \emptyset, A} Q_A + \sum_{x \in \partial_0 \Gamma} (h + \epsilon \eta_x) \kappa_x. \quad (4.10)$$

Then for any η in the full measure set $\mathcal{N}$ on which the limits in Theorem 3.1 are defined, this implies that

$$\lim_{L, K \rightarrow \infty; K \geq L} \left(F_{\Gamma_K,B}^{h,\delta,\Gamma_L} - F_{\Gamma_K,B}^h \right) / |\Gamma_L| = \mathcal{F}(h + \delta, \beta) - \mathcal{F}(h, \beta). \quad (4.11)$$

each of these differences is a concave function of δ , so the above mentioned convexity argument gives

$$\text{LIM}_{L, K \rightarrow \infty; K \geq L} \frac{\partial}{\partial \delta} F_{\Gamma_K,B}^{h,\delta,\Gamma_L} / |\Gamma_L| \in \left[\frac{\partial \mathcal{F}}{\partial h-}, \frac{\partial \mathcal{F}}{\partial h+} \right], \quad (4.12)$$

and the desired results follow immediately from Eq. (4.8). $\square$

In particular, the results which are formulated in Theorems 4.1 and 4.2 as statements of differentiability of the free energy directly imply the following.

Theorem 4.4: *In dimensions $d \leq 2$, under the assumptions of Theorem 4.1 the system almost certainly does not exhibit (short or long) long range order.*

Furthermore, for systems with continuous symmetry which meet the assumptions of Theorem 4.2, in dimensions $d \leq 4$ almost certainly there is no (short or long) long range order at $h = 0$.

V. THE FREE-ENERGY-DIFFERENCE FUNCTIONAL

The above discussion reduces our main results to Theorems 4.1 and 4.2. A key tool for the proof of these theorems is a quantity which describes the differences in the finite volume content of free energies between two of the system's equilibrium states, which are constructed to have the two extremal mean magnetizations, $\frac{\partial \mathcal{F}}{\partial h+}$ and $\frac{\partial \mathcal{F}}{\partial h-}$, assuming they are not the same. The difference, which is denoted below as G_L , will be shown to satisfy contradictory bounds: (i) an absolute upper bound on $|G_L|$ which is due to the observation that the two free energies cannot differ by more than the magnitude of the interaction across the finite volume boundary, (ii) a lower bound on the fluctuations of this quantity which reflects the idea that with a systematic difference in the magnetization the states' free energies will respond differently to the fluctuating random fields.

More specifically, the free energy differences (properly defined) will be shown to satisfy

$$|G_L| \leq CL^{d-1} + DL^{d/2}, \quad (5.1)$$

with $D = 0$ for finite range interactions. In the presence of continuous symmetry for finite range interactions the bound is improved to

$$|G_L| \leq CL^{d-2}. \quad (5.2)$$

For the opposite bound it will be shown that if there is a first order transition then $G_L/L^{d/2}$ converges to a normally distributed random variable with variance $b > 0$:

$$\frac{G_L}{L^{d/2}} \rightarrow N(0, b). \quad (5.3)$$

The construction of a quantity with the above characteristics is the main subject of this section. The contradiction between the two bounds yields the main result.

Estimates similar to the above form the basis of the argument of Imry and Ma,¹ and the more precise argument outlined above is similar to the one which was employed for the analysis of in the classical case.^{7,8} However, the technique introduced there for the construction of the auxiliary quantity G_L employed probability measures over states, defining what has since been called metastates.²⁰ Unfortunately, although metastates can be defined for quantum systems,^{21,22} not all the steps taken in Ref. 8 for the construction of G_L have such an extension. The progress made in this work is enabled by the observation that one can define a suitable quantity G_L in terms of just free energy differences, avoiding more delicate issues of quantum states. This makes it possible to formulate a proof parallel to the classical case (and indeed the nonrigorous arguments which inspired it).

As a simple illustration of the concept one may first consider the zero temperature case of a system such as the random field Ising model with two distinct states, labeled by $+$ and $-$, of different mean magnetizations and with only the “canonical dependence” on η (cf. Ref. 8). In that case, for G_L one may take the difference in the two states’ energy contents in finite volumes

$$\begin{aligned} E_L(\eta) &:= E_{+,L}(\eta) - E_{-,L}(\eta) = \epsilon \left\langle \sum_{x \in \Gamma_L} \eta_x \kappa_x \right\rangle_+ - \epsilon \left\langle \sum_{x \in \Gamma_L} \eta_x \kappa_x \right\rangle_- \\ &\approx 2\epsilon \left(\sum_{x \in \Gamma_L} \eta_x \right) M \rightarrow 2\epsilon ML^{d/2} N(0, 1), \end{aligned} \quad (5.4)$$

where the last step, in which the central limit theorem is invoked, is valid only to the extent that one may ignore the adaptation of the system to the disorder. (A more precise statement is possible taking into account the dependence of the $\pm$ states on η (Ref. 8)). The above form provides only a suggestive example in the context of a special (and classical) system at $T = 0$. We turn now to a more general definition which would be suitable for our purpose.

Our choice of $G_L(\eta)$ for the general case is based on the following free energy “second difference” (in the field h , and disorder η_{Γ_L}):

$$\widehat{G}_{L,K}^\delta(\eta_L) := \frac{1}{2} \text{Av} [F_K^{h+\delta}(\eta) - F_K^{h+\delta}(r_L(\eta)) - F_K^{h-\delta}(\eta) + F_K^{h-\delta}(r_L(\eta)) | \eta_L], \quad (5.5)$$

where $r_L\eta$ is the field obtained by setting $\eta_x = 0$ for all $x \in \Gamma_L$, η_L is the restriction of η to Γ_L and $\text{Av} [\cdot | \eta_L]$ is the conditional average over the random terms outside Γ_L , i.e., with η_L held constant. It should be noted that F_K (which is defined in (3.5)) refers to the free energy with the periodic boundary conditions. This choice assures translation covariance, which is mentioned below (condition 4. in Lemma (5.1)) and used in the argument.

