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# Energy-Level Statistics of Model Quantum Systems: Universality and Scaling in a Lattice-Point Problem

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We investigate the statistics of the number $N(R, S)$ of lattice points, $n \in \mathbf{Z}^2$, in an annular domain $\Pi(R, w) = (R+w)A \setminus RA$, where $R, w > 0$. Here $A$ is a fixed convex set with smooth boundary and $w$ is chosen so that the area of $\Pi(R, w)$ is $S$. The statistics comes from $R$ being taken as random (with a smooth density) in some interval $[c_1 T, c_2 T]$, $c_2 > c_1 > 0$. We find that in the limit $T \to \infty$ the variance and distribution of $\Delta N = N(R; S) - S$ depend strongly on how $S$ grows with $T$. There is a saturation regime $S/T \to \infty$, as $T \to \infty$, in which the fluctuations in $\Delta N$ coming from the two boundaries of $\Pi$ are independent. Then there is a scaling regime, $S/T \to z$, $0 < z < \infty$, in which the distribution depends on $z$ in an almost periodic way going to a Gaussian as $z \to 0$. The variance in this limit approaches $z$ for "generic" $A$, but can be larger for "degenerate" cases. The former behavior is what one would expect from the Poisson limit of a distribution for annuli of finite area.

**KEY WORDS:** Energy-level statistics; integrable quantum systems; lattice point problem.

## 1. INTRODUCTION

This paper continues our study of the distribution of integer lattice points inside a "random" region on the plane.[8,17,6,7,9,10] While the question can be thought of as number-theoretic, our motivation comes primarily from the interest in the distribution of eigenvalues of quantum systems. The latter problem has received a great deal of attention in recent years, with

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particular emphasis on the question of how the statistics of eigenvalues relates to the nature of the corresponding classical dynamical system.

One of the striking conjectures is a universality of the local statistics of eigenvalues of generic quantum Hamiltonians: for integrable systems the local statistics is Poissonian, while for hyperbolic systems it is the Wigner statistics of the ensemble of Gaussian matrices. This conjecture is based on a number of analytical, numerical, and experimental results (see, e.g., refs. 1, 12, 20, and 19) and the task of theory is to explain them and to find out their range of validity. To see the connection between the lattice-point problem and the statistics of eigenvalues in the integrable case, consider a simple model system, a free particle in a rectangular box with periodic boundary conditions. In this case the eigenvalues are

$$E_n = \frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2}, \qquad n = (n_1, n_2) \in \mathbf{Z}^2$$

where $2\pi a_1$, $2\pi a_2$ are the sides of the box. The problem then is: what is the statistics of the numbers $E_n$? This is clearly the same problem as finding the statistics of lattice points inside a suitable ellipse. More generally, we may consider integrable systems with eigenvalues

$$E_n = I(n_1 - \alpha_1, n_2 - \alpha_2), \qquad n = (n_1, n_2) \tag{1.1}$$

where (possibly after some rescaling[3]) $I(x_1, x_2)$ is a homogeneous function of second order and ask about the statistics of $E_n$. Now, in the sequence $\{E_n, n \in \mathbf{Z}^2\}$ there is nothing random, so the first question is, what do we mean by statistics of the sequence $E_n$? To describe it, let us consider the sequence of energy levels $E_n$ in the interval $[E, E+S]$ with $E \gg S \gg 1$ (the average spacing between levels is of order of 1). Assuming that $E$ is uniformly distributed on an interval $c_1 T \leqslant E \leqslant c_2 T$, we may consider the sequence $X_E = \{E_n - E\colon E \leqslant E_n < E + S\}$ as a random one. The question is: does the limit distribution of $X_E$ exist, when $T \to \infty$ and $S$ is a prescribed function of $T$. Is this limit distribution Poisson?

This problem was considered originally in the work of Berry and Tabor[3] (see also the review paper[1]), where convincing physical arguments were presented in favor of a Poisson limit distribution. In particular, the distribution of the distances between neighboring levels was found numerically to be exponential, which fits to Poisson statistics. Sinai[30] and Major[25] (see also ref. 5) studied rigorously the Poisson limit distribution in a model lattice problem. They showed that for a typical (in a probability sense) oval in a plane the number of integral points in a random narrow strip of a fixed area between two enlarged ovals has a Poisson distribution. Major proved also some other results in this direction, and he showed that a typical oval from the probability spaces which were used in his work and in the work of Sinai does not belong to $C^2$. For typical smooth, say $C^4-$, ovals the Poisson conjecture for the number of integral points in a random narrow strip of a fixed area remains open; see ref. 18 for some related results in this direction.

In a different direction Casati *et al.*[14] argued that the sequence of levels cannot be considered as truly random. In fact, Casati *et al.*[13] found numerically a saturation of rigidity at large energies, which gave an estimate of the range of the applicability of the Poisson conjecture (the rigidity is a statistical characteristic which estimates how well the counting function of the energy levels is approximated locally by a linear function; it was introduced by Dyson and Mehta in their studies of ensembles of Gaussian matrices[26]).

Berry[2] carried out an analytical analysis of the saturation of the rigidity. He showed that the rigidity has a crossover at the scale of $S$ of the order $E^{1/2}$, where $E$ is the energy, from linear Poisson-like behavior to a saturation. Berry's computations were not completely rigorous, and he also used tacitly some assumptions on nondegeneracy of the spectrum. In the present work we carry out a rigorous study of the statistics for all cases in which $S \to \infty$ as $E^\delta$ for $\delta \geqslant 1/2$.

