PROCEEDINGS OF THE ROYAL SOCIETY A

Proc. R. Soc. A (2010) **466**, 3203–3224
doi:10.1098/rspa.2009.0635
Published online 20 May 2010

Normal typicality and von Neumann's quantum ergodic theorem

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We discuss the content and significance of John von Neumann's quantum ergodic theorem (QET) of 1929, a strong result arising from the mere mathematical structure of quantum mechanics. The QET is a precise formulation of what we call normal typicality, i.e. the statement that, for typical large systems, every initial wave function ψ_0 from an energy shell is 'normal': it evolves in such a way that $|\psi_t\rangle\langle\psi_t|$ is, for most t, macroscopically equivalent to the micro-canonical density matrix. The QET has been mostly forgotten after it was criticized as a dynamically vacuous statement in several papers in the 1950s. However, we point out that this criticism does not apply to the actual QET, a correct statement of which does not appear in these papers, but to a different (indeed weaker) statement. Furthermore, we formulate a stronger statement of normal typicality, based on the observation that the bound on the deviations from the average specified by von Neumann is unnecessarily coarse and a much tighter (and more relevant) bound actually follows from his proof.

Keywords: ergodicity in quantum statistical mechanics; equilibration; thermalization; generic Hamiltonian; typical Hamiltonian; macro-state

1. Introduction

Quantum statistical mechanics has many similarities to the classical version and also some differences. Two facts true in the quantum but not in the classical case, *canonical typicality* and (what we call) *normal typicality*, follow from just the general mathematical structure of quantum mechanics. Curiously, both were discovered early on in the history of quantum mechanics, in fact both in the 1920s, and subsequently forgotten until recently. Canonical typicality was basically anticipated, though not clearly articulated, by Schrödinger (1927) and rediscovered a few years ago by several groups independently (Gemmer *et al.* 2004; Goldstein *et al.* 2006; Popescu *et al.* 2006). Normal typicality, the topic of this paper, was discovered, clearly articulated and rigorously proved by John von Neumann (von Neumann 1929) as a 'quantum ergodic theorem' (QET). In the 1950s, though, the QET was heavily criticized in two influential

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Received 30 November 2009
Accepted 19 April 2010

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papers (Farquhar & Landsberg 1957; Bocchieri & Loinger 1958) as irrelevant to quantum statistical mechanics, and indeed as dynamically vacuous. The criticisms (repeated in Bocchieri & Loinger 1959; Farquhar 1961, 1964; Landsberg 1961, 2005) have led many to dismiss von Neumann's QET (e.g. Ludwig 1958; van Hove 1959, p. 273; Prosperi & Scotti 1960; Jancel 1969; Pechukas 1984; Toda *et al.* 1991, p. 227). We show here that these criticisms are invalid. They actually apply to a statement different from (indeed weaker than) the original theorem. The dismissal of the QET is therefore unjustified. Furthermore, we also formulate two new statements about normal typicality, see theorems 6.2 and 8.1, which in fact follow from von Neumann's proof. (We provide further discussion of von Neumann's QET article in a subsequent work (Goldstein *et al.* in press).)

In recent years, there has been a renewed strong interest in the foundations of quantum statistical mechanics (see Gemmer *et al.* 2004; Goldstein *et al.* 2006, 2010; Popescu *et al.* 2006; Reimann 2008; Rigol *et al.* 2008; Linden *et al.* 2009); von Neumann's work, which has been mostly forgotten, has much to contribute to this topic.

The QET concerns the long-time behaviour of the quantum state vector

$$\psi_t = \exp(-\mathrm{i}Ht)\psi_0, \tag{1.1}$$

(where we have set $\hbar = 1$) of a macroscopic quantum system, e.g. one with more than 10^{20} particles, enclosed in a finite volume. Suppose that ψ_t belongs to a 'micro-canonical' subspace $\mathscr{H}$ of the Hilbert space $\mathscr{H}_{\mathrm{total}}$, corresponding to an energy interval that is large on the microscopic scale, i.e. contains many eigenvalues, but small on the macroscopic scale, i.e. different energies in that interval are not discriminated macroscopically. Thus, the dimension of $\mathscr{H}$ is finite but huge, in fact exponential in the number of particles. We use the notation

$$D = \dim \mathscr{H} \tag{1.2}$$

(=S_a in von Neumann (1929), S in Farquhar & Landsberg (1957) and Bocchieri & Loinger (1958)). The micro-canonical density matrix ρ_{mc} is then $1/D$ times the identity operator on $\mathscr{H}$, and the micro-canonical average of an observable A on $\mathscr{H}$ is given by

$$\mathrm{tr}(\rho_{\mathrm{mc}} A) = \frac{\mathrm{tr}\, A}{D} = \mathbb{E}\langle\varphi|A|\varphi\rangle, \tag{1.3}$$

where φ is a random vector with uniform distribution over the unit sphere of $\mathscr{H}$

$$\{\varphi \in \mathscr{H} \mid \|\varphi\| = 1\}, \tag{1.4}$$

and $\mathbb{E}$ means the expectation value. In the following, we denote the time average of a function $f(t)$ by a bar

$$\overline{f(t)} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t\, f(t). \tag{1.5}$$

Despite the name, the property described in the QET is not precisely analogous to the standard notion of ergodicity as known from classical mechanics and the mathematical theory of dynamical systems. That is why we prefer to call quantum

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systems with the relevant property 'normal' rather than 'ergodic'. Nevertheless, to formulate a quantum analogue of ergodicity was von Neumann's motivation for the QET. It is characteristic of ergodicity that time averages coincide with phase-space averages. Put differently, letting X_t denote the phase point at time t of a classical Hamiltonian system, δ_{X_t} the delta measure concentrated at that point and μ_{mc} the micro-canonical (uniform) measure on an energy surface, ergodicity is equivalent to

$$\overline{\delta_{X_t}} = \mu_{\mathrm{mc}} \tag{1.6}$$

for almost every X_0 on this energy surface. In quantum mechanics, if we regard a pure state $|\psi_t\rangle\langle\psi_t|$ as analogous to the pure state δ_{X_t} and ρ_{mc} as analogous to μ_{mc}, the statement analogous to equation (1.6) reads

$$\overline{|\psi_t\rangle\langle\psi_t|} = \rho_{\mathrm{mc}}. \tag{1.7}$$

As pointed out by von Neumann (1929), the left-hand side always exists and can be computed as follows. Let $\{\phi_\alpha\}$ be an orthonormal basis of eigenvectors of H with eigenvalues E_α. If ψ_0 has coefficients $c_\alpha = \langle\phi_\alpha|\psi_0\rangle$

$$\psi_0 = \sum_{\alpha=1}^{D} c_\alpha |\phi_\alpha\rangle, \tag{1.8}$$

then

$$\psi_t = \sum_{\alpha=1}^{D} \mathrm{e}^{-\mathrm{i}E_\alpha t} c_\alpha |\phi_\alpha\rangle, \tag{1.9}$$

and thus

$$\overline{|\psi_t\rangle\langle\psi_t|} = \sum_{\alpha,\beta} \overline{\mathrm{e}^{-\mathrm{i}(E_\alpha - E_\beta)t}} c_\alpha c_\beta^* |\phi_\alpha\rangle\langle\phi_\beta|. \tag{1.10}$$

Suppose that H is non-degenerate; then $E_\alpha - E_\beta$ vanishes only for $\alpha = \beta$, so the time-averaged exponential is $\delta_{\alpha\beta}$, and we have that

$$\overline{|\psi_t\rangle\langle\psi_t|} = \sum_{\alpha} |c_\alpha|^2 |\phi_\alpha\rangle\langle\phi_\alpha|. \tag{1.11}$$

While the case (1.7) occurs only for those special wave functions that have $|c_\alpha|^2 = 1/D$ for all α, in many cases it is true of *all* initial wave functions ψ_0 on the unit sphere of $\mathscr{H}$ that $\overline{|\psi_t\rangle\langle\psi_t|}$ is *macroscopically equivalent* to ρ_{mc}.