When the double limit $\lim_{\delta \searrow 0} \lim_{K \rightarrow \infty} \widehat{G}_{L,K}^\delta(\eta_L)$, exists (for all η_L), the quantity defined by it has properties we desire of $G_L(\eta_L)$. (The order of the limits is important here: taking $\delta \searrow 0$ after the infinite volume limit serves to construct the general version of the $+$ and $-$ states.) Inconveniently, the limits are not generally known (or expected in full generality) to exist. However, compactness arguments can be applied to prove sufficient convergence along subsequences. The essential statement is the following. We use here the ℓ^1 -Lipschitz seminorm $||| \cdot |||$ (in lieu of a

uniform bound on the derivative) which is defined as

$$|||f||| := \sup_{\substack{\eta, \eta' \in \mathcal{E} \\ 0 < \|\eta - \eta'\|_1 < \infty}} \frac{|f(\eta) - f(\eta')|}{\|\eta - \eta'\|_1}, \quad (5.6)$$

with $\|\eta\|_1 := \sum_{x \in \mathbb{Z}^d} |\eta_x|$.

Lemma 5.1: For any $\beta \leq \infty$, there are sequences $K_j \rightarrow \infty$ and $\delta_i \rightarrow 0$ such that the limits

$$G_L(\eta_L) := \lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} \left(\widehat{G}_{L, K_j}^{\delta_i} - \text{Av} \left[\widehat{G}_{L, K_j}^{\delta_i} \right] \right) \quad (5.7)$$

exist for all L and η , and have the following properties:

1. $\text{Av } G_L(\eta_L) = 0$,
2. $G_L(\eta_L)$ depends on the values of η_x only for $x \in \Gamma_L$,
3. $|||G_L||| \leq \epsilon$,
4. $\text{Av} [G_L \eta_\Lambda] = G_{L'}(T_x \eta)$ whenever $T_{-x} \Gamma_{L'} = \Lambda \subset \Gamma_L$,
5. G_1 (that is, G_L with $L = 1$, which is a function of one variable) has a distributional derivative G'_1 satisfying $|||\frac{\partial G_1}{\partial \eta_0}||| \leq \beta \epsilon^2$, and
6. $\text{Av } G'_1(\eta_0) = \frac{\epsilon}{2} \left(\frac{\partial \mathcal{F}}{\partial h_+} - \frac{\partial \mathcal{F}}{\partial h_-} \right)$.

Furthermore, if the system satisfies the weak FKG condition with respect to κ , then G_1 is monotone. For $\beta = \infty$ the same results hold assuming that the distribution of η is absolutely continuous.

The proof is postponed to Sec. VIII, as it is based on compactness arguments paralleling those in Ref. 8. Before turning to it, in Secs. VI and VII we will show that any such $G_L(\eta_L)$ must obey the upper bound (5.1), and where appropriate (5.2), and also satisfies (5.3).

VI. UPPER BOUNDS ON THE FREE ENERGY DIFFERENCE

We will frequently use the following estimate to control the effects of changes in the Hamiltonian on the free energy.

Proposition 6.1 (from Ref. 11): For any Hermitian matrices C, D of the same finite size,

$$|\log \text{Tr } e^C - \log \text{Tr } e^D| \leq \|C - D\| \quad (6.1)$$

where $\|\cdot\|$ is the operator norm.

As a simple application, this means that the change in free energy for the system defined by the Hamiltonian (3.1) when boundary conditions are added is bounded in terms of the total of the norms of all terms in the interaction which cross the boundary.

A. A general surface bound

The following lemma shows generally that the free energy difference due to random field fluctuations in a finite region can be bounded by the norm of the interaction terms crossing the surface of that region, and that for short range interactions this is proportional to the area of that surface.

Lemma 6.2 Assuming the interactions are either of finite range or satisfy (A1) (of the Appendix), for any pair of sequences $\delta_i \rightarrow 0$, $K_j \rightarrow \infty$ for which the following sequence of functions converges for all η_{Γ_L} :

$$G_L(\eta_L) = \lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} \left(\widehat{G}_{L, K_j}^{\delta_i} - \text{Av} \left[\widehat{G}_{L, K_j}^{\delta_i} \right] \right) \quad (6.2)$$

the limiting function satisfies (with some $C, D < \infty$):

$$|G_L| \leq CL^{d-1} + DL^{d/2}. \quad (6.3)$$

Proof: We will show that

$$|\widehat{G}_{L,K}^\delta| \leq cL^{d-1} + bL^{d/2} + O(\delta L^d) \quad (6.4)$$

with c, b independent of L, K , and δ . The desired result holds then with $C = 2c, D = 2b$.