## 1.1. Informal Statement of Results

We will consider averages related to the classical limit theorems of probability theory. Let $N(E, S)$ be the number of $E_n$ in $(E, E+S]$. The question is, what is the asymptotics of $\operatorname{Var} N(E, S)$ and what is the limit distribution of $[N(E, S) - \langle N(E, S)\rangle]/[\operatorname{Var} N(E, S)]^{1/2}$? This problem has an obvious interpretation as a lattice problem. Referring to (1.1),

$$N(E, S) = \#\{n \mid E < I(n_1 - \alpha_1, n_2 - \alpha_2) \leqslant E + S\}$$

is the number of lattice points in the annulus

$$E < I(x_1 - \alpha_1, x_2 - \alpha_2) \leqslant E + S$$

What we prove in the present article can be informally summarized as follows. Let

$$N(E) = \#\{E_n : E_n \leqslant E\}$$

Then obviously

$$N(E, S) = N(E+S) - N(E)$$

We prove that if $E \to \infty$, $S/E \to 0$, and $S/E^{1/2} \to \infty$, then $N(E+S)$ and $N(E)$ are asymptotically independent, so

$$\operatorname{Var} N(E, S) \sim \operatorname{Var} N(E+S) + \operatorname{Var} N(E)$$

It was shown in ref. 6 that

$$\operatorname{Var} N(E) \sim V_0 E^{1/2}$$

Hence

$$\operatorname{Var} N(E, S) \sim 2V_0 E^{1/2}$$

Similarly, the distribution of

$$\frac{N(E, S) - \langle N(E, S) \rangle}{E^{1/4}}$$

converges to the distribution of a difference of two independent identically distributed random variables, whose distributions coincide with the limit distribution of $F(E) = [N(E) - \langle N(E) \rangle]/E^{1/4}$. The existence of a limit distribution of $F(E)$ for the circle problem was proved in refs. 22 and 8. Results for general ovals were proved in ref. 6 and properties of this limit distribution were studied in ref. 7. It was shown that in a generic case this limit distribution possesses a density which decays at infinity roughly as $\exp(-Cx^4)$. [The distribution of $F(E)$ and parameters $V_0$, $C$, etc., depend on $I$ and on $\alpha$,[9,7] but we do not indicate this explicitly.]

In the regime $E \to \infty$ and $S/E^{1/2} \to z > 0$, we prove a scaling behavior of the variance,

$$\operatorname{Var} N(E, S) \sim E^{1/2} V(E^{-1/2} S)$$

and we compute the scaling function $V(z)$ as an infinite series. This gives $V(z)$ as an almost periodic function, so it is oscillating and has no limit at infinity. We show that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T V(z)\, dz = 2V_0$$

and in a generic case (for instance, for almost all ellipses),

$$V(z) \sim z \tag{1.2}$$

as $z \to 0$. This implies that when $z \to 0$,

$$\operatorname{Var} N(E, S) \sim S \tag{1.3}$$

which is consistent with a Poisson distribution. The relation (1.2) is violated in degenerate cases. For instance, we show that for the circle with center at any rational point $\alpha = (\alpha_1, \alpha_2)$ the behavior of $V(z)$ is given by

$$V(z) \sim Cz\,|\log z|, \qquad z \to 0 \tag{1.4}$$

This anomalous behavior of $V(z)$ is related to an arithmetic degeneracy of the circle problem: for some $k \in \mathbf{N}$ there are many representations of $k$ as sum of two squares, so that there are many lattice points at the circle $\{|x| = k^{1/2}\}$. On the average, the number of representations grows as $\log k$, which shows up in the log correction to linear asymptotics of $V(z)$ as $z \to 0$. Note that recently Luo and Sarnak[28] found deviations from the Wigner statistics for an "arithmetic" hyperbolic system (see also the earlier physical paper[11]), which are related as well to an (arithmetic) degeneracy of the problem. For a circle with center at a Diophantine point $\alpha$ in $[0, 1]^2$, the behavior is normal, satisfying (1.3).

We prove also the existence of a limit distribution of

$$\frac{N(E, S) - \langle N(E, S)\rangle}{[\operatorname{Var} N(E, S)]^{1/2}}$$

in the regime $S/T \to z$. The limit distribution is not Gaussian and in a generic case its density decays at infinity roughly as $\exp(-Cx^4)$. However, when $z \to 0$ this limit distribution converges to a standard Gaussian distribution.

## 1.2. Precise Formulation of Problem and Results

Let $I(x)$ be a homogeneous convex function of order 2 on the plane, so that

$$I(\lambda x) = \lambda^2 I(x) > 0, \qquad \forall \lambda > 0, \quad x \in \mathbf{R}^2\backslash\{0\} \tag{1.5}$$

$$\left(\frac{\partial^2 I(x)}{\partial x_i\,\partial x_j}\right)_{i,j=1,2} > 0, \qquad \forall x \in \mathbf{R}^2\backslash\{0\} \tag{1.6}$$

We will assume in addition that

$$I(x) \in C^7(\mathbf{R}^2\backslash\{0\}) \tag{1.7}$$

Let $\alpha \in \mathbf{R}^2$. Consider

$$N_0(E; \alpha) = \#\{n \in \mathbf{Z}^2 \colon I(n - \alpha) \leqslant E\}$$

which gives the number of lattice points in the convex region $\{x \in \mathbf{R}^2 \colon I(x - \alpha) \leqslant E\}$. We are interested in the behavior of $N_0(E; \alpha)$ when $\alpha$ is fixed and $E \to \infty$.

In what follows we will use the parameter $R = E^{1/2}$ instead of $E$, and we define

$$N(R; \alpha) = N_0(R^2; \alpha)$$

Then $N(R; \alpha)$ has a geometric interpretation as the number of lattice points inside the convex oval $\alpha + R\gamma$, where $\gamma = \{x \in \mathbf{R}^2 : I(x) = 1\}$. For large $R$, $N(R; \alpha)$ is approximately equal to the area of the interior of $\alpha + R\gamma$, which is

$$\text{Area}\{\text{Int}(\alpha + R\gamma)\} = AR^2, \qquad A = \text{Area}\{\text{Int}\,\gamma\} \tag{1.8}$$

so the problem is the behavior of the error function

$$\Delta N(R; \alpha) = N(R; \alpha) - AR^2 \tag{1.9}$$

Figure 1 presents $\Delta N(R; \alpha)$ for the circle centered at $\alpha = 0$, so that this is the error function of the classical circle problem. $\Delta N(R; \alpha)$ behaves very irregularly also for other $\alpha$, so we may think of $\Delta N(R; \alpha)$ as a random function of $R$, and ask, what are the statistical properties of $\Delta N(R; \alpha)$? By statistical properties we mean some averages of functions of $\Delta N(R; \alpha)$, obtained by weighing $R$ according to some weight. The statistical properties of $\Delta N(R; \alpha)$ were studied in refs. 22 and 8 for a circle and in refs. 6 and 7 for a general oval curve of the following class, which includes (1.5)–(1.7):

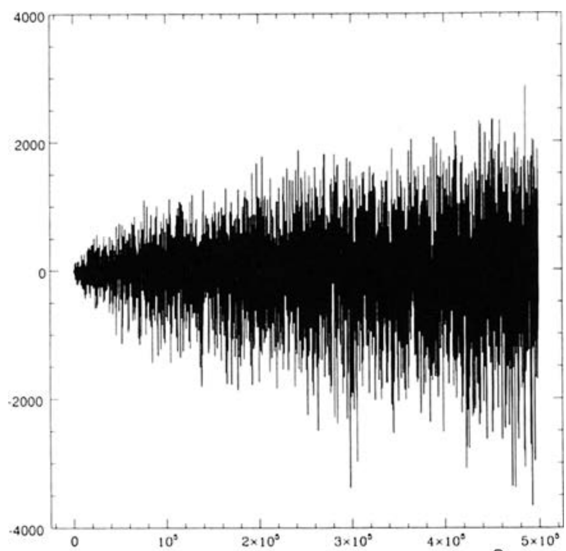


Fig. 1. Error function of the circle problem.