What we mean here by macroscopic equivalence corresponds in the work of von Neumann (1929) to a decomposition of $\mathscr{H}$ into mutually orthogonal subspaces $\mathscr{H}_\nu$,

$$\mathscr{H} = \bigoplus_{\nu} \mathscr{H}_\nu, \tag{1.12}$$

such that each $\mathscr{H}_\nu$ corresponds to a different macro-state ν. We call the $\mathscr{H}_\nu$ the 'macro-spaces' and write $\mathscr{D}$ for the family $\{\mathscr{H}_\nu\}$ of subspaces, called a

'macro-observer' in von Neumann's paper, and P_ν for the projection to $\mathscr{H}_\nu$. We use the notation

$$d_\nu = \dim \mathscr{H}_\nu \tag{1.13}$$

($= s_{\nu,a}$ in von Neumann (1929), s_ν in Farquhar & Landsberg (1957) and Bocchieri & Loinger (1958)).[1]

As a simple example, we may consider, for a gas consisting of $n > 10^{20}$ atoms enclosed in a box $\Lambda \subset \mathbb{R}^3$, the following 51 macro-spaces $\mathscr{H}_0, \mathscr{H}_2, \mathscr{H}_4, \ldots, \mathscr{H}_{100}$: $\mathscr{H}_\nu$ contains the quantum states for which the number of atoms in the left half of Λ lies between $\nu - 1$ per cent of n and $\nu + 1$ per cent of n. Note that in this example $\mathscr{H}_{50}$ has the overwhelming majority of dimensions.[2]

Given $\mathscr{D}$, we say that two density matrices ρ and ρ' are *macroscopically equivalent*, in symbols

$$\rho \overset{\mathscr{D}}{\sim} \rho', \tag{1.14}$$

if and only if

$$\mathrm{tr}(\rho P_\nu) \approx \mathrm{tr}(\rho' P_\nu) \tag{1.15}$$

for all ν. (The sense of $\approx$ will be made precise later.) For example, $|\psi\rangle\langle\psi| \overset{\mathscr{D}}{\sim} \rho_{\mathrm{mc}}$ if and only if

$$\|P_\nu \psi\|^2 \approx \frac{d_\nu}{D} \tag{1.16}$$

for all ν. This is, in fact, the case for most vectors ψ on the unit sphere of $\mathscr{H}$, provided the d_ν are sufficiently large, as follows (see equation (4.6)), from the following easy geometrical fact (e.g. von Neumann 1929, p. 55; see also appendix II of Jancel (1969)).

Lemma 1.1. *If $\mathscr{H}_\nu$ is any fixed subspace of dimension d_ν and φ is a random vector with uniform distribution on the unit sphere, then*

$$\mathbb{E}\|P_\nu \varphi\|^2 = \frac{d_\nu}{D}, \quad \mathrm{Var}\|P_\nu \varphi\|^2 = \mathbb{E}\left(\|P_\nu \varphi\|^2 - \frac{d_\nu}{D}\right)^2 = \frac{1}{d_\nu}\left(\frac{d_\nu}{D}\right)^2 \frac{(D - d_\nu)}{(D+1)}. \tag{1.17}$$

[1] Von Neumann motivated the decomposition (1.12) by beginning with a family of operators corresponding to coarse-grained macroscopic observables and arguing that by 'rounding' the operators, the family can be converted to a family of operators $M_1, \ldots, M_k$ that commute with each other, have pure point spectrum and have huge degrees of degeneracy. (This reasoning has inspired research about whether for given operators $A_1, \ldots, A_k$ whose commutators are small one can find approximations $M_i \approx A_i$ that commute exactly; the answer is, for $k \geq 3$ and general $A_1, \ldots, A_k$, no (Choi 1988).) A macro-state can then be characterized by a list $\nu = (m_1, \ldots, m_k)$ of eigenvalues m_i of the M_i, and corresponds to the subspace $\mathscr{H}_\nu \subseteq \mathscr{H}$ containing the simultaneous eigenvectors of the M_i with eigenvalues m_i; that is, $\mathscr{H}_\nu$ is the intersection of the respective eigenspaces of the M_i and d_ν is the degree of simultaneous degeneracy of the eigenvalues $m_1, \ldots, m_k$. For a notion of macro-spaces that does not require that the corresponding macro-observables commute, see De Roeck *et al.* (2006), in particular §2.1.1. (Concerning the main results discussed below, theorems 5.3 and 6.2, a plausible guess is that normal typicality extends to non-commuting families $A_1, \ldots, A_k$—of observables that may also fail to commute with ρ_{mc}—provided that the observables have a sufficiently small variance in the sense of lemma 1.1, i.e. that $\mathrm{Var}(\langle\varphi|A|\varphi\rangle)$ be small. We shall, however, not elaborate on this here.)

[2] Actually, these subspaces form an orthogonal decomposition of $\mathscr{H}_{\mathrm{total}}$ rather than of the energy shell $\mathscr{H}$, since the operator of particle number in the left half of Λ fails to map $\mathscr{H}$ to itself. Thus, certain approximations that we do not want to describe here are necessary in order to obtain an orthogonal decomposition of $\mathscr{H}$.

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Returning to the time average, we obtain that $\overline{|\psi_t\rangle\langle\psi_t|} \overset{\mathscr{D}}{\sim} \rho_{\mathrm{mc}}$ if and only if

$$\sum_{\alpha} |c_\alpha|^2 \langle\phi_\alpha|P_\nu|\phi_\alpha\rangle \approx \frac{d_\nu}{D} \tag{1.18}$$

for all ν. Condition (1.18) is satisfied for every $\psi_0 \in \mathscr{H}$ with $\|\psi_0\| = 1$ if

$$\langle\phi_\alpha|P_\nu|\phi_\alpha\rangle \approx \frac{d_\nu}{D} \tag{1.19}$$

for every α and ν, a condition on H and $\mathscr{D}$ that von Neumann showed is typically obeyed, in a sense which we shall explain. The analogy between $\overline{|\psi_t\rangle\langle\psi_t|} \overset{\mathscr{D}}{\sim} \rho_{\mathrm{mc}}$ and ergodicity lies in the fact that the time average of a pure state in a sense agrees with the micro-canonical ensemble, with the two differences that the agreement is only an approximate agreement on the macroscopic level, and that it typically holds for *every*, rather than *almost every*, pure state.

However, even more is true for many quantum systems: not just the time average but even $|\psi_t\rangle\langle\psi_t|$ itself is macroscopically equivalent to ρ_{mc} for most times t in the long run, i.e.

$$\|P_\nu\psi_t\|^2 \approx \frac{d_\nu}{D} \tag{1.20}$$

for all ν for most t. Such a system, defined by H, $\mathscr{D}$ and ψ_0, we call *normal*, a terminology inspired by the concept of a *normal real number* (Normal number 2009). Above we have stressed the continuity with the standard notion of ergodicity. Yet, normality is in part stronger than ergodicity (it involves no time averaging) and in part weaker (it involves only macroscopic equivalence); in short, it is a different notion.

Suppose now, as in the example between equations (1.13) and (1.14), that one of the macro-spaces, $\mathscr{H}_\nu = \mathscr{H}_{\mathrm{eq}}$, has the overwhelming majority of dimensions

$$\frac{d_{\mathrm{eq}}}{D} \approx 1. \tag{1.21}$$

It is then appropriate to call this macro-state the thermal equilibrium state and write $\nu = eq$. We say that the system is in thermal equilibrium at time t if and only if $\|P_{\mathrm{eq}}\psi_t\|^2$ is close to 1, or, put differently, if and only if

$$\|P_{\mathrm{eq}}\psi_t\|^2 \approx \frac{d_{\mathrm{eq}}}{D}. \tag{1.22}$$

Thus, if a system is normal, then it is in equilibrium most of the time. Of course, if it is not in equilibrium initially, the waiting time until it first reaches equilibrium is not specified, and may be longer than the present age of the universe.[3]

The case that one of the $\mathscr{H}_\nu$ has the overwhelming majority of dimensions is an important special case but was actually not considered by von Neumann; it is discussed in detail in Goldstein *et al.* (2010). Von Neumann (and many other authors) had a different understanding of thermal equilibrium; he would have said a system is in thermal equilibrium at time t if and only if equation (1.20)

[3]Furthermore, due to the quasi-periodicity of the time dependence of any density matrix (not just a pure one) of our system, it will keep keep on returning to (near) its initial state.

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holds for all ν, so that $|\psi_t\rangle\langle\psi_t| \overset{\mathscr{D}}{\sim} \rho_{\mathrm{mc}}$. Here we disagree with him, as well as with his suggestion that the further theorem in von Neumann (1929), which he called the 'quantum H-theorem' and which is a close cousin of the QET, is a quantum analogue of Boltzmann's H-theorem. Yet other definitions of thermal equilibrium have been used in Reimann (2008) and Linden *et al.* (2009); see §6 of Goldstein *et al.* (2010) for a comparative overview and Goldstein *et al.* (in press) for a broader overview of such definitions.