To see this, we let Λ_L be the smallest cube in $\mathbb{Z}^d$ so that κ_x acts within Λ_L for all $x \in \Gamma_L$, and then split the Hamiltonian into contributions from Λ_L and $\Gamma_K \setminus \Lambda_L$ and terms connecting the two. We consider the quantity

$$F_{K|L}^h(\eta) := -\frac{1}{\beta} \log \text{Tr} \exp \left(-\beta \left[H_{\Lambda_L,0}^{h,\eta} + H_{\Gamma_K \setminus \Lambda_L,*}^{h,\eta} \right] \right), \quad (6.5)$$

where the subscript 0 refers to free boundary conditions, and the subscript * refers to periodic boundary conditions on the edge of Γ_K and free boundary conditions on the edge of Λ_L . The Hamiltonian in Eq. (6.5) differs from $H_{\Gamma_K}^{h,\eta}$ only in the absence of terms connecting Λ_L to the rest of Γ_K , which as in Eq. (4.10) we denote by $V_{\Lambda_L}^\eta$. Then by Proposition 6.1

$$|F_{K|L}^h(\eta) - F_K^h(\eta)| \leq \|V_{\Lambda_L}^\eta\|, \quad (6.6)$$

and therefore

$$\begin{aligned} & |F_K^{h+\delta}(\eta) - F_K^{h+\delta}(r_L(\eta)) - F_K^{h-\delta}(\eta) + F_K^{h-\delta}(r_L(\eta))| \\ & \leq |F_{K|L}^{h+\delta}(\eta) - F_{K|L}^{h+\delta}(r_L(\eta)) - F_{K|L}^{h-\delta}(\eta) + F_{K|L}^{h-\delta}(r_L(\eta))| + 4 \|V_{\Lambda_L}^\eta\|. \end{aligned} \quad (6.7)$$

Each $F_{K|L}$ is a sum of a part from Λ_L and a part from the rest of Γ_L ; the latter cancel each other exactly, and the former can be arranged in two pairs which differ only in the constant field term which has norm of order δL^d . Taking the average conditioned on η_{Γ_L} and noting that the definition of Λ_L makes $V_{\Lambda_L}^\eta$ independent of η_{Γ_L} ,

$$|\widehat{G}_{L,K}^\delta(\eta_L)| \leq 2 \text{Av} \|V_{\Lambda_L}^\eta\| + O(\delta L^d) \quad (6.8)$$

for all η , and since the δ term is uniform in K ,

$$|G_L(\eta_L)| \leq 2 \text{Av} \|V_{\Lambda_L}^\eta\|. \quad (6.9)$$

This reduces Inequality (6.3) to a comparable bound on $V_{\Lambda_L}^\eta$, i.e.,

$$\text{Av} \|V_{\Lambda_L}^\eta\| \leq cL^{d-1} + bL^{d/2}. \quad (6.10)$$

For finite range interactions it is obvious that $\text{Av} \|V_{\Lambda_L}^\eta\| \leq cL^{d-1}$, and the Appendix discusses more general conditions under which Inequality (6.10) holds. $\square$

B. An improved bound for systems with continuous symmetry

For systems with continuous symmetry the stronger bound of Inequality (5.2) can be obtained through the analysis of soft deformations. The argument is similar to the one which was carried out in the classical context in Ref. 8. Although the estimate may at first sight appear to be simply a calculation of the Bloch spin wave energy, that alone would not yield the desired result since the first order term is potentially much larger than the required bound. However, by comparing the distortion energies of opposite deformations one finds that in any situation for one of the distortions the first order term is of the desired sign. Therefore, the analysis can be limited to the second order term – and that yields the claimed bound.

We need to modify the definition of G_L slightly to accommodate random vector fields. The argument will be based on the effect of the fluctuations of a single component, described by expressions where the others have been averaged out. We let

$$\widehat{G}_{L,K}^{\delta\hat{e}}(\vec{\eta}_L) = \frac{1}{2} \text{Av} \left[F_K^{\delta\hat{e}}(\vec{\eta}) - F_K^{\delta\hat{e}}(r_L(\vec{\eta})) - F_K^{-\delta\hat{e}}(\vec{\eta}) + F_K^{-\delta\hat{e}}(r_L(\vec{\eta})) | \hat{e} \cdot \vec{\eta}_L \right], \quad (6.11)$$

$$G_L(\vec{\eta}_L) = \lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} \left(\widehat{G}_{L,K_j}^{\delta_i \hat{e}}(\vec{\eta}) - \text{Av} \left[\widehat{G}_{L,K_j}^{\delta_i \hat{e}}(\hat{e} \cdot \vec{\eta}) \right] \right), \quad (6.12)$$

where $\hat{e}$ is an arbitrary unit vector; note that the rotation symmetry of the system and of the distribution of $\vec{\eta}$ mean that the right-hand side of the last expression is independent of $\hat{e}$.

Lemma 6.3: For any system with continuous symmetry and isotropic disorder as described in Sec. III, of interactions which are either of finite range or satisfy (A7), at $\vec{h} = 0$:

$$|G_L(\vec{\eta}_L)| \leq CL^{d-2}, \quad (6.13)$$

with $C < \infty$, uniformly in L and $\vec{\eta}_L$.

Proof: We shall derive (6.13) through uniform bounds on

$$g_{L,M}^{\delta\hat{e}}(\vec{\eta}) := \text{Av} \left[F_K^{\delta\hat{e}}(\vec{\eta}) - F_K^{-\delta\hat{e}}(\vec{\eta}) | \hat{e} \cdot \vec{\eta}_L \right], \quad (6.14)$$

for which we shall show that assumption (A7) implies

$$|g_{L,K}^{\delta\hat{e}}(\vec{\eta})| = O(L^{d-2}). \quad (6.15)$$

The two quantities are related through

$$G_L(\vec{\eta}_L) = \frac{1}{2} \lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} \left(g_{L,K_j}^{\delta_i \hat{e}}(\vec{\eta}) - \text{Av} g_{L,K_j}^{\delta_i \hat{e}}(\hat{e} \cdot \vec{\eta}) \right), \quad (6.16)$$

and hence (6.15) implies the desired statement.