**Class Γ.** $\gamma \in \Gamma$ if $\gamma$ is a $C^7$-smooth convex closed curve without self-intersection in a plane, such that the origin lies inside $\gamma$ and the curvature of $\gamma$ is positive at every $x \in \gamma$.

Let us normalize $\Delta N(R; \alpha)$ to

$$F(R; \alpha) = R^{-1/2} \Delta N(R; \alpha) \tag{1.10}$$

and denote by $C_b(\mathbf{R}^1)$ the space of bounded continuous functions on $\mathbf{R}^1$. Let

$$0 \leqslant c_1 < c_2 \tag{1.11}$$

be fixed numbers, and $\varphi(c) \geqslant 0$ be a fixed bounded density on $[c_1, c_2]$ with normalization

$$\frac{1}{c_2 - c_1} \int_{c_1}^{c_2} \varphi(c)\, dc = 1 \tag{1.12}$$

**Theorem A.**[6] Assume $\gamma \in \Gamma$. Then there exists a probability measure $\nu_\alpha(dt)$ on $\mathbf{R}^1$ such that $\forall g(t) \in C_b(\mathbf{R}^1)$ and $\forall \varphi(c)$,

$$\lim_{T \to \infty} \frac{1}{(c_2 - c_1) T} \int_{c_1 T}^{c_2 T} g[F(R; \alpha)]\, \varphi(R/T)\, dR = \int_{-\infty}^{\infty} g(t)\, \nu_\alpha(dt) \tag{1.13}$$

In addition,

$$\lim_{T \to \infty} \frac{1}{(c_2 - c_1) T} \int_{c_1 T}^{c_2 T} F(R; \alpha)\, \varphi(R/T)\, dR = \int_{-\infty}^{\infty} t \nu_\alpha(dt) = 0 \tag{1.14}$$

and

$$\lim_{T \to \infty} \frac{1}{(c_2 - c_1) T} \int_{c_1 T}^{c_2 T} F(R; \alpha)^2\, \varphi(R/T)\, dR = \int_{-\infty}^{\infty} t^2 \nu_\alpha(dt) \tag{1.15}$$

Note that $\nu_\alpha(dt)$ does not depend on $\varphi(c)$. $\varphi(c) \equiv 1$ corresponds to a uniform distribution of $R$ on $[c_1 T, c_2 T]$, and $\varphi(c) = 2(c_1 + c_2)^{-1} c$ corresponds to a uniform distribution of $E = R^2$ on $[(c_1 T)^2, (c_2 T)^2]$.

Theorem A shows that typical values of $\Delta N(R; \alpha)$ are of order $R^{1/2}$, and $\nu_\alpha(dt)$ is a limit distribution of $R^{-1/2} \Delta N(R; \alpha)$ assuming that $R$ is nicely distributed on $[c_1 T, c_2 T]$ and $T \to \infty$.

In the present work we are interested in the statistics of the increment $\Delta N(R + w; \alpha) - \Delta N(R; \alpha)$. This increment has a clear geometric meaning as a difference between the number of lattice points in the annular strip $\Pi(R, w; \alpha)$ between two ovals, $\alpha + R\gamma$ and $\alpha + (R + w)\gamma$, and the area of $\Pi(R, w; \alpha)$. Our aim is to find the statistics of $\Delta N(R + w; \alpha) - \Delta N(R; \alpha)$. To formulate the problem precisely, we fix the area $S$ of $\Pi(R, w; \alpha)$. We will assume, for normalization, that

$$\text{Area}\{\text{Int}\,\gamma\} = 1 \tag{1.16}$$

Then $w > 0$ is a positive solution of the quadratic equation

$$2wR + w^2 = S \tag{1.17}$$

$$w = (S/R)\{1 + [1 + (S/R^2)]^{1/2}\}^{-1} = [S/(2R)][1 + O(S/R^2)] \tag{1.18}$$

Let

$$\Delta N(R, S; \alpha) = \Delta N(R + w; \alpha) - \Delta N(R; \alpha) \tag{1.19}$$

with $w$ given by (1.18). Then

$$\Delta N(R, S; \alpha) = N_0(E + S; \alpha) - N_0(E; \alpha) - S, \qquad E = R^2$$

and the first problem we are interested in concerns the asymptotics of the second moment of $\Delta N(R, S; \alpha)$,

$$D(T, S; \alpha) = \frac{1}{(c_2 - c_1)T} \int_{c_1 T}^{c_2 T} [\Delta N(R, S; \alpha)]^2 \varphi(R/T)\, dR \tag{1.20}$$

as $T \to \infty$. Our aim is to prove the following scaling of $D(T, S; \alpha)$:

**Scaling Behavior.**

$$\text{(I)} \quad \lim_{S/T^2 \to 0,\, S/T \to \infty} T^{-1} D(T, S; \alpha) = V(\alpha) > 0 \tag{1.21}$$

$$\text{(II)} \quad \lim_{T \to \infty,\, S/T \to z} T^{-1} D(T, S; \alpha) = V(z; \alpha) > 0, \qquad \forall z > 0 \tag{1.22}$$

$$\text{(III)} \quad \lim_{z \to 0} z^{-1} V(z; \alpha) = 1 \qquad \text{for typical } \gamma \tag{1.23}$$

In fact we prove (I) and (II) for all $\gamma \in \Gamma$ and we compute $V(\alpha)$ and $V(z; \alpha)$. (III) will be shown to hold for generic $\gamma$, i.e., when $\gamma$ has no symmetries which give rise to multiplicities of the eigenvalues. In general, the behavior of $V(z; \alpha)$ as $z \to 0$ will depend on the relevant group of symmetry. Observe that (III) implies that if $S/T$ is a fixed small number and $T$ is sufficiently big, then $S^{-1} D(T, S; \alpha)$ is close to 1. For a Poisson point random field of density 1 in the plane the variance of the number of points in a domain of area $S$ is equal to $S$, so Eqs. (1.21)–(1.23) describe a transition from a Poisson-like asymptotics at (III) (for typical $\gamma$) through a scaling at (II) to a saturation at (I). A very interesting problem is to extend (III) to

$$\text{(III')} \qquad \lim_{T\to\infty,\, S/T\to 0} S^{-1}D(T, S; \alpha) = 1$$

It is generally accepted by physicists that this should hold generically; see ref. 2 for arguments in this direction. We in fact believe that (III') holds for every oval $\gamma$ satisfying some Diophantine hypothesis, but we will not discuss this point in the present article.