The QET provides conditions under which a system is normal for *every* initial state vector ψ_0. Note that statements about *most* initial state vectors ψ_0 are much weaker; for example, most state vectors ψ_0 are in thermal equilibrium by lemma 1.1, so a statement about most ψ_0 need not convey any information about systems starting out in non-equilibrium. Furthermore, the QET asserts normal typicality, i.e. that typical macroscopic systems are normal for every ψ_0; more precisely, that for most choices of $\mathscr{D}$ (or H), macroscopic systems are normal for every ψ_0. It thus provides reason to believe that macroscopic systems in practice are normal.

Informal statement of the QET (*for fully precise statements, see theorems* 5.3, 6.2 *and* 8.1): Following von Neumann, we say that a Hamiltonian H with non-degenerate eigenvalues $E_1, \ldots, E_D$ *has no resonances* if and only if

$$E_\alpha - E_\beta \neq E_{\alpha'} - E_{\beta'} \quad \text{unless} \begin{cases} \text{either} & \alpha = \alpha' \text{ and } \beta = \beta' \\ \text{or} & \alpha = \beta \text{ and } \alpha' = \beta'. \end{cases} \tag{1.23}$$

In words, this means that also the energy differences are non-degenerate. Let $\mathscr{H}$ be any Hilbert space of finite dimension D, and let H be a self-adjoint operator on $\mathscr{H}$ with no degeneracies and no resonances. If the natural numbers d_ν are sufficiently large (precise conditions will be given later) and $\sum_\nu d_\nu = D$, then most families $\mathscr{D} = \{\mathscr{H}_\nu\}$ of mutually orthogonal subspaces $\mathscr{H}_\nu$ with $\dim \mathscr{H}_\nu = d_\nu$ are such that for every wave function $\psi_0 \in \mathscr{H}$ with $\|\psi_0\| = 1$ and every ν, equation (1.20) holds most of the time in the long run.

When we say that a statement $p(x)$ is true 'for most x' we mean that

$$\mu\{x|p(x)\} \geq 1 - \delta, \tag{1.24}$$

where $0 < \delta \ll 1$ and μ is a suitable probability measure; we will always use the appropriate *uniform* measure, as specified explicitly in §2. (When we speak of 'most of the time in the long run', the meaning is a bit more involved since there is no uniform probability measure on the half axis $[0, \infty)$; see §2.)

Let $p(\mathscr{D}, \psi_0)$ be the statement that for every ν, equation (1.20) holds most of the time in the long run. The misunderstanding of the QET starting in the 1950s consists of mixing up the statement

$$\text{for most } \mathscr{D}\text{: for all } \psi_0\text{: } p(\mathscr{D}, \psi_0), \tag{1.25}$$

which is part of the QET, with the inequivalent statement

$$\text{for all } \psi_0\text{: for most } \mathscr{D}\text{: } p(\mathscr{D}, \psi_0). \tag{1.26}$$

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To see that these two statements are indeed inequivalent, let us illustrate the difference between 'for most x: for all y: $p(x, y)$' and 'for all y: for most x: $p(x, y)$' by two statements about a company

$$\text{Most employees are never ill} \tag{1.27}$$

and

$$\text{On each day, most employees are not ill} \tag{1.28}$$

Here, x ranges over employees, y over days, and $p(x, y)$ is the statement 'employee x is not ill on day y'. It is easy to understand that statement (1.27) implies statement (1.28), and statement (1.28) does not imply statement (1.27), as there is the (very plausible) possibility that most employees are sometimes ill, but not on the same day.

Von Neumann's proof establishes equation (1.25), while the proofs in Farquhar & Landsberg (1957) and Bocchieri & Loinger (1958) establish only the weaker version (equation (1.26)). von Neumann (1929) also made clear in a footnote on p. 58 of his article which version he intended:

> Note that what we have shown is not that for every given ψ or A the ergodic theorem and the H-theorem hold for most $\omega_{\lambda,\nu,a}$, but instead that they hold universally for most $\omega_{\lambda,\nu,a}$, i.e., for all ψ and A. The latter is of course much more than the former.

Here, A is not important right now while $\omega_{\lambda,\nu,a}$ corresponds to $\mathscr{D}$ in our notation. So the quotation means that what von Neumann has shown is not equation (1.26) but equation (1.25) for a certain p.

The remainder of this paper is organized as follows. In §2, we make explicit which measures are used in the role of μ. Section 3 contains a discussion of the method of appeal to typicality. In §4, we give the precise definition of normality. Section 5 contains a precise formulation of von Neumann's theorem and an outline of his proof. Section 6 contains our stronger version of the QET with tighter bounds on the deviations. In §7, we show that the versions of the QET in Farquhar & Landsberg (1957) and Bocchieri & Loinger (1958) differ from the original as described above. In §8, we provide another version of the QET, assuming typical H instead of typical $\mathscr{D}$. Finally, in §9, we compare von Neumann's result with the recent literature.

2. Measures of most

Let us specify which measure μ is intended in equation (1.24) when referring to most wave functions, most unitary matrices, most orthonormal bases, most Hamiltonians, most subspaces or most decompositions $\mathscr{D}$. It is always the appropriate uniform probability measure.

For wave functions ψ, μ is the (normalized, $(2D-1)$-dimensional) surface area measure on the unit sphere in Hilbert space $\mathscr{H}$.

For unitary matrices $U = (U_{\alpha\beta})$, the uniform probability distribution over the unitary group $U(D)$ is known as the *Haar measure*. It is the unique normalized measure that is invariant under multiplication (either from the left or from the right) by any fixed unitary matrix.

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For orthonormal bases, the Haar measure defines a probability distribution (the *uniform distribution*) over the set of orthonormal bases of $\mathscr{H}$, $ONB(\mathscr{H})$, as follows. Fix first some orthonormal basis $\phi_1, \ldots, \phi_D$ for reference. Any other orthonormal basis $\omega_1, \ldots, \omega_D$ can be expanded into the ϕ_β

$$\omega_\alpha = \sum_{\beta=1}^{D} U_{\alpha\beta}\phi_\beta, \tag{2.1}$$

where the coefficients $U_{\alpha\beta}$ form a unitary matrix. Conversely, for any given unitary matrix $U = (U_{\alpha\beta})$, equation (2.1) defines an orthonormal basis; thus, a random Haar-distributed U defines a random orthonormal basis (ω_α), whose distribution we call the uniform distribution. It is independent of the choice of the reference basis ϕ because the Haar measure is invariant under right multiplication by a fixed unitary matrix. Note also that the marginal distribution of any single basis vector ω_α is the uniform distribution on the unit sphere in $\mathscr{H}$.

For Hamiltonians, we will regard the eigenvalues as fixed and consider the uniform measure for its eigenbasis. This is the same distribution as that of $H = UH_0U^{-1}$ when U has uniform distribution and H_0 is fixed.

For subspaces, we will regard the dimension d as fixed; the measure over all subspaces of dimension d arises from the measure on $ONB(\mathscr{H})$ as follows. If the random orthonormal basis $\omega_1, \ldots, \omega_D$ has a uniform distribution, we consider the random subspace spanned by $\omega_1, \ldots, \omega_d$ and call its distribution uniform.

For decompositions $\mathscr{D} = \{\mathscr{H}_\nu\}$, we will regard the number N of subspaces as fixed, as well as their dimensions d_ν; the measure over decompositions arises from the measure on $ONB(\mathscr{H})$ as follows. Given the orthonormal basis $\omega_1, \ldots, \omega_D$, we let $\mathscr{H}_\nu$ be the subspace spanned by those ω_α with $\alpha \in J_\nu$, where the index sets J_ν form a partition of $\{1, \ldots, D\}$ with $\#J_\nu = d_\nu$; we also regard the index sets J_ν as fixed.

The Haar measure is also invariant under the inversion $U \mapsto U^{-1}$. A consequence is what we will call the 'unitary inversion trick': If ϕ is any fixed orthonormal basis and ω a random orthonormal basis with uniform distribution then the joint distribution of the coefficients $U_{\alpha\beta} = \langle\phi_\beta|\omega_\alpha\rangle$ is the same as if ω were any fixed orthonormal basis and ϕ random with uniform distribution. The reason is that in the former case the matrix U is Haar-distributed, and in the latter case U^{-1} is Haar-distributed, which yields the same distribution of U. As a special case, considering only one of the ω_α and calling it ψ, we obtain that if ϕ is any fixed orthonormal basis and ψ a random vector with uniform distribution then the joint distribution of the coefficients $\langle\phi_\beta|\psi\rangle$ is the same as if ψ were any fixed unit vector and ϕ random with uniform distribution.