Let ρ be the generator (in $so(N)$) of a rotation in a plane containing $\hat{e}$, and for each $x \in \mathbb{Z}^d$ let ρ_x be the generator of the corresponding rotation in the single-site algebra $\mathcal{A}_x$ (Unlike in Ref. 10, we will use the “mathematician’s” convention that rotations are given by $e^{\theta\rho}$, so that ρ is an antihermitian matrix and ρ_x is an antihermitian operator). We introduce the slowly varying angles

$$\theta_x := \begin{cases} 0, & x \in \Gamma_L \\ \frac{\|x\|_1 - L}{L} \pi, & 0 < d_L(x) < L \\ \pi, & d_L(x) \geq L \end{cases}, \quad (6.17)$$

where $d_L(x)$ is the distance from x to Γ_L in the largest-component metric, i.e.,

$$d_L(x) := \min_{y \in \Gamma_L} \|x - y\|_\infty. \quad (6.18)$$

We also introduce the associated rotations on fields and on $\mathcal{A}_\Gamma$ defined by

$$\widehat{R}_x := e^{\theta_x \rho} \quad (6.19)$$

$$(R_\theta(\vec{\eta}))_x \equiv R_x \vec{\eta}_x \quad (6.20)$$

$$\widehat{R}_\theta = \bigotimes_{x \in \Gamma} e^{\theta_x \rho_x}. \quad (6.21)$$

where Γ is the relevant finite subset of $\mathbb{Z}^d$ which can be inferred from the context.

$\widehat{R}_\theta$ is unitary, and so we can rewrite the free energy $F_K^{-\delta\hat{e}}(\vec{\eta})$ appearing in (6.14) as

$$F_K^{-\delta\hat{e}}(\vec{\eta}) = -\frac{1}{\beta} \log \text{Tr} \exp \left(-\beta \widehat{R}_\theta^{-1} H_{\Gamma_K}^{-\delta\hat{e}, \vec{\eta}} \widehat{R}_\theta \right). \quad (6.22)$$

At this stage we could carry out a spin-wave analysis, writing

$$\text{Av} [F_K^{-\delta\hat{e}}(\vec{\eta})|\hat{e} \cdot \vec{\eta}_L] = -\frac{1}{\beta} \text{Av} \left[\log \text{Tr} \exp \left(-\beta [H_{\Gamma_K}^{\delta\hat{e},\vec{\eta}} + \Delta H_\theta] \right) \right] |\hat{e} \cdot \vec{\eta}_L|, \quad (6.23)$$

which by Lemma 6.1 implies

$$|g_{L,K}^{\delta\hat{e}}(\vec{\eta})| \leq \|\Delta H_\theta\|. \quad (6.24)$$

However, this will not lead to the desired result, since $\|\Delta H_\theta\|$ includes “first order” terms (in $\nabla\theta$) which scale as L^{d-1} as well as the “second order” terms (and higher) which scale as L^{d-2} . To cancel the former we rewrite $F_K^{-\delta\hat{e}}(\vec{\eta})$ using an opposite rotation and average the two expressions, to obtain

$$\begin{aligned} \text{Av} [F_K^{-\delta\hat{e}}(\vec{\eta})|\hat{e} \cdot \vec{\eta}_L] &= -\frac{1}{2\beta} \text{Av} \left[\log \text{Tr} \exp \left(-\beta [H_{\Gamma_K}^{\delta\hat{e},\vec{\eta}} + \Delta H_\theta] \right) \right. \\ &\quad \left. + \log \text{Tr} \exp \left(-\beta [H_{\Gamma_K}^{\delta\hat{e},\vec{\eta}} + \Delta H_{-\theta}] \right) \right] |\hat{e} \cdot \vec{\eta}_L|. \end{aligned} \quad (6.25)$$

A combination of the Cauchy-Schwarz inequality, the Golden-Thompson inequality²³ and Lemma 6.1 yields the general relation

$$\begin{aligned} \log \text{Tr} e^A - \frac{1}{2} \log \text{Tr} e^B - \frac{1}{2} \log \text{Tr} e^C &= \log \left(\frac{\text{Tr} e^A}{(\text{Tr} e^B)^{1/2} (\text{Tr} e^C)^{1/2}} \right) \leq \log \left(\frac{\text{Tr} e^A}{\text{Tr} e^{B/2} e^{C/2}} \right) \\ &\leq \log \text{Tr} e^{A-(B+C)/2} \leq \left\| A - \frac{B+C}{2} \right\| \end{aligned} \quad (6.26)$$

for arbitrary Hermitian matrices A, B, C . Applying this to Eq. (6.25) gives

$$g_{L,K}^{\delta\hat{e}}(\vec{\eta}) \leq \frac{1}{2} \|\Delta H_\theta + \Delta H_{-\theta}\|. \quad (6.27)$$

We now write out the terms in the rotated Hamiltonian in Eq. (6.22) as

$$\begin{aligned} &\hat{R}_\theta^{-1} H_\Gamma^{\delta\hat{e},\vec{\eta}} \hat{R}_\theta \\ &= \hat{R}_\theta^{-1} \left(\sum_A P_\Gamma(Q_A) + \sum_{x \in \Gamma} (-\delta\hat{e} + \epsilon\vec{\eta}_x) \cdot P_\Gamma(\vec{k}_x) \right) \hat{R}_\theta, \end{aligned} \quad (6.28)$$

where $P_\Gamma(Q_A)$ and $P_\Gamma(\vec{k}_x)$ denote the terms appearing in the Hamiltonian with periodic boundary conditions, obtained by mapping each site outside Γ to the corresponding site within Γ .

Since $\vec{k}$ are vector operators (recall Eq. (3.12)),

$$\vec{\eta}_x \cdot (\hat{R}_\theta^{-1} \vec{k}_x \hat{R}_\theta) = \vec{\eta}_x \cdot R_\theta(\vec{k})_x = [R_\theta^{-1}(\vec{\eta})_x] \cdot \vec{k}_x; \quad (6.29)$$

inside Γ_L there is no rotation, and outside we are performing an average with respect to an isotropic distribution, so this term makes no contribution to ΔH .