The second problem concerning $\Delta N(R, S; \alpha)$ we are interested in is the existence and the scaling for a limit distribution of appropriately normalized $\Delta N(R, S; \alpha)$. This will be considered below.

*Notation.* For $\xi \in \mathbf{R}^2$, $\xi \neq 0$, consider the unique point $x(\xi) \in \gamma$, where the outer normal vector to $\gamma$, $n_{x(\xi)}$, coincides with $|\xi|^{-1}\xi$. Denote

$$Y(\xi) = \xi \cdot x(\xi) \tag{1.24}$$

where $a \cdot b = a_1 b_1 + a_2 b_2$ for $a, b \in \mathbf{R}^2$. The curve $\gamma^* = \{\xi\colon Y(\xi) = 1\}$ is known as the polar of $\gamma$. Note that $(1/2)\, Y^2(\xi)$ is the Legendre transform of $\frac{1}{2} I(x)$,

$$(1/2)\, Y^2(\xi) = -\inf_{x \in \mathbf{R}^2} \left(\tfrac{1}{2} I(x) - x \cdot \xi\right) \tag{1.25}$$

Let $0 < Y_1 < Y_2 < \cdots$ be all possible values of $Y(n)$ with $n \in \mathbf{Z}^2 \backslash \{0\}$. Define

$$u_\alpha(k) = \sum_{n \in \mathbf{Z}^2\colon Y(n) = Y_k} e(n \cdot \alpha)\, |n|^{-3/2} \, [\rho(n)]^{1/2} \tag{1.26}$$

where $\rho(\xi)$ is the radius of curvature of $\gamma$ at $x(\xi)$ and

$$e(t) = \exp(2\pi i t) \tag{1.27}$$

Let

$$J_\varphi = \frac{1}{(c_2 - c_1)} \int_{c_1}^{c_2} c\varphi(c)\, dc \tag{1.28}$$

**Theorem 1.1.** Assume $\gamma \in \Gamma$. Then

$$\lim_{S/T\to\infty,\, S/T^2\to 0} T^{-1} \frac{1}{(c_2 - c_1) T} \int_{c_1 T}^{c_2 T} [\Delta N(R, S; \alpha)]^2 \, \varphi(R/T)\, dR$$
$$= V(\alpha) = J_\varphi W(\alpha) \tag{1.29}$$

with

$$W(\alpha)=\pi^{-2}\sum_{k=1}^{\infty}|u_\alpha(k)|^2 \tag{1.30}$$

**Corollary.** If $\gamma\in\Gamma$, then

$$\lim_{\Delta c\to 0}\lim_{S/T\to\infty,\, S/T^2\to 0} T^{-1}\frac{1}{T\,\Delta c}\int_T^{T(1+\Delta c)}[\Delta N(R,S;\alpha)]^2\,dR=W(\alpha) \tag{1.31}$$

**Theorem 1.2.** If $\gamma\in\Gamma$, then

$$\lim_{T\to\infty,\, S/T\to z} T^{-1}\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T}[\Delta N(R,S;\alpha)]^2\,\varphi(R/T)\,dR$$
$$=V(z;\alpha)=\frac{1}{(c_2-c_1)}\int_{c_1}^{c_2}c\varphi(c)\,W(z/c;\alpha)\,dc \tag{1.32}$$

with

$$W(z;\alpha)=\pi^{-2}\sum_{k=1}^{\infty}|u_\alpha(k)|^2\,[1-\cos(\pi Y_k z)] \tag{1.33}$$

**Corollary.** If $\gamma\in\Gamma$, then

$$\lim_{\Delta c\to 0}\lim_{T\to\infty,\, S/T\to z} T^{-1}\frac{1}{T\,\Delta c}\int_T^{T(1+\Delta c)}[\Delta N(R,S;\alpha)]^2\,dR=W(z;\alpha) \tag{1.34}$$

The scaling function $W(z;\alpha)$ represents a local averaging of $[\Delta N(R,S;\alpha)]^2$. Comparing (1.31) with (1.34), one would expect that $\lim_{z\to\infty}W(z;\alpha)=W(\alpha)$. Formula (1.33) shows, however, that this is not true: $W(z;\alpha)$ is an almost periodic function of $z$, so it oscillates at infinity. What is true is that on the average $W(z;\alpha)$ converges to $W(\alpha)$, that is,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T W(z;\alpha)\,dz=W(\alpha) \tag{1.35}$$

Another interesting problem is to find the asymptotics of $W(z;\alpha)$ when $z\to 0$. We shall prove that it is universal in a generic situation. To that end, we introduce the following class of ovals:

**Class $\Gamma_0$.** $\gamma\in\Gamma_0$ if $\gamma\in\Gamma$ and $Y(m)=Y(n)$ if and only if $m=n$.

Since $\Gamma_0$ is defined through a countable number of inequalities $\{Y(m)\neq Y(n), m\neq n\}$, we may think of $\Gamma_0$ as of the set of generic ovals.

In the case when the oval $\gamma$ is symmetric with respect to the origin, the condition that $Y(m) = Y(n)$ only if $m = n$ is certainly violated, since then $Y(-n) = Y(n)$.

To cover this case we introduce the following class of ovals. Let $G$ be the group of all isometries $i: \mathbf{R}^2 \to \mathbf{R}^2$ of a plane such that $i(0) = 0$ and $i(\mathbf{Z}^2) = \mathbf{Z}^2$. The group $G$ consists of eight elements. Let $H$ be a subgroup of $G$. We say that an oval $\gamma$ is invariant with respect to $H$ if $g\gamma = \gamma$ for every $g \in H$.