The concept of 'most times' is a little more involved because it involves a limiting procedure. Let $\delta' > 0$ be given; we say that a statement $p(t)$ *holds for* $(1-\delta')$-*most* t (in the long run) if and only if

$$\liminf_{T\to\infty} \frac{1}{T} \left|\left\{0 < t < T \,\middle|\, p(t) \text{ holds}\right\}\right| \geq 1 - \delta', \tag{2.2}$$

where $|M|$ denotes the size (Lebesgue measure) of the set $M \subseteq \mathbb{R}$. (So this concept of 'most' does not directly correspond to a probability distribution.)

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3. The method of appeal to typicality

We would like to clarify the status of statements about 'most' $\mathscr{D}$ (or, for that matter, most H or most ψ_0), and in so doing elaborate on von Neumann's method of appeal to typicality. Fierz (1955, p. 711) criticized this method as follows:[4]

> The physical justification of the hypothesis [that all observers are equally probable] is of course questionable, as the assumption of equal probability for all observers is entirely without reason. Not every macroscopic observable in the sense of von Neumann will really be measurable. Moreover, the observer will try to measure exactly those quantities which appear characteristic of a given system.

In the same vein, Pauli (1956) wrote in a private letter to Fierz:

> As far as assumption B [that all observers are equally probable] is concerned [...] I consider it *now* not only as lacking in plausibility, but *nonsense*.

Concerning these objections, we first note that it is surely informative that normality holds for some $\mathscr{D}$s, let alone that it holds in fact for most $\mathscr{D}$s, with 'most' understood in a mathematically natural way. But we believe that more should be said.

When employing the method of appeal to typicality, one usually uses the language of probability theory. When we do so, we do not mean to imply that any of the objects considered is random in reality. What we mean is that certain sets (of wave functions, of orthonormal bases, etc.) have certain sizes (e.g. close to 1) in terms of certain natural measures of size. That is, we describe the behaviour that is *typical* of wave functions, orthonormal bases, etc. However, since the mathematics is equivalent to that of probability theory, it is convenient to adopt that language. For this reason, we do not mean, when using a normalized measure μ, to make an 'assumption of *a priori* probabilities', even if we use the word 'probability.' Rather, we have in mind that, if a condition is true of most $\mathscr{D}$, or most H, this fact may *suggest* that the condition is also true of a concrete given system, unless we have reasons to expect otherwise.

Of course, a theorem saying that a condition is true of the vast majority of systems does not *prove* anything about a concrete given system; if we want to know for sure whether a given system is normal for every initial wave function, we need to check the relevant condition, which is equation (5.6). Nevertheless, a typicality theorem is, as we have suggested, illuminating; at the very least, it is certainly useful to know which behaviour is typical and which is exceptional. Note also that the terminology of calling a system 'typical' or 'atypical' might easily lead us to wrongly conclude that an atypical system will not be normal. A given system may have some properties that are atypical and nevertheless satisfy the condition (5.6) implying that the system is normal for every initial wave function.

[4]This quotation was translated from German by R. Tumulka.

The method of appeal to typicality belongs to a long tradition in physics, which includes also Wigner's work on random matrices of the 1950s. In the words of Wigner (1967):

> One [...] deals with a specific system, with its proper (though in many cases unknown) Hamiltonian, yet pretends that one deals with a multitude of systems, all with their own Hamiltonians, and averages over the properties of these systems. Evidently, such a procedure can be meaningful only if it turns out that the properties in which one is interested are the same for the vast majority of the admissible Hamiltonians.

This method was used by Wigner to obtain specific new and surprising predictions about detailed properties of complex quantum systems in nuclear physics. Here the method of appeal to typicality is used to establish much less, *viz.* approach to thermal equilibrium.

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4. Bounds on deviations

Two different definitions of normality are relevant to our discussion. Consider a system for which $\mathscr{H}, H, \mathscr{D}$ and ψ_0 are given. Let N denote the number of macro-spaces $\mathscr{H}_\nu$, and let $\varepsilon > 0$ and $\delta' > 0$ also be given.

Definition 4.1. The system is ε-δ'-normal in von Neumann's (1929) sense if and only if, for $(1-\delta')$-most t in the long run

$$\left| \|P_\nu \psi_t\|^2 - \frac{d_\nu}{D} \right| < \varepsilon \sqrt{\frac{d_\nu}{ND}} \tag{4.1}$$

for all ν.

Let us connect this to how von Neumann formulated the property considered in the QET, which is: for $(1-\delta')$-most t in the long run,

$$\left| \langle \psi_t | A | \psi_t \rangle - \frac{\operatorname{tr} A}{D} \right| < \varepsilon \sqrt{\frac{\operatorname{tr}(A^2)}{D}} \tag{4.2}$$

for every real-linear combination ('macro-observable') $A = \sum_\nu \alpha_\nu P_\nu$. The quantity $\operatorname{tr} A/D = \operatorname{tr}(\rho_{\mathrm{mc}} A)$ is the micro-canonical average of the observable A. The quantity $\sqrt{\operatorname{tr}(A^2)/D} = \sqrt{\operatorname{tr}(\rho_{\mathrm{mc}} A^2)}$ was suggested by von Neumann as a measure of the magnitude of the observable A in the micro-canonical average. To see that equation (4.2) is more or less equivalent to equation (4.1), note first that equation (4.2) implies, by setting one $\alpha_\nu = 1$ and all others to zero, that

$$\left| \|P_\nu \psi_t\|^2 - \frac{d_\nu}{D} \right| < \varepsilon \sqrt{\frac{d_\nu}{D}}. \tag{4.3}$$

This is only slightly weaker than equation (4.1), namely by a factor of $\sqrt{N}$, when N is much smaller than D/d_ν, as would be the case for the $\mathscr{H}_\nu$ considered by von Neumann. Conversely, equation (4.1) for every ν implies equation (4.2) for every A. This follows from

$$\sum_\nu |x_\nu| \le \sqrt{N} \sqrt{\sum_\nu |x_\nu|^2}, \tag{4.4}$$

a consequence of the Cauchy–Schwarz inequality, by setting $x_\nu = \alpha_\nu \varepsilon \sqrt{d_\nu/ND}$.

Definition 4.2. The system is ε-δ'-normal in the strong sense if and only if, for $(1-\delta')$-most t in the long run

$$\left| \|P_\nu \psi_t\|^2 - \frac{d_\nu}{D} \right| < \varepsilon \frac{d_\nu}{D} \tag{4.5}$$

for all ν.

In the cases considered by von Neumann equation (4.5) is a much stronger inequality than equation (4.1). The motivation for considering equation (4.5) is twofold. On the one hand, lemma 1.1 implies that for most wave functions φ, the deviation of $\|P_\nu \varphi\|^2$ from d_ν/D is actually smaller than d_ν/D. (Indeed, the Chebyshev inequality yields for $X = \|P_\nu \varphi\|^2$ that

$$\mu\left(|X - d_\nu/D| < \varepsilon \frac{d_\nu}{D} \right) \geq 1 - \frac{VarX}{(\varepsilon d_\nu/D)^2} \geq 1 - \frac{1}{\varepsilon^2 d_\nu}, \tag{4.6}$$

which tends to 1 as $d_\nu \to \infty$.) On the other hand, strong normality means that $\|P_\nu \psi_t\|^2$ actually is *close* to d_ν/D, as the *relative error* is small. In contrast, the bound in equation (4.1) is greater than the value to be approximated, and so would not justify the claim $\|P_\nu \psi_t\|^2 \approx d_\nu/D$.

The basic (trivial) observation about normality is the following lemma.

Lemma 4.3. *For arbitrary $\mathscr{H}, H, \mathscr{D}, \psi_0$ with $\|\psi_0\| = 1$ and any $\varepsilon > 0$ and $\delta' > 0$, if*

$$G = G(H, \mathscr{D}, \psi_0, \nu) := \overline{\left| \|P_\nu \psi_t\|^2 - \frac{d_\nu}{D} \right|^2} < \varepsilon^2 \frac{d_\nu}{ND} \frac{\delta'}{N} =: \text{bound}_1 \tag{4.7}$$

for every ν then the system is ε-δ'-normal in von Neumann's sense. If

$$G < \varepsilon^2 \frac{d_\nu^2}{D^2} \frac{\delta'}{N} =: \text{bound}_2 \tag{4.8}$$

for every ν then the system is ε-δ'-normal in the strong sense.