As for the fixed field terms, we have

$$\hat{e} \cdot (\hat{R}_\theta^{-1} \vec{k}_x \hat{R}_\theta) = (R_x^{-1} \hat{e}) \cdot \vec{k}_x. \quad (6.30)$$

The choices of ρ and θ were intended precisely to make $R_x \hat{e} = -\hat{e}$ for $d_L(x) > L$; and for the remaining $(3L)^d$ sites we have $\|\hat{e} \cdot (\hat{R}_\theta^{-1} \vec{k}_x \hat{R}_\theta) + \hat{e} \cdot \vec{k}_x\| \leq 2$, so these terms make a contribution to ΔH which is uniformly bounded in norm by $2(3L)^d \delta$.

We are left with the terms arising from the transformation of the nonrandom interaction. For legibility we let $\Gamma = \Gamma_K$ for the rest of this section. For any A and any (arbitrarily chosen) $x \in A \cap \Gamma$,

$$\begin{aligned}\widehat{R}_{-\theta} P_\Gamma(Q_A) \widehat{R}_\theta &= \left(\bigotimes_{y \in A \cap \Gamma} e^{-(\theta_y - \theta_x) \rho_y} e^{-\theta_x \rho_y} \right) P_\Gamma(Q_A) \left(\bigotimes_{z \in A \cap \Gamma} e^{-\theta_x \rho_z} e^{(\theta_z - \theta_x) \rho_z} \right) \\ &= \left(\bigotimes_{y \in A \cap \Gamma} e^{-(\theta_y - \theta_x) \rho_y} \right) P_\Gamma(Q_A) \left(\bigotimes_{z \in A \cap \Gamma} e^{(\theta_z - \theta_x) \rho_z} \right),\end{aligned}\quad (6.31)$$

(using the rotation invariance of Q_A). Expanding the exponentials, we obtain

$$\begin{aligned}\widehat{R}_{-\theta} P_\Gamma(Q_A) \widehat{R}_\theta &= P_\Gamma(Q_A) + \sum_{y \in A \cap \Gamma} (\theta_x - \theta_y) (\rho_y P_\Gamma(Q_A) - P_\Gamma(Q_A) \rho_y) \\ &\quad + O\left(\frac{(\text{diam } A)^2 |A|^2}{L^2} \|Q_A\|\right),\end{aligned}\quad (6.32)$$

where the estimate of the higher order terms uses

$$|\theta_x - \theta_y| \leq \frac{\pi \|x - y\|_\infty}{L} \leq \frac{\pi \text{diam } A}{L} \quad (6.33)$$

and the observation that the n th order term in the expansion is potentially a sum of $|A|^n$ terms, as well as $\|P_\Gamma(Q_A)\| \leq \|Q_A\|$. The first order terms are odd in θ , and will cancel in $\Delta H_\theta + \Delta H_{-\theta}$, with the leading term being second order. What appears there is

$$\sum_{A \cap \Gamma \neq \emptyset} (\widehat{R}_{-\theta} P_\Gamma(Q_A) \widehat{R}_\theta - P_\Gamma(Q_A)) = O\left(L^d \sum_{A \ni 0} \frac{1}{|A|} \frac{(\text{diam } A)^2 |A|^2}{L^2} \|Q_A\|\right) = O(L^{d-2}), \quad (6.34)$$

where for non-finite-range interactions the last equality requires the assumption (A7), which is presented in the Appendix.

Then the right-hand side of Inequality (6.27) is

$$\|\Delta H_\theta + \Delta H_{-\theta}\| = O(L^{d-2}) + O(\delta L^d). \quad (6.35)$$

This provides only an upper bound on $g_{L,M}^{\delta\widehat{e}}(\vec{\eta})$, rather than a bound on its absolute value. However, it is obvious from the Definition (6.14) of g that $g_{L,M}^{\delta\widehat{e}}(\vec{\eta}) = -g_{L,M}^{-\delta\widehat{e}}(\vec{\eta})$, so the needed lower bound follows automatically. This proves (6.14), and thus, thorough (6.16), the claimed bound (6.13). $\square$

VII. STOCHASTIC LOWER BOUNDS ON THE LOCAL FREE ENERGY DIFFERENCE

To prove the Imry and Ma¹ phenomenon, Aizenman and Wehr⁸ employed a somewhat generalized form of the central limit theorem which is suitable for the families of random variables presently under consideration.^{8,24} We will use the following reformulation of that statement. Since the proof is in the literature it will be omitted here; one incorporating a correction due to Bovier²⁵ can be found in Ref. 15.

Recalling that T_x is the operation of translation by $x \in \mathbb{Z}^d$ and $\text{Av}[\cdot|\eta_\Lambda]$ is the conditional average conditioned on the values of $\{\eta_x\}_{x \in \Lambda} =: \eta_\Lambda$, we present

Proposition 7.1: Let η_x be a collection of i.i.d. random variables (indexed by $x \in \mathbb{Z}^d$) with $\text{Av} |\eta_x|^{2+\delta} < \infty$ for some $\delta > 0$, and let G_L be a family of real functions indexed by $L \in \mathbb{N}$, each with the following properties:

1. $\text{Av } G_L(\eta) = 0$,
2. $|||G_L||| \leq \epsilon$ (with $|||\cdot|||$ the Lipschitz seminorm defined by (5.6)),

3. $G_L(\eta)$ depends on the values of η_x only for $x \in \Gamma_L$, and
4. $\text{Av}[G_L(\eta)|\eta_\Lambda] = G_{L'}(T_x\eta_\Lambda)$ whenever $T_{-x}\Gamma_{L'} = \Lambda \subset \Gamma_L$.