**Class $\Gamma_0(H)$.** $\gamma \in \Gamma_0(H)$ if $\gamma \in \Gamma$ is invariant with respect to $H$, and $Y(m) = Y(n)$ only if $m = gn$ for some $g \in H$.

For $\gamma \in \Gamma_0(H)$ and $\alpha \in \mathbf{R}^2$, consider a subgroup $H_\alpha \subset H$ such that $g \in H_\alpha$ if $g\alpha = \alpha + n$ with some $n \in \mathbf{Z}^2$. Let

$$m_\alpha(H) = |H_\alpha| \tag{1.36}$$

be the number of elements in $H_\alpha$.

**Theorem 1.3.** If $\gamma \in \Gamma_0(H)$, then

$$\lim_{z \to 0} z^{-1} W(z; \alpha) = m_\alpha(H) \tag{1.37}$$

*Remarks.* (1) If $\gamma \in \Gamma_0$, (1.37) reduces to $\lim_{z \to 0} z^{-1} W(z; \alpha) = 1$. (2) Note that if $m \in \alpha + R\gamma$ and $g \in H_\alpha$, so that $g\alpha = \alpha + n$, then $gm \in g\alpha + Rg\gamma = \alpha + n + R\gamma$, hence $gm - n \in \alpha + R\gamma$. For a typical $m \in \mathbf{Z}^2$ the points $gm - n$, $g \in H_\alpha$, are different, and therefore $m_\alpha(H)$ is the multiplicity of integer points on $\alpha + R\gamma$, caused by the symmetry.

A circle with center at the origin is invariant with respect to $G$. However, the circle does not belong to $\Gamma_0(G)$, since for some $k \in \mathbf{N}$ there exist many different representations of $k$ as a sum of two squares, which are not related to any symmetry. The number of such representations grows, on the average, as $\log k$ and this shows up in the behavior of $W(z; 0)$.

A vector $\alpha \in \mathbf{R}^2$ is called Diophantine if there exist $C$, $N > 0$ such that for all $n \in \mathbf{Z}^2 \backslash \{0\}$,

$$|n \cdot \alpha| \geqslant C\, |n|^{-N}$$

**Theorem 1.4.** If $\gamma$ is a circle with the center at the origin, then

$$\lim_{z \to 0} (z\, |\log z|)^{-1}\, W(z; \alpha) = C_\alpha > 0 \qquad \text{for every rational } \alpha \in \mathbf{Q}^2$$

$$\lim_{z \to 0} z^{-1} W(z; \alpha) = 1 \qquad \text{for every Diophantine } \alpha$$

*Remarks.* (1) From the proof of Theorem 1.4 below an explicit formula for $C_\alpha$ follows. For instance, for $\alpha=0$, $C_\alpha=6\pi^{-1}$. (2) Theorem 1.4 can be extended to any ellipse with rational ratio of squared half-axes $a_1^2/a_2^2$.

Theorems 1.3 and 1.4 are illustrated in Figs. 2–4, which present the scaling function $W(z;\alpha)$, respectively, for a circle with $\alpha=(0,0)$, for an ellipse with ratio of axes $1/\pi$ and $\alpha=(0,0)$, and finally for the same ellipse with $\alpha=(0.1,0.1)$. The behavior of $W(z;\alpha)$ as $z\to 0$ is readily seen in the figures to be consistent with that given in Theorems 1.3 and 1.4; the slope being infinite in Fig. 2, 4 in Fig. 3, and 1 in Fig. 4.

Now we consider the existence of a limit distribution of

$$F(R,S;\alpha)=R^{-1/2}\,\Delta N(R,S;\alpha)$$

Let $\nu_\alpha(dt)$ be the limit distribution of $F(R;\alpha)$ (see Theorem A above). Denote by $\nu_\alpha^-(dt)$ the distribution obtained by reflection of $\nu_\alpha(dt)$, so that

$$\int_a^b \nu_\alpha^-(dt)=\int_{-b}^{-a}\nu_\alpha(dt)$$

**Theorem 1.5.** If $\gamma\in\Gamma$, then $\forall g(t)\in C_b(\mathbf{R}^1)$,

$$\lim_{S/T\to\infty,\,S/T^2\to 0}\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T} g(F(R,S;\alpha))\,\varphi(R/T)\,dR=\int_{-\infty}^{\infty} g(t)\,\mu_\alpha(dt)$$

with $\mu_\alpha=\nu_\alpha * \nu_\alpha^-$, where $*$ denotes convolution.

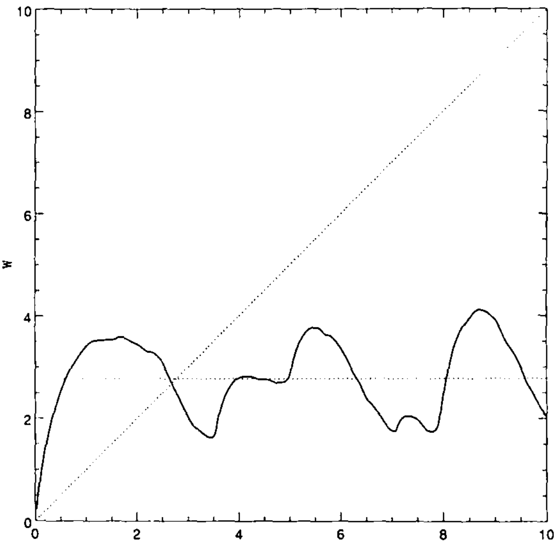


Fig. 2. Scaling function of the circle problem.

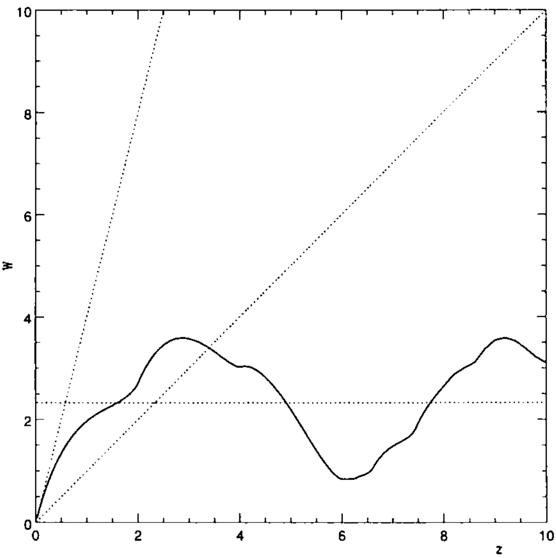


Fig. 3. Scaling function for an ellipse with $a_1/a_2 = 1/\pi$, $\alpha = (0., 0.)$.