Proof. If a non-negative quantity $f(t)$ (such as the $|\cdots|^2$ above) is greater than or equal to $a := \varepsilon^2 d_\nu/ND > 0$ for more than the fraction $b := \delta'/N > 0$ of the time interval $[0, T]$ then its average over $[0, T]$ must be greater than ab. By assumption (4.7), this is not the case for any ν when T is sufficiently large. But $|\cdots|^2 \geq a$ means violating equation (4.1). Therefore, for sufficiently large T, the fraction of the time when equation (4.1) is violated for any ν is no greater than δ'; thus, equation (2.2) holds with $p(t)$ given by $\forall \nu$: equation (4.1).

In the same way, one obtains equation (4.5) from equation (4.8). ■

5. Von Neumann's QET

We now describe von Neumann's result. To evaluate the expression G, let $\phi_1, \ldots, \phi_D$ be an orthonormal basis of $\mathscr{H}$ consisting of eigenvectors of the Hamiltonian H with eigenvalues $E_1, \ldots, E_D$, and expand ψ_0 in

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that basis

$$\psi_0 = \sum_{\alpha=1}^{D} c_\alpha \phi_\alpha, \quad \psi_t = \sum_{\alpha=1}^{D} \mathrm{e}^{-\mathrm{i}E_\alpha t} c_\alpha \phi_\alpha. \tag{5.1}$$

Inserting this into G and multiplying out the square, one obtains

$$G = \sum_{\alpha,\alpha',\beta,\beta'} \overline{\mathrm{e}^{\mathrm{i}(E_\alpha - E_{\alpha'} - E_\beta + E_{\beta'})t}} c_\alpha^* c_{\alpha'} c_\beta c_{\beta'}^* \langle \phi_\alpha | P_\nu | \phi_\beta \rangle \langle \phi_{\alpha'} | P_\nu | \phi_{\beta'} \rangle^*$$
$$- 2\frac{d_\nu}{D} \Re \sum_{\alpha,\beta} \overline{\mathrm{e}^{\mathrm{i}(E_\alpha - E_\beta)t}} c_\alpha^* c_\beta \langle \phi_\alpha | P_\nu | \phi_\beta \rangle + \frac{d_\nu^2}{D^2}. \tag{5.2}$$

If H is non-degenerate, then $E_\alpha - E_\beta$ vanishes only for $\alpha = \beta$, so the time-averaged exponential in the last line is $\delta_{\alpha\beta}$. Furthermore, if H has no resonances then the time-averaged exponential in the first line of equation (5.2) becomes $\delta_{\alpha\alpha'}\delta_{\beta\beta'} + \delta_{\alpha\beta}\delta_{\alpha'\beta'} - \delta_{\alpha\alpha'}\delta_{\beta\beta'}\delta_{\alpha\beta}$, and we have that

$$G = \sum_{\alpha,\beta} |c_\alpha|^2 |c_\beta|^2 (|\langle \phi_\alpha | P_\nu | \phi_\beta \rangle|^2 + \langle \phi_\alpha | P_\nu | \phi_\alpha \rangle \langle \phi_\beta | P_\nu | \phi_\beta \rangle)$$
$$- \sum_{\alpha} |c_\alpha|^4 \langle \phi_\alpha | P_\nu | \phi_\alpha \rangle^2 - 2\frac{d_\nu}{D} \sum_{\alpha} |c_\alpha|^2 \langle \phi_\alpha | P_\nu | \phi_\alpha \rangle + \frac{d_\nu^2}{D^2} \tag{5.3}$$

$$= \sum_{\alpha \neq \beta} |c_\alpha|^2 |c_\beta|^2 |\langle \phi_\alpha | P_\nu | \phi_\beta \rangle|^2 + \left(\sum_{\alpha} |c_\alpha|^2 \langle \phi_\alpha | P_\nu | \phi_\alpha \rangle - \frac{d_\nu}{D} \right)^2 \tag{5.4}$$

$$\leq \max_{\alpha \neq \beta} |\langle \phi_\alpha | P_\nu | \phi_\beta \rangle|^2 + \max_{\alpha} \left(\langle \phi_\alpha | P_\nu | \phi_\alpha \rangle - \frac{d_\nu}{D} \right)^2 \tag{5.5}$$

using $\sum |c_\alpha|^2 = 1$. This calculation proves the following.

Lemma 5.1. *For arbitrary $\mathscr{H}$ and $\mathscr{D}$, for any H without degeneracies and resonances, and for any $\varepsilon > 0$ and $\delta' > 0$, if, for every ν,*

$$\max_{\alpha \neq \beta} |\langle \phi_\alpha | P_\nu | \phi_\beta \rangle|^2 + \max_{\alpha} \left(\langle \phi_\alpha | P_\nu | \phi_\alpha \rangle - \frac{d_\nu}{D} \right)^2 < \mathrm{bound}_{1,2} \tag{5.6}$$

then, for every $\psi_0 \in \mathscr{H}$ with $\|\psi_0\| = 1$, the system is ε-δ'-normal in von Neumann's sense (respectively, in the strong sense).

Note that every initial wave function behaves normally, provided H and $\mathscr{D}$ together satisfy condition (5.6). Now von Neumann's QET asserts that for any given H and any suitable given values of the d_ν, most $\mathscr{D}$ will satisfy equation (5.6). It is convenient to think of $\mathscr{D}$ as arising from a uniformly distributed orthonormal basis $\omega_1, \ldots, \omega_D$ in the sense that $\mathscr{H}_\nu$ is spanned by those ω_α with $\alpha \in J_\nu$, as described in §2. The coefficients $U_{\alpha\beta} = \langle \phi_\beta | \omega_\alpha \rangle$ of ω_α relative to the eigenbasis of H then form a Haar-distributed unitary matrix, and

$$\langle \phi_\alpha | P_\nu | \phi_\beta \rangle = \sum_{\gamma \in J_\nu} \langle \phi_\alpha | \omega_\gamma \rangle \langle \omega_\gamma | \phi_\beta \rangle = \sum_{\gamma \in J_\nu} U_{\gamma\alpha} (U_{\gamma\beta})^*. \tag{5.7}$$

Let log denote the natural logarithm.

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Lemma 5.2 (von Neumann 1929**).** *There is a (big) constant $C_1 > 1$ such that whenever the two natural numbers D and d_ν satisfy*

$$C_1 \log D < d_\nu < \frac{D}{C_1}, \tag{5.8}$$

and U is a Haar-distributed random unitary $D \times D$ matrix, then

$$\mathbb{E} \max_{\alpha\neq\beta=1}^{D} \left| \sum_{\gamma=1}^{d_\nu} U_{\gamma\alpha} (U_{\gamma\beta})^* \right|^2 \leq \frac{\log D}{D}, \tag{5.9}$$

and

$$\mathbb{E} \max_{\alpha=1}^{D} \left(\sum_{\gamma=1}^{d_\nu} |U_{\gamma\alpha}|^2 - \frac{d_\nu}{D} \right)^2 \leq \frac{9 d_\nu \log D}{D^2}. \tag{5.10}$$

To express that $\mu\{x|p(x)\} \geq 1-\delta$, we also say that $p(x)$ holds for $(1-\delta)$-most x. Putting together lemma 5.1 (for bound_1) and lemma 5.2, we have the following:[5]

Theorem 5.3 (von Neumann's QET 1929**).** *Let $\varepsilon > 0$, $\delta > 0$ and $\delta' > 0$. Suppose the numbers D, N and $d_1, \ldots, d_N$ are such that $d_1 + \ldots + d_N = D$ and, for all ν,*

$$\max\left(C_1, \frac{10N^2}{\varepsilon^2 \delta' \delta}\right) \log D < d_\nu < \frac{D}{C_1}, \tag{5.11}$$

where C_1 is the constant of lemma 5.2. For arbitrary $\mathscr{H}$ of dimension D and any H without degeneracies and resonances, $(1-\delta)$-most orthogonal decompositions $\mathscr{D} = \{\mathscr{H}_\nu\}$ of $\mathscr{H}$ with $\dim \mathscr{H}_\nu = d_\nu$ are such that for every wave function $\psi_0 \in \mathscr{H}$ with $\|\psi_0\| = 1$ the system is ε-δ'-normal in von Neumann's sense.