Then

$$G_L(\eta)/L^{d/2} \longrightarrow N(0, b) \quad (7.1)$$

in distribution as $L \rightarrow \infty$, for some b satisfying

$$\text{Av } G_1^2 \leq b^2 \leq 2\epsilon^2(\text{Av } |\eta_0|)^2. \quad (7.2)$$

Assumption 4 in essence states that the different functions involved are essentially the same quantity at different scales, and that it is in some sense translation invariant. More precisely: the family forms a stationary Martingale, in a multidimensional sense of this condition.

In order to use this result, we will need to establish some control over the conditions under which $\text{Av } G_1^2 > 0$. To do this, we employ the following criterion (proven in Appendix III of Ref. 8):

Proposition 7.2: Let ν be a Borel probability measure on $\mathbb{R}$, and let g be a continuous function with $|||g||| = 1$ and

$$\int g'(t)\nu(dt) > 0. \quad (7.3)$$

Then any of the following is a sufficient condition for $\int g(t)^2\nu(dt) > 0$:

1. ν is absolutely continuous.
2. g is differentiable with $|||g'|||$ finite, and ν has no isolated point masses.
3. g is differentiable with $|||g'|||$ finite, $g' \geq 0$, and ν is not concentrated at a single point.

To apply this we note that G_1 is a function of one variable (η_0) which satisfies the conditions given above. In particular, it will be shown that the criteria 2. and sometimes 3. are satisfied at finite temperatures, in which case $|||\frac{\partial G_1}{\partial \eta_0}|||$ is uniformly bounded. In addition, we will see that the weak FKG property implies monotonicity of G_1 .

VIII. CONCLUSION OF THE PROOFS OF THE MAIN RESULTS

A. Existence of an infinite-system limit for free energy fluctuations

Having established the properties we require of the infinite-system free energy fluctuation G_L , we now proceed to show that such a suitable quantity exists. This will be done in two overlapping cases. We will have occasion to use the following equicontinuity arguments:

Lemma 8.1: Let $f_{ij} : \mathbb{R}^N \rightarrow \mathbb{R}$ be a family of functions labeled by $i, j \in \mathbb{N}$, each satisfying $|||f_{ij}||| \leq 1$ and $f_{ij}(0) = 0$. Then there are subsequences i_k, j_l such that

$$f(z) = \lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} f_{i_k j_l}(z) \quad (8.1)$$

exists for all $z \in \mathbb{R}^N$. Furthermore the convergence is uniform on any compact domain $\Xi \subset \mathbb{R}^N$.

Lemma 8.2: Let $f_{ij} : \mathbb{R}^N \rightarrow \mathbb{R}$ be as in Lemma 8.1 above, without assuming $f_{ij}(0) = 0$. Then the same result holds assuming instead that there is some c such that $|f_{ij}(z)| \leq c < \infty$ for all $z \in \mathbb{R}^N$, $i, j \in \mathbb{N}$.

These two lemmata are proven by applying the Arzelà-Ascoli theorem to obtain compactness of the space of sequences involved, then repeatedly applying the diagonal subsequence trick to obtain a single pair of sequences simultaneously satisfying all of the uniformity criteria.¹⁵

1. Finite temperature

We now begin with the proof of Theorem 5.1 in the case $\beta < \infty$. First, note that it is obvious that if the specified limits exist, then they will have the properties labeled 1 and 2 in that theorem. To define the sequence used we proceed as follows. For $x \in \Gamma_L$, let

$$\phi_{L,K,\delta}^x(\eta) := \frac{\partial \widehat{G}_{L,K}^\delta}{\partial \eta_x} = \frac{\epsilon}{2} \text{Av} \left[\langle \kappa_x \rangle_K^{h+\delta}(\eta) - \langle \kappa_x \rangle_K^{h-\delta}(\eta) \middle| \eta_L \right], \quad (8.2)$$

then evidently

$$|\phi_{L,K,\delta}^x(\eta)| \leq \epsilon, \quad (8.3)$$

and

$$\begin{aligned} \left| \frac{\partial \phi_{L,K,\delta}^x}{\partial \eta_y} \right| &= \frac{\beta \epsilon^2}{2} \left| \langle \kappa_x \kappa_y \rangle_K^{h+\delta}(\eta) - \langle \kappa_x \kappa_y \rangle_K^{h-\delta}(\eta) \right. \\ &\quad \left. - \langle \kappa_x \rangle_K^{h+\delta}(\eta) \langle \kappa_y \rangle_K^{h+\delta}(\eta) + \langle \kappa_x \rangle_K^{h-\delta}(\eta) \langle \kappa_y \rangle_K^{h-\delta}(\eta) \right| \leq 2\beta \epsilon^2. \end{aligned} \quad (8.4)$$

For $\beta < \infty$, this means that for each L , $\phi_{L,K,\delta}^x$ is an equicontinuous family of functions of the L^d variables η_{Γ_L} . We can apply Lemma 8.2 to find a decreasing sequence $\delta_i \rightarrow 0$ and an increasing sequence $K_j \rightarrow \infty$ (by applying the diagonal subsequence trick, we can choose them to be independent of x and L) so that

$$\psi_L^x(\eta) := \lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} \phi_{L,K_j,\delta_i}^x(\eta) \quad (8.5)$$

exists, and by uniformity of convergence

$$G_L(\eta_L) := \lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} \left(\widehat{G}_{L,K_j}^{\delta_i}(\eta_L) - \text{Av} \widehat{G}_{L,K_j}^{\delta_i} \right) \quad (8.6)$$

also exists with

$$\frac{\partial G_L}{\partial \eta_x} = \psi_L^x(\eta) \quad (8.7)$$

for all $L, \eta, x \in \Gamma_L$. Property 3 and the relation of FKG to monotonicity then follow from Eq. (8.2), and property 5 follows from Eq. (8.4).