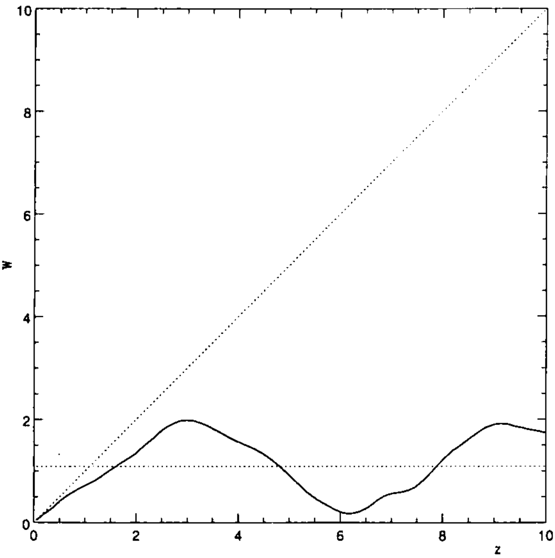


Fig. 4. Scaling function for an ellipse with $a_1/a_2 = 1/\pi$, $\alpha = (0.1, 0.1)$.

A simple explanation of Theorem 1.5 is that when $S/T \to \infty$, i.e., when the annulus is "thick," $F(R+w;\alpha)$ and $F(R;\alpha)$ are independent random variables in the limit $T \to \infty$, assuming that $R$ is uniformly distributed on $[c_1 T, c_2 T]$, so

$$F(R, S; \alpha) \approx F(R+w; \alpha) - F(R; \alpha)$$

is a difference of independent random variables.

**Theorem 1.6.** If $\gamma \in \Gamma$, then for every $z > 0$ there exists a probability measure $\mu_\alpha(dt; z)$, which depends continuously, in the weak topology, on $z$, such that $\forall g(t) \in C_b(\mathbf{R}^1)$,

$$\lim_{T \to \infty, S/T \to z} \frac{1}{(c_2 - c_1)T} \int_{c_1 T}^{c_2 T} g(F(R, S; \alpha))\, \varphi(R/T)\, dR$$

$$= \frac{1}{(c_2 - c_1)} \int_{c_1}^{c_2} dc\, \varphi(c) \int_{-\infty}^{\infty} g(t)\, \mu_\alpha(dt; z/c)$$

**Corollary.**

$$\lim_{\Delta c \to 0} \lim_{T \to \infty, S/T \to z} \frac{1}{T\, \Delta c} \int_T^{T(1+\Delta c)} g(F(R, S; \alpha))\, dR = \int_{-\infty}^{\infty} g(t)\, \mu_\alpha(dt; z)$$

**Theorem 1.7.** For every $g(t) \in C_b(\mathbf{R}^1)$, $\int_{-\infty}^{\infty} g(t)\, \mu_\alpha(dt; z)$ is a continuous almost periodic function in $z$, and

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T dz \int_{-\infty}^{\infty} g(t)\, \mu_\alpha(dt; z) = \int_{-\infty}^{\infty} g(t)\, \mu_\alpha(dt)$$

Our next goal is to describe properties of the measures $\mu_\alpha(dt; z)$. To do this, we need some conditions of incommensurability of the frequencies $Y(n)$. Define $M \subset \mathbf{Z}^2$ as $M = M_0 \cup M_1$ with

$$M_0 = \{(0, 1), (1, 0), (0, -1), (-1, 0)\}$$

$$M_1 = \{n = (n_1, n_2) \mid |n_1|, |n_2| \text{ are positive and relatively prime}\}$$

In some respects $M$ plays the role of the set of prime numbers on the lattice $\mathbf{Z}^2$. Define now:

**Class $\Gamma_1$.** An oval $\gamma$ belongs to $\Gamma_1$ if $\gamma \in \Gamma$ and the numbers $\{Y(n); n \in M\}$ are linearly independent over $\mathbf{Q}$.

It is clear that $\Gamma_1 \subset \Gamma_0$, so $\Gamma_1$ does not contain symmetric ovals. To cover the case of symmetric ovals let us consider a subgroup $H \subset G$.

Consider any fundamental domain $M(H) \subset M$ for the action of $H$ on $M$, i.e., for every $n \in M$ there exists a unique $m \in M(H)$ such that $n = gm$ for some $g \in H$. Define:

**Class $\Gamma_1(H)$.** An oval $\gamma$ belongs to $\Gamma_1(H)$ if $\gamma$ is invariant with respect to $H$, and the numbers $\{Y(n); n \in M(H)\}$ are linearly independent over $\mathbf{Q}$.

As an example consider

$$g_0: (x_1, x_2) \to -(x_1, x_2), \qquad g_1: (x_1, x_2 \to (-x_1, x_2)$$
$$g_2: (x_1, x_2) \to (x_1, -x_2)$$

and

$$H_0 = \{\mathrm{Id}, g_0\}$$
$$H_{12} = \{\mathrm{Id}, g_0, g_1, g_2\}$$

which are, respectively, the symmetry group with respect to the origin and the symmetry group with respect to the coordinate axes. A general class of ovals which belong to $\Gamma_1(H_0)$ and $\Gamma_1(H_{12})$ is described in ref. 7. A characteristic example is an ellipse with transcendental ratio of half-axes. In general, the condition that the numbers $Y(n)$, $n \in M(H)$, are linearly independent over $\mathbf{Q}$ (or, equivalently, over $\mathbf{Z}$), is equivalent to a countable number of inequalities

$$\sum_{k=1}^{N} r_k Y(n_k) \neq 0, \qquad r_k \in \mathbf{Z}$$

and this condition can be viewed as a condition of a "generic" situation.

**Theorem 1.8.** If $\gamma \in \Gamma_1(H)$ for some $H \subset G$, then for every $z > 0$, $\mu_\alpha(dt; z)$ possesses a density

$$p_\alpha(t; z) = \frac{\mu_\alpha(dt; z)}{dt}$$

which is an analytic (entire) function of $t \in \mathbf{C}$, and for real $t$, $\forall \varepsilon > 0$,

$$0 \leqslant p_\alpha(t; z) \leqslant C \exp(-\lambda t^4) \tag{1.38}$$

$$P_\alpha(-t; z), 1 - P_\alpha(t; z) \geqslant C'_\varepsilon \exp(-\lambda'_\varepsilon t^{4+\varepsilon}), \qquad t \geqslant 0 \tag{1.39}$$

where $P_\alpha(t; z) = \int_{-\infty}^{t} p_\alpha(t'; z)\, dt'$ and $C$, $\lambda$, $C'_\varepsilon$, $\lambda'_\varepsilon > 0$.