Proof. Regard $\mathscr{D}$ as random with uniform distribution and let X be the left-hand side of equation (5.6). Using equation (5.11), it follows from lemma 5.2 that $\mathbb{E}X \leq 10 \log D/D$. By Markov's inequality,

$$\mathbb{P}(X \geq \text{bound}_1) \leq \frac{\mathbb{E}X}{\text{bound}_1} \leq \frac{10 \log D}{D \, \text{bound}_1} < \delta, \tag{5.12}$$

using equation (5.11) again. Theorem 5.3 then follows from lemma 5.1. ■

6. Strong version

It is an unsatisfactory feature of the QET that all d_ν are assumed to be much smaller (by at least a factor C_1) than D, an assumption excluding that one of the macro-states ν corresponds to thermal equilibrium. However, this assumption can be removed, and even the strong sense of normality can be concluded. An inspection of von Neumann's proof of lemma 5.2 reveals that it actually proves the following.

[5]For clarity, we have modified von Neumann's statement a bit.

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Lemma 6.1 (von Neumann 1929). *There is a (big) constant $C_2 > 1$ such that whenever the two natural numbers D and d_ν satisfy*

$$C_2 < d_\nu < D - C_2, \tag{6.1}$$

and U is a Haar-distributed random unitary $D \times D$ matrix then, for every $0 < a < d_\nu^2/D^2 C_2$,

$$\mathbb{P}\left(\max_{\alpha\neq\beta=1}^{D} \left| \sum_{\gamma=1}^{d_\nu} U_{\gamma\alpha} (U_{\gamma\beta})^* \right|^2 \geq a \right) \leq \frac{D^2}{2} \exp\left(-4a(D-1)\right) \tag{6.2}$$

and

$$\mathbb{P}\left(\max_{\alpha=1}^{D} \left(\sum_{\gamma=1}^{d_\nu} |U_{\gamma\alpha}|^2 - \frac{d_\nu}{D} \right)^2 \geq a \right) \leq \frac{D^3}{\sqrt{2\pi d_\nu}(D-d_\nu)} \exp\left(-\Theta \frac{D^2 a}{2 d_\nu} \right), \tag{6.3}$$

with $\Theta = 1 - 2/3\sqrt{C_2}$.

From this, we can obtain, with lemma 5.1, the following stronger version of the QET, which von Neumann did not mention.

Theorem 6.2. *Theorem* 5.3 *remains valid if one replaces 'normal in von Neumann's sense' by 'normal in the strong sense' and equation (5.11) by*

$$\max\left(C_2, \sqrt{\left(\frac{3N}{\varepsilon^2\delta'}\right) D \log D} \right) < d_\nu < D - C_2 \tag{6.4}$$

and

$$\varepsilon^2\delta' < \frac{2N}{C_2}, \quad \frac{D}{\log D} > \frac{100N}{\varepsilon^2\delta'} \quad \text{and} \quad D > \frac{1}{\delta}, \tag{6.5}$$

where C_2 is the constant of lemma 6.1.

Proof. Set $a = \text{bound}_2/2 = (\varepsilon^2\delta'/2N)(d_\nu/D)^2$ in equations (6.2) and (6.3). The first assumption in equation (6.5) ensures that the condition $a < d_\nu^2/D^2 C_2$ in lemma 6.1 is satisfied. Assumption (6.4) includes

$$d_\nu^2 > \frac{3N}{\varepsilon^2\delta'} D \log D \tag{6.6}$$

$$> \frac{N}{\varepsilon^2\delta'} D(2\log D - \log\delta) \tag{6.7}$$

using $\log D > -\log\delta$ from the third assumption in equation (6.5). Now equation (6.7) implies that $4a(D-1) > 2aD \geq 2\log D - \log\delta$, so that the right-hand side of equation (6.2) is less than $\delta/2$. Furthermore, from the

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second assumption in equation (6.5), we have that $1 > 100N \log D/\varepsilon^2\delta' D$, which yields with equation (6.6) that $d_v^2 > (300N^2/\varepsilon^4\delta'^2)\log^2 D$, and thus $d_v > (16N/\Theta\varepsilon^2\delta')\log D$, using $\Theta > 16/\sqrt{300}$ (which follows from $C_2 \geq 121$). Because of $\log D > -\log\delta$, we have that

$$d_v > \frac{4N}{\Theta\varepsilon^2\delta'}(3\log D - \log\delta), \tag{6.8}$$

which implies that $\Theta D^2 a/2d_v = \Theta(\varepsilon^2\delta'/4N)\,d_v > 3\log D - \log\delta$, so also the right-hand side of equation (6.3) is less than $\delta/2$. Thus, equation (5.6) is fulfilled for bound$_2$ with probability at least $1-\delta$. ■

The stronger conclusion requires the strong assumption that $\sqrt{D\log D} \ll d_v$, whereas von Neumann's version needed $\log D \ll d_v \ll D$.

Concerning a thermal equilibrium macro-state with $d_{\mathrm{eq}}/D \geq 1-\varepsilon$, theorem 6.2 provides conditions under which most subspaces $\mathscr{H}_{\mathrm{eq}}$ of dimension d_{eq} are such that, for every $\psi_0 \in \mathscr{H}$ with $\|\psi_0\| = 1$, the system will be in thermal equilibrium for most times. More precisely, theorem 6.2 implies the following. Let $\varepsilon > 0$, $\delta > 0$, and $\delta' > 0$. Suppose that the number D is so big that equation (6.5) holds with $N = 2$ and that d_{eq} is such that

$$1 - \varepsilon \leq \frac{d_{\mathrm{eq}}}{D} \leq 1 \tag{6.9}$$

and

$$\max\left(C_2, \sqrt{(6/\varepsilon^2\delta')D\log D}\right) < d_{\mathrm{eq}} < D - \max\left(C_2, \sqrt{(6/\varepsilon^2\delta')D\log D}\right). \tag{6.10}$$

For arbitrary $\mathscr{H}$ of dimension D and any Hamiltonian H without degeneracies and resonances, $(1-\delta)$-most subspaces $\mathscr{H}_{\mathrm{eq}}$ of $\mathscr{H}$ with $\dim\mathscr{H}_{\mathrm{eq}} = d_{\mathrm{eq}}$ are such that for every wave function $\psi_0 \in \mathscr{H}$ with $\|\psi_0\| = 1$, the relation

$$\|P_{\mathrm{eq}}\psi_t\|^2 > 1 - 2\varepsilon \tag{6.11}$$

holds for $(1-\delta')$-most t. In this statement, however, the conditions can be relaxed (in particular, H may have resonances, and the upper bound on d_{eq} in equation (6.10) can be replaced with D), and the statement can be obtained through a proof that is much simpler than von Neumann's (see Goldstein *et al.* 2010).

7. Misrepresentations

We now show that the statements presented as the QET in Farquhar & Landsberg (1957) and Bocchieri & Loinger (1958) differ from the original theorem (in fact, in inequivalent ways) and are dynamically vacuous.

It is helpful to introduce the symbol $\forall\!\!\!\forall$ to denote 'for most'. It can be regarded as a quantifier like the standard symbols $\forall$ (for all) and $\exists$ (for at least one). So, if $p(x)$ is a statement containing the free variable x then we write $\forall\!\!\!\forall x : p(x)$

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when we mean $\mu\{x|p(x)\} \geq 1-\delta$, assuming that it is clear from the context which measure μ and which magnitude of δ are intended. With this notation, the misunderstanding as described in equation (1.26) versus equation (1.25) can be expressed by saying that the quantifiers $\bigvee x$ and $\forall y$ do not commute:

$$\bigvee x \forall y : p(x,y) \quad \not\Leftrightarrow \quad \forall y \bigvee x : p(x,y). \tag{7.1}$$

The two expressions are not equivalent. Indeed, the set of x's (whose measure is close to 1) is allowed to depend on y if the quantifiers are of the form $\forall y \bigvee x$ but not if they are of the form $\bigvee x \forall y$. That is, if they are of the form $\bigvee x \forall y$, then there exists a set M of x's, not depending on y, with $\mu_x(M) \geq 1-\delta$ such that $\forall x \in M \forall y : p(x,y)$. Thus the first expression in equation (7.1) is stronger than the second

$$\bigvee x \forall y : p(x,y) \quad \Rightarrow \quad \forall y \bigvee x : p(x,y). \tag{7.2}$$

This should be contrasted with situations in which quantifiers do commute, for example $\forall x \forall y \Leftrightarrow \forall y \forall x$ and $\bigvee x \bigvee y \Leftrightarrow \bigvee y \bigvee x$ (though the bound δ on the exceptions may become worse[6]). An exceptional case, in which $\bigvee x$ and $\forall y$ do commute, occurs when the variable y assumes only a very limited number n (e.g. $n=10$) of possible values: Then $\forall y \bigvee x : p$ implies $\bigvee x \forall y : p$ with, however, the bound δ on the exceptions worse by a factor of n, $\delta \to n\delta$. In our case, however, $y = \psi_0$ varies in an infinite set.