To show property 6, we use Eqs. (8.7) and (8.5) to write

$$\text{Av} G'_1 = \text{Av} \psi_1^0 = \text{Av} \lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} \phi_{1,K_j,\delta_i}^0. \quad (8.8)$$

Recalling the definition of ϕ and using dominated convergence to exchange limits and averages we have

$$\text{Av} G'_1 = \lim_{i \rightarrow \infty} \lim_{K \rightarrow \infty} \frac{\epsilon}{2} \text{Av} \left[\langle \kappa_0 \rangle_K^{h+\delta_i} - \langle \kappa_0 \rangle_K^{h-\delta_i} \right] \quad (8.9)$$

and since the random fields are i.i.d. we can apply the Birkhoff ergodic theorem²⁶ to replace the disorder average with a volume average, which is related to the derivative of the free energy by Proposition 4.3, giving

$$\text{Av} G'_1 = \frac{\epsilon}{2} \left(\frac{\partial \mathcal{F}}{\partial h_+} - \frac{\partial \mathcal{F}}{\partial h_-} \right). \quad (8.10)$$

Note that Eq. (8.2) implies that, for any $L' < L \leq K$,

$$\phi_{L',K,\delta}^x = \text{Av} \left[\phi_{L,K,\delta}^x \middle| \eta_{\Gamma_{L'}} \right], \quad (8.11)$$

and we apply the conditional form of the dominated convergence theorem to the limits used to define ψ to obtain

$$\psi_K^x = \text{Av} \left[\psi_L^x \middle| \eta_K \right]. \quad (8.12)$$

Since $\widehat{G}_{L,K_j}^{\delta_i}$ was defined in terms of periodic boundary conditions and the limits used to obtain ϕ are independent of x , we also have

$$\phi_{L,K,\delta}^x(T_y \eta) = \phi_{L,K,\delta}^{x-y}(\eta), \quad (8.13)$$

and taking the limit (8.5) we have

$$\psi_L^x(T_y \eta) = \psi_L^{x-y}(\eta). \quad (8.14)$$

Together with Eq. (8.12), this lets us obtain

$$\text{Av} [\psi_L^y | \eta_\Lambda] = \text{Av} [\psi_L^{y+x} \circ T_x | \eta_{L'}] \quad (8.15)$$

or in other words (in light of (8.7))

$$\text{Av} \left[\frac{\partial G_L}{\partial \eta_y} \middle| \eta_\Lambda \right] = \frac{\partial}{\partial \eta_y} G_{L'}(T_x \eta). \quad (8.16)$$

The uniform bounds proven above allow us to take the partial derivative with respect to η_y outside the conditional average,²⁷ and the resulting expression can then be integrated to prove property 4.

2. Absolutely continuous random fields

With $\beta = \infty$, it is still obvious from Eq. (5.5) that $\widehat{G}_{L,K}^\delta(0) = 0$; Inequality (8.3) implies $|||\widehat{G}_{L,K}^\delta||| \leq \epsilon$, so we can apply Lemma 8.1 to obtain (8.6), with $|||G_L||| \leq \epsilon$. $\text{Av } G_L = 0$ is obvious. We can apply the diagonal subsequence trick to obtain sequences independent of L , which implies that the consistency condition 4 of Theorem 7.1 is satisfied.

The difference arises in proving that $\text{Av } G'_1 = \epsilon M$. Without equicontinuity of the derivatives, there is no reason to expect that functions constructed like ϕ of Sec. VIII A 1 will converge uniformly, or that the pointwise limit will be differentiable. This problem can be resolved by applying the following convergence criteria:

Lemma 8.3: Let g be a measurable function and g_n a sequence of measurable functions such that $\|g_n\|_\infty \leq 1$, $\|g\|_\infty \leq 1$, and

$$\lim_{n \rightarrow \infty} \int_a^b g_n(t) dt = \int_a^b g(t) dt \quad (8.17)$$

for all $a, b \in \mathbb{R}$. Then for any finite measure ν which is absolutely continuous with respect to the Lebesgue measure,

$$\lim_{n \rightarrow \infty} \int g_n d\nu = \int g d\nu. \quad (8.18)$$

This can be proven by elementary measure theory techniques.¹⁵

Thanks to Rademacher's theorem,²⁸ Lipschitz continuity guarantees that f_n have distributional derivatives, i.e., functions g_n satisfying

$$\int_a^b g_n(t) dt = f_n(b) - f_n(a) \quad (8.19)$$

for any $a, b \in \mathbb{R}$; and furthermore that $\|g_n\|_\infty = |||f_n||| \leq 1$. We therefore have

Corollary 8.4: Let f_n be a sequence of functions $\mathbb{R} \rightarrow \mathbb{R}$ such that $f_n \rightarrow f$ pointwise, and $|||f_n||| \leq 1$. Then f has a distributional derivative f' satisfying

$$\lim_{n \rightarrow \infty} \int g_n(t) d\nu(t) = \int f'(t) d\nu(t). \quad (8.20)$$

Applying Corollary 8.4 twice to $\widehat{G}_{1,K_j}^{\delta_i}(\eta_L) - \text{Av } \widehat{G}_{1,K_j}^{\delta_i}$ and G_1 , we obtain

$$\text{Av } G'_1 = \lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} \text{Av} \left(\widehat{G}_{1,K_j}^{\delta_i} \right)'(\eta), \quad (8.21)$$

and applying Proposition 4.3 and the Birkhoff ergodic theorem to the right-hand side as in the finite temperature case we obtain

$$\text{Av } G'_1 = \frac{\epsilon}{2} \left(\frac{\partial \mathcal{F}}{\partial h_+} - \frac{\partial \mathcal{F}}{\partial h_-} \right), \quad (8.22)$$

and the proof of Theorem 5.1 is complete.