**Theorem 1.9.** The previous theorem holds for a circle with the center at the origin, with a slightly weaker estimate, instead of (1.38):

$$\begin{aligned} 0 \leqslant p_\alpha(t; z) &\leqslant C_\varepsilon \exp(-\lambda_\varepsilon t^{4-\varepsilon}) \\ P_\alpha(-t; z), 1 - P_\alpha(t; z) &\geqslant C'_\varepsilon \exp(-\lambda'_\varepsilon t^{4+\varepsilon}), \qquad t \geqslant 0, \quad \forall \varepsilon > 0 \end{aligned} \tag{1.40}$$

If $\mu(dt)$ is a distribution of a random variable $\xi$ with zero mean, denote by $\tilde{\mu}(dt)$ a distribution of the normalized random variable $\xi/(\operatorname{Var} \xi)^{1/2}$. Then

$$\int_{-\infty}^{\infty} t^2 \tilde{\mu}(dt) = 1$$

Now we describe the limit behavior of the measure $\tilde{\mu}_\alpha(dt; z)$ when $z \to 0$.

**Theorem 1.10.** If $\gamma \in \Gamma_1(H)$ for some $H \subset G$, then $\lim_{z \to 0} \tilde{\mu}_\alpha(dt; z)$ is a standard Gaussian distribution.

**Theorem 1.11.** If $\gamma$ is a circle with the center at the origin, then

$$\lim_{z \to 0} \tilde{\mu}_\alpha(dt; z)$$

is a standard Gaussian distribution.

The proof of the above results makes use of the fact, first noted by Heath-Brown,[22] that the parameter $R$ in $F(R; \alpha)$ can be thought of as a time parameter in a flow on an infinite-dimensional torus. Statistical properties of $F(R; \alpha)$ are therefore related to ergodic properties of almost periodic functions. These in turn can be obtained by suitable approximations as quasiperiodic functions, i.e., by flows on a finite-dimensional torus, which is a part of standard ergodic theory. To carry out this program we devote the next section to the derivation of some general result on the ergodic properties of almost periodic functions in the Besicovitch space $B^2$. Before doing that, however, we restate a theorem from ref. 6 which shows that $F(R; \alpha)$ belong to $B^2$.

**Theorem B.** If $\gamma \in \Gamma$, then for every $\alpha \in \mathbf{R}^2$, $F(R; \alpha)$, as a function of $R$, belongs to the Besicovitch space $B^2$. A Fourier expansion of $F(R; \alpha)$ in $B^2$ is given by the formula (see Section 2)

$$F(R; \alpha) = \pi^{-1} \sum_{n \in \mathbf{Z}^2 \setminus \{0\}} |n|^{-3/2} [\rho(n)]^{1/2} \cos[2\pi R Y(n) + \phi(n; \alpha)] \tag{1.41}$$

with

$$\phi(n; \alpha) = 2\pi n \cdot \alpha - \frac{3\pi}{4}$$

Using the notation (1.26), we can rewrite the Fourier series (1.41) as

$$F(R;\alpha)=\pi^{-1}\sum_{k=1}^{\infty}|u_\alpha(k)|\cos\left(2\pi Y_k R+\theta_\alpha(k)-\frac{3\pi}{4}\right),\qquad \theta_\alpha(k)=\arg u_\alpha(k) \tag{1.42}$$

We shall derive Theorems 1.1, 1.2, and 1.5–1.7 from Theorem B and some general results on almost periodic functions in $B^2$. These general results are formulated and proved in the next section. To prove Theorems 1.3, 1.4, and 1.8–1.11 we need more refined arguments.

## 2. SOME GENERAL RESULTS ON ALMOST PERIODIC FUNCTIONS

In this section we will prove some preparatory results on a limit distribution of the values of an almost periodic function. We will use the Besicovitch space $B^2$ of almost periodic functions. A function $F(R)$ on $\{0<R<\infty\}$ belongs to $B^2$ if for every $\varepsilon>0$ there exists a trigonometric polynomial

$$P_\varepsilon(R)=\sum_{n=1}^{N_\varepsilon} a_{n\varepsilon}\exp(i\lambda_{n\varepsilon}R) \tag{2.1}$$

such that

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T |F(R)-P_\varepsilon(R)|^2\,dR<\varepsilon \tag{2.2}$$

For $F(R)\in B^2$ we can define[4]

$$\|F(R)\|_{B^2}=\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T |F(R)|^2\,dR\right)^{1/2} \tag{2.3}$$

It is to be noted that $\|\cdot\|_{B^2}$ is only a seminorm and not a norm, i.e., $\|F(R)\|_{B^2}=0$ does not imply $F(R)\equiv 0$. For instance, if $\lim_{R\to\infty}F(R)=0$, then $\|F(R)\|_{B^2}=0$. The Fourier coefficients of $F(R)\in B^2$ are defined as

$$a(\lambda)=\lim_{T\to\infty}\frac{1}{T}\int_0^T F(R)\exp(-i\lambda R)\,dR \tag{2.4}$$

It is known that $a(\lambda)\neq 0$ at most only for countably many $\lambda=\lambda_n$, $n\in\mathbf{N}$, and that $\|F(R)\|_{B^2}=0$ if all $a(\lambda_n)=0$. We shall use the notation

$$F(R)=\sum_{n=1}^{\infty} a(\lambda_n)\exp(i\lambda_n R) \tag{2.5}$$

which shows that $a(\lambda_n)$ are the Fourier coefficients of $F(R)$.