In this symbolic notation, and leaving out some details, theorems 5.3 and 6.2 can be paraphrased as

$$\forall H \bigvee \mathscr{D} \forall \psi_0 \bigvee t \forall \nu : \|P_\nu \psi_t\|^2 \approx \frac{d_\nu}{D}, \tag{7.5}$$

where $\forall H$ should be taken to mean 'for all Hamiltonians without degeneracies and resonances', and $\approx$ should be understood either in the wide sense of equation (4.1) for theorem 5.3, or in the sense of equation (4.5) for theorem 6.2. Let us now look at what Farquhar & Landsberg (1957) and Bocchieri & Loinger (1958) write.

We focus first on the article of Bocchieri & Loinger (1958). As we show presently, their version of the QET has a different order of quantifiers, with fatal consequences. It also differs in a second way from the original as it deals with

[6]More precisely, if

$$\mu_x\big\{x|\mu_y\{y|p(x,y)\} \geq 1-\delta_y\big\} \geq 1-\delta_x \tag{7.3}$$

then, for every $\varepsilon_x > 0$,

$$\mu_y\big\{y|\mu_x\{x|p(x,y)\} \geq 1-\varepsilon_x\big\} \geq 1-\varepsilon_y \tag{7.4}$$

with $\varepsilon_y \geq (\delta_x + \delta_y - \delta_x\delta_y)/\varepsilon_x$. (For example, equation (7.4) holds for $\varepsilon_x = \varepsilon_y = \sqrt{\delta_x + \delta_y}$.) To see this, note that equation (7.3) implies that, relative to the product measure $\mu_x \otimes \mu_y$, at least the fraction $(1-\delta_x)(1-\delta_y)$ of all pairs (x,y) satisfies $p(x,y)$; thus,

$$\int \mu_y(\mathrm{d}y)\mu_x\{x|p(x,y)\} = \mu_x \otimes \mu_y\{(x,y)|p(x,y)\} \geq 1-(\delta_x + \delta_y - \delta_x\delta_y),$$

and this implies equation (7.4).

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the strong sense of normality instead of von Neumann's sense; this, of course, is a strengthening of von Neumann's statement. Finally, their version drops von Neumann's hypotheses on the Hamiltonian (no degeneracy, no resonance); this, of course, is a difference that Bocchieri and Loinger were aware of and emphasized as evidence that von Neumann made unnecessary hypotheses.

Indeed, in Bocchieri & Loinger (1958), the statement 'These relations constitute von Neumann's ergodic theorem' (p. 670) is preceded by their eqn (13), which in our notation reads

$$\mathbb{E}\overline{\|P_\nu\psi_t\|^2} = \frac{d_\nu}{D}; \quad \frac{\mathbb{E}\overline{\left|\|P_\nu\psi_t\|^2 - d_\nu/D\right|^2}}{d_\nu^2/D^2} \ll 1, \tag{7.6}$$

where the average $\mathbb{E}$ is taken over $\mathscr{D}$ relative to the uniform distribution.[7] From this, it follows that for all ψ_0 it is true for most $\mathscr{D}$ that $\|P_\nu\psi_t\|^2 \approx d_\nu/D$ for most t, with deviation small compared with d_ν/D. Moreover, as equation (7.6) holds for all H, and, via equation (4.8), the conclusion can be shown to hold simultaneously for all ν, the version of Bocchieri & Loinger (1958) can be written, in analogy to equation (7.5), as

$$\forall H\, \forall\psi_0\, \bigvee\mathscr{D}\, \bigvee t\, \forall\nu : \|P_\nu\psi_t\|^2 \approx \frac{d_\nu}{D}. \tag{7.7}$$

This statement is not only inequivalent to von Neumann's, it is also *dynamically vacuous*. By this, we mean that it follows from a statement that does not refer to any time other than 0. Indeed, the relations (7.6) are proved in Bocchieri & Loinger (1958) by first proving for any fixed ψ that[8]

$$\mathbb{E}\|P_\nu\psi\|^2 = \frac{d_\nu}{D}; \quad \frac{\mathbb{E}\left|\|P_\nu\psi\|^2 - d_\nu/D\right|^2}{d_\nu^2/D^2} \ll 1, \tag{7.8}$$

which is equation (7.6) without the procedure of time averaging, then setting $\psi = \psi_t$ and taking the time average on both relations, and finally commuting the time average and the average $\mathbb{E}$ over $\mathscr{D}$, which is always allowed by Fubini's theorem. In the notation using the symbol $\bigvee$, equation (7.8) yields

$$\forall\psi\, \bigvee\mathscr{D}\, \forall\nu : \|P_\nu\psi\|^2 \approx \frac{d_\nu}{D}. \tag{7.9}$$

This fact is the non-dynamical reason why equation (7.7) is true: since equation (7.9) applies to every ψ, it applies in particular to ψ_t for any H, ψ_0, and t. That is, equation (7.9) implies

$$\forall H\, \forall\psi_0\, \forall t\, \bigvee\mathscr{D}\, \forall\nu : \|P_\nu\psi_t\|^2 \approx \frac{d_\nu}{D}, \tag{7.10}$$

[7]More precisely, their proof shows that for every $\eta > 0$ and every H, if every $d_\nu > 1/\eta$ then, for all ψ_0 and ν, $\mathbb{E}\overline{\left|\|P_\nu\psi_t\|^2 - d_\nu/D\right|^2} < \eta d_\nu^2/D^2$.

[8]In fact, these expectation values are independent of ψ, by the invariance of the Haar measure.

and since $\forall t \bigvee \mathscr{D} \Rightarrow \bigvee t \bigvee \mathscr{D} \Rightarrow \bigvee \mathscr{D} \bigvee t$, equation (7.10) implies equation (7.7). Thus, equation (7.7) is dynamically vacuous. This fact was essentially the criticism put forward against the QET in Bocchieri & Loinger (1958).[9]

We turn to the article of Farquhar & Landsberg (1957). As we show presently, their version of the QET differs from the original in the same ways as the version of Bocchieri & Loinger (1958), as well as in that it concerns only the *time average* of $\|P_\nu\psi_t\|^2$, while the original QET concerns the value of $\|P_\nu\psi_t\|^2$ for most t.

Indeed, the result on which their version of the QET is based is expressed in their eqn (2.17), which holds for every H and $D \geq 3$ and reads in our notation as

$$\frac{\mathbb{E}\left|\overline{\|P_\nu\psi_t\|^2} - d_\nu/D\right|^2}{d_\nu^2/D^2} < \frac{2(D-d_\nu)}{d_\nu D}. \tag{7.11}$$

For large d_ν, this yields

$$\frac{\mathbb{E}\left|\overline{\|P_\nu\psi_t\|^2} - d_\nu/D\right|^2}{d_\nu^2/D^2} \ll 1, \tag{7.12}$$

and thus

$$\forall H\ \forall\psi_0\ \bigvee \mathscr{D}\ \forall \nu : \overline{\|P_\nu\psi_t\|^2} \approx \frac{d_\nu}{D}. \tag{7.13}$$

This result concerns only the time average of $\|P_\nu\psi_t\|^2$ but provides no control over the time variance, and so does not inform us about the behavior for most t. Moreover, equation (7.13) has the wrong order of quantifiers. Finally, since equation (7.12) follows from the inequality in equation (7.6) using $\overline{f(t)}^2 \leq \overline{f(t)^2}$, it is a logical consequence of a dynamically vacuous statement and, thus, is itself dynamically vacuous.