B. Concluding the proofs of Theorems 4.1 and 4.2

In summary, we have established that under the hypotheses of Theorems 4.1 and 4.2 the free energy fluctuations $G_L(\eta)$ are uniformly bounded above by CL^{d-2} , and correspondingly $CL^{d/2}$, and that if there is a 1st order phase transition (i.e., a discontinuity in $\frac{\partial \mathcal{F}}{\partial h}$) then $G_L(\eta)$ has asymptotically normal fluctuations on a scale of $L^{d/2}$. That is a contradiction even in the marginal dimensions (where the powers match) as a normally distributed random variable will take with positive probability values which are arbitrarily large on any given scale. An alternative way to present this case is to note that assuming the existence of a 1st order phase transition, by Proposition 7.1 we get

$$\lim_{L \rightarrow \infty} \text{Av exp}(tG_L/L^{d/2}) = \exp(t^2 b^2/2). \quad (8.23)$$

At the same time, if $\|G_L\|_\infty \leq AL^{d/2}$ then for all positive t :

$$\text{Av } e^{tG_L/L^{d/2}} \leq e^{tA}. \quad (8.24)$$

Clearly, if $b \neq 0$, the two relations are incompatible at sufficiently large t . Finally, Proposition 7.2 allows to conclude that in any of the cases under consideration, $b = 0$ implies $M = 0$, and the proof is complete.

ACKNOWLEDGMENTS

We thank S. Chakravarty, R. Fisch, and M. Schechter for discussions; M.A. also wishes to thank the Departments of Physics and Mathematics at the Weizmann Institute of Science for hospitality. The work was supported in part by National Science Foundation (NSF) (Grant Nos. DMS-0602360 (M.A.) and DMR 08-02120 (R.L.G. and J.L.L.), ERC starting Grant No. CoMBoS-239694 (R.L.G.), and by AFOSR Grant No. FA9550-10-1-0131 (R.L.G. and J.L.L.).

APPENDIX: DECAY CONDITIONS FOR LONG RANGE INTERACTIONS

In this appendix we turn to the general class of interactions under which the results derived in this work apply.

The estimate which is relevant for Theorem 4.1 is valid if

$$\sum_{\substack{A \ni 0 \\ \text{diam } A \leq L}} \text{diam } A \frac{|\partial A|}{|A|} \|Q_A\| \leq c' L^{(2-d)/2} \quad (A1)$$

for some constant c' . In case of two body interaction, that is when Q_A is non-zero only when A is a two point set, this condition is met if

$$\|Q_{\{x,y\}}\| \leq \text{Const.}/|x-y|^{3d/2}. \quad (A2)$$

As noted in the proof, the essential estimate which is required for Lemma 6.2, and therefore Theorem 4.1, is that

$$\text{Av } \|V_{\Lambda_L}^\eta\| \leq cL^{d-1} + bL^{d/2}, \quad (A3)$$

with some finite constants b and c . The contribution in $V_{\Lambda_L}^\eta$ of terms involving the random fields η consists of a collection of $O(L^{d-1})$ independent summands with finite average norm. Thus, the

assumption (A3) requires only a bound of the same form on the nonrandom part of the interaction. The following simple calculation⁸ shows that (A1) provides a sufficient condition for that.

By the triangle inequality

$$\|V_L^{0,0}\| \leq \sum_{A: A \cap \Gamma_L \notin \{\emptyset, A\}} \|Q_A\|. \quad (\text{A4})$$

In this sum the terms with diameter L or less contribute at most

$$\sum_{\substack{A \ni 0 \\ \text{diam } A \leq L}} 2dL^{d-1} \frac{\text{diam } A}{|A|} \leq 2dc'L^{d/2}, \quad (\text{A5})$$

and the remaining portion is bounded by

$$\begin{aligned} & \sum_{\substack{A \ni 0 \\ \text{diam } A \leq L}} L^d \frac{|\partial A|}{|A|} \|Q_A\| \\ & \leq L^{d-1} \sum_{\substack{A \ni 0 \\ \text{diam } A \leq L}} \text{diam } A \frac{|\partial A|}{|A|} \|Q_A\| \leq c'L^{d/2}. \end{aligned} \quad (\text{A6})$$

Putting the two parts back together we have Inequality (A3) with $C' = (2d + 1)c'$.

The recent result of Cassandro, Orlandi, and Picco²⁹ allows to conclude that at $d = 1$ the condition (A2) does indeed provide the threshold decay rate for the validity of Theorem 4.1. They prove that the phase transition is stable under weak disorder in a family of one dimensional Ising spin systems with long range interactions with decay rates arbitrarily close to $3/2$, more explicitly with $\|Q_{\{x,y\}}\| \approx \text{Const.}/|x - y|^{3/2 - \gamma}$ at any $\gamma \in (0, 0.08)$. (Curiously, while the proof of the complimentary result does not extent beyond this narrow range of values of γ , the lower end of the interval coincides with (A2).) The question whether the above criterion is optimal also for $d = 2$ is of interest, as there are some notable systems with inverse cube interactions in two dimensions.³⁰

For the stronger conclusion which is derived here for interactions with continuous symmetries the relevant assumption is not (A3) but

$$\sum_{A \ni 0} (\text{diam } A)^2 |A| \|Q_A\| < \infty, \quad (\text{A7})$$

which is used in (6.34), in the proof of the free energy estimate of Lemma 6.3. For pair interactions this reduces to the statement

$$\sum_{x \in \mathbb{Z}^d} \|Q_{\{0,x\}}\| |x|^2 < \infty, \quad (\text{A8})$$

found (in slightly different notation) in Ref. 10.

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