For $F(R) \in B^2$ we have the Parseval identity[4]

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |F(R)|^2 \, dR = \sum_{n=1}^{\infty} |a(\lambda_n)|^2 \tag{2.6}$$

and

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \left| F(R) - \sum_{n=1}^{N} a(\lambda_n) \exp(i\lambda_n R) \right|^2 dR = 0 \tag{2.7}$$

If $F(R)$ is real-valued, then $a(-\lambda_n) = \overline{a(\lambda_n)}$, and the Fourier series (2.5) can be rewritten as

$$F(R) = \sum_{n=1}^{\infty} b(\lambda_n) \cos(\lambda_n R + \phi_n), \qquad \lambda_n \geqslant 0 \tag{2.8}$$

Then the Parseval identity has the form

$$(\|F(R)\|_{B^2})^2 = b(0)^2 + (1/2) \sum_{n:\, \lambda_n \neq 0} b(\lambda_n)^2 \tag{2.9}$$

We have also a more general formula:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(R+t)\, F(R)\, dR = b(0)^2 + (1/2) \sum_{n:\, \lambda_n \neq 0} b(\lambda_n)^2 \cos(\lambda_n t) \tag{2.10}$$

The Schwarz inequality and (2.2) imply

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \min\{1, |F(R) - P_\varepsilon(R)|\}\, dR < \varepsilon \tag{2.11}$$

We shall use the following theorem from ref. 6.

**Theorem C.** If $F(R) \in B^2$, then there exists a probability distribution $\nu(dx)$ on $\mathbf{R}^1$, with a finite variance $\int_{-\infty}^{\infty} x^2 \nu(dx)$, such that for every probability density $\varphi(x)$ on $[0, 1]$ and every bounded continuous function $g(x)$ on $\mathbf{R}^1$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T g(F(R))\, \varphi(R/T)\, dR = \int_{-\infty}^{\infty} g(x)\, \nu(dx) \tag{2.12}$$

In addition,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(R)\, dR = \int_{-\infty}^{\infty} x\nu(dx) = a(0)$$

and

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |F(R)|^2\,dR=\int_{-\infty}^{\infty} x^2\nu(dx)$$

The distribution $\nu(dx)$ defined by (2.12) is called the distribution of $F(R)$.

The definition of the space $B^2$ and all the discussed properties of almost periodic functions from the space $B^2$, including Theorem C, admit a straightforward extension to vector-valued functions $F(R)=(F_1(R),\ldots, F_k(R))$.

We shall call a joint distribution of $k$ almost periodic functions $F_1(R),\ldots, F_k(R)$ the distribution of the vector function $F(R)=(F_1(R),\ldots, F_k(R))$. Here, in principle, $F_j(x)$ can also be a vector-valued almost periodic functions, but we shall not use this case.

It is noteworthy that if the Fourier frequencies

$$\{\lambda_n^{(1)}\},\ldots, \{\lambda_n^{(k)}\}$$

of $F_1(R),\ldots, F_k(R)$ are linearly independent over $\mathbf{Q}$, i.e., if

$$\sum_{n=1}^{N_1} r_n^{(1)}\lambda_n^{(1)}+\cdots+\sum_{n=1}^{N_k} r_n^{(k)}\lambda_n^{(k)}=0, \qquad r_n^{(i)}\in\mathbf{Q}$$

implies

$$\sum_{n=1}^{N_1} r_n^{(1)}\lambda_n^{(1)}=\cdots=\sum_{n=1}^{N_k} r_n^{(k)}\lambda_n^{(k)}=0$$

then a joint distribution of $F_1(R),\ldots, F_k(R)$ is a product of the distributions of $F_1(R),\ldots, F_k(R)$, so that $F_1(R),\ldots, F_k(R)$ are independent.

We shall prove the following theorem:

**Theorem 2.1**. Let $F(R)\in B^2$ and $w=w_S(R)>0$ be a positive solution of Eq. (1.17). Then for every $\varphi(c)\in L^\infty([c_1, c_2])$ which satisfies (1.12) and every continuous function $g(x)=g(x_1, x_2)$ on $\mathbf{R}^2$, such that $g(x_1, x_2)=O(x_1^2+x_2^2)$, when $x_1^2+x_2^2\to\infty$,

$$\lim_{S/T\to\infty,\, S/T^2\to 0}\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T} g(F(R+w_S(R)), F(R))\,\varphi(R/T)\,dR$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1, x_2)\,\nu(dx_1)\,\nu(dx_2) \tag{2.13}$$

where $\nu(dx)$ is the distribution of $F(R)$.

The condition $S/T\to\infty$ implies that $w_S(T)\to\infty$. Theorem 2.1 shows that in this case, assuming also that $S/T^2\to 0$, $F(R+w_S(R))$ and $F(R)$ are asymptotically independent. A heuristic explanation of this result is that the averaging in two different scales, $R$ and $w_S(R)$, is independent in the limit $T\to\infty$.

*Proof of Theorem 2.1.* We will prove (2.13) first in a particular case, when $g(x)=g(x_1, x_2)\in C_0^\infty(\mathbf{R}^2)$, i.e., $g(x)$ is a $C^\infty$-function with compact support, and $\varphi(c)\equiv 1$. We will then consider the general case.

The condition $g(x)\in C_0^\infty(\mathbf{R}^2)$ implies that for all $x$, $y\in\mathbf{R}^2$,

$$|g(x)-g(y)|\leqslant C_0\min\{1, |x-y|\} \tag{2.14}$$

with some $C_0>0$. So from (2.11),

$$\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T}|g(F(R+w_S(R)), F(R)) - g(P_\varepsilon(R+w_S(R)), P_\varepsilon(R))|\,dR\leqslant C_1\varepsilon \tag{2.15}$$

with some $C_1>0$.

We can choose frequencies $\omega_1,..., \omega_k$, which are linearly independent over $\mathbf{Q}$, such that all $\lambda_{n\varepsilon}$ in (2.1) are linear combinations of $\omega_1,..., \omega_k$, with integer coefficients. Then

$$P_\varepsilon(R)=\sum_{n\in M}a_{n\varepsilon}\exp(in\omega R) \tag{2.16}$$

where $M\subset\mathbf{Z}^k$ is a finite set of multi-indices $n=(n_1,..., n_k)$ and $n\omega=n_1\omega_1+\cdots+n_k\omega_k$. Define

$$A_\varepsilon(t_1,..., t_k)=\sum_{n\in M}a_{n\varepsilon}\exp(int) \tag{2.17}$$

with $t=(t_1,..., t_k)$ and $nt=n_1t_1+\cdots+n_kt_k$. Then

$$P_\varepsilon(R)=A_\varepsilon(\omega_1R,..., \omega_kR) \tag{2.18}$$

hence

$$\begin{aligned}\frac{1}{(c_2-c_1)T}&\int_{c_1T}^{c_2T}g(P_\varepsilon(R+w_S(R)), P_\varepsilon(R))\,dR\\ &=\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T}g(A_\varepsilon(\omega_