8. Typical Hamiltonian

Normality for most $\mathscr{D}$s is more or less equivalent to normality for most Hs. Indeed, by the 'unitary inversion trick' described in §2, one can trade the typicality assumption on $\mathscr{D}$ in the QET for a typicality assumption on H, without any essential modification of the proof. This is because the relevant condition (5.6)

[9]The exact nature of the criticism, though, remained a bit unclear in Bocchieri & Loinger (1958), as they did not make explicit what it means for a statement to be dynamically vacuous. They pointed out that equation (7.6) is valid for every Hamiltonian, including $H = 0$, and that the proof of equation (7.6) by means of equation (7.8) did not, in fact, require that $\psi_t = \exp(-\mathrm{i}Ht)\psi_0$, but only that $\psi_t = f_t(\psi_0)$ for an arbitrary measure-preserving mapping f_t from the unit sphere to itself. These facts strongly suggest that equation (7.6) is dynamically vacuous, but should *per se* not be regarded as a proof; for example, the Poincaré recurrence theorem is valid for every Hamiltonian, or in fact for every measure-preserving flow f_t on the unit sphere in a finite-dimensional Hilbert space, but clearly has dynamical content. That is why we defined a 'dynamically vacuous statement' to be a logical consequence of a statement that does not refer to time.

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involves only

$$\langle\phi_\alpha|P_\nu|\phi_\beta\rangle = \sum_{\gamma\in J_\nu}\langle\phi_\alpha|\omega_\gamma\rangle\langle\omega_\gamma|\phi_\beta\rangle\,, \tag{8.1}$$

where we can either regard ϕ as fixed and ω as random (as von Neumann did) or vice versa. With this change, the (strong) QET reads as follows.

Theorem 8.1. *Let $\varepsilon>0$, $\delta'>0$ and $\delta>0$. Suppose the numbers D, N and $d_1+\cdots+d_N=D$ satisfy equations (6.4) and (6.5). Suppose further that the real numbers $E_1,\ldots,E_D$ are all distinct and have no resonances as defined in equation (1.23). For arbitrary $\mathscr{H}$ of dimension D and any orthogonal decomposition $\mathscr{D}=\{\mathscr{H}_\nu\}$ with $\dim\mathscr{H}_\nu=d_\nu$, $(1-\delta)$-most operators H with eigenvalues $E_1,\ldots,E_D$ are such that for every wave function $\psi_0\in\mathscr{H}$ with $\|\psi_0\|=1$ the system is ε-δ'-normal in the strong sense.*

This means, in the notation of equation (7.5), that

$$\forall\mathscr{D}\bigvee H\,\forall\psi_0\bigvee t\,\forall\nu:\|P_\nu\psi_t\|^2\approx\frac{d_\nu}{D}. \tag{8.2}$$

It would be nice also to have a similar theorem asserting that normality for all ψ_0 is typical even within a smaller class of Hamiltonians, say those of the form

$$H=-\sum_{i=1}^{n}\frac{\hbar^2\nabla_i^2}{2m_i}+\sum_{i=1}^{n}U(x_i)+\sum_{i\neq j}V(x_i-x_j), \tag{8.3}$$

where the pair potential V is allowed to be any function from a suitable class. Here, n denotes the number of particles, $x_i\in\mathbb{R}^3$ the coordinate of particle i, ∇_i the derivative relative to x_i, m_i the mass of particle i and U the external potential. However, such a theorem seems presently out of reach.

As a corollary of equation (8.2), one obtains for $\nu=\mathrm{eq}$ that

$$\forall\mathscr{H}_{\mathrm{eq}}\bigvee H\,\forall\psi_0\bigvee t:\|P_{\mathrm{eq}}\psi_t\|^2\approx 1, \tag{8.4}$$

where $\forall\mathscr{H}_{\mathrm{eq}}$ should be taken to mean 'for all subspaces $\mathscr{H}_{\mathrm{eq}}$ of dimension d_{eq}' (which is greater than $(1-\varepsilon')D$). In fact, this conclusion remains true (Goldstein *et al.* 2010) under weaker technical assumptions (H may have resonances, and equation (6.4) can be replaced by $(1-\varepsilon')D<d_{\mathrm{eq}}\leq D$).

As a corollary of equation (8.4), for a typical Hamiltonian every energy eigenfunction is in thermal equilibrium, i.e. close to $\mathscr{H}_{\mathrm{eq}}$. (This statement could, of course, be obtained more directly. The condition that every energy eigenfunction is in equilibrium is a special case, for $\nu=\mathrm{eq}$, of the condition $\langle\phi_\alpha|P_\nu|\phi_\alpha\rangle\approx d_\nu/D$ for all α, which is part of condition (5.6), which by lemma 5.2 is typically obeyed.)

We can be a bit more general than either theorem 6.2 or theorem 8.1 and say that what is needed to obtain strong normality is that the unitary matrix $U_{\alpha\beta}=\langle\phi_\beta|\omega_\alpha\rangle$ relating the energy eigenbasis ϕ_β to a basis ω_α aligned with $\mathscr{D}$ be like most unitary matrices in that they satisfy equation (5.6). This means, more or less, that the energy eigenbasis and $\mathscr{D}$ should be unrelated. By the way, this is connected to the reason why $\mathscr{H}$ was physically interpreted as a 'micro-canonical' space, i.e. one corresponding to an 'energy shell'. For a more comprehensive Hilbert space including states of macroscopically different energies, the energy eigenbasis and $\mathscr{D}$

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would no longer be unrelated. Indeed, a sufficiently coarse-grained version of the Hamiltonian should be among the macroscopic observables and thus be diagonal in the ω_α basis.

9. Comparison with the recent literature

The results of Tasaki (1998), Reimann (2008) and Linden *et al.* (2009) also concern conditions under which a quantum system will spend most of the time in 'thermal equilibrium'. For the sake of comparison, their results, as well as von Neumann's, can be described in a unified way as follows. Let us say that a system with initial wave function $\psi(0)$ *equilibrates* relative to a class $\mathscr{A}$ of observables if for most times τ,

$$\langle\psi(\tau)|A|\psi(\tau)\rangle \approx \mathrm{Tr}\left(\overline{|\psi(t)\rangle\langle\psi(t)|}A\right) \quad \text{for all } A\in\mathscr{A}. \tag{9.1}$$

We then say that the system *thermalizes* relative to $\mathscr{A}$ if it equilibrates and, moreover,

$$\mathrm{Tr}\left(\overline{|\psi(t)\rangle\langle\psi(t)|}A\right) \approx \mathrm{Tr}(\rho_{\mathrm{mc}}A) \quad \text{for all } A\in\mathscr{A}, \tag{9.2}$$

with ρ_{mc} the micro-canonical density matrix (in our notation, $1/D$ times the projection P to $\mathscr{H}$). With these definitions, the results of Tasaki (1998), Reimann (2008), Linden *et al.* (2009) can be formulated by saying that, under suitable hypotheses on H and $\psi(0)$ and for large enough D, a system will equilibrate, or even thermalize, relative to a suitable class $\mathscr{A}$. Von Neumann's QET establishes thermalization for a family $\mathscr{A}$ of commuting observables, the algebra generated by $\{P_\nu\}$.

Tasaki (1998) as well as Linden *et al.* (2009) consider a system coupled to a heat bath, $\mathscr{H}_{\mathrm{total}}=\mathscr{H}_{\mathrm{sys}}\otimes\mathscr{H}_{\mathrm{bath}}$, and take $\mathscr{A}$ to contain all operators of the form $A_{\mathrm{sys}}\otimes 1_{\mathrm{bath}}$. Tasaki considers a rather special class of Hamiltonians and establishes thermalization assuming that

$$\max_\alpha |\langle\phi_\alpha|\psi(0)\rangle|^2 \ll 1, \tag{9.3}$$

a condition that implies that many eigenstates of H contribute to $\psi(0)$ appreciably and that can (more or less) equivalently be rewritten as

$$\sum_\alpha \left|\langle\phi_\alpha|\psi(0)\rangle\right|^4 \ll 1. \tag{9.4}$$

Under assumption (9.4) on $\psi(0)$, Linden *et al.* establish equilibration for H satisfying equation (1.23). They also establish a result in the direction of thermalization under the additional hypothesis that the dimension of the energy shell of the bath is much greater than $\dim\mathscr{H}_{\mathrm{sys}}$.

Reimann's mathematical result (2008) can be described in the above scheme as follows. Let $\mathscr{A}$ be the set of all observables A with (possibly degenerate) eigenvalues between 0 and 1 such that the absolute difference between any two eigenvalues is at least (say) 10^{-1000}. He establishes equilibration for H satisfying equation (1.23), assuming that $\psi(0)$ satisfies equation (9.4).

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We thank Detlef Dürr and Tony Short for helpful discussions. S.G. was supported in part by National Science Foundation [grant DMS-0504504]. N.Z. is supported in part by Istituto Nazionale di Fisica Nucleare. J. L. L. Lebowitz is supported in part by NSF (grant DMR 08-02120) and by AFOSR (grant AF-FA 09550-07).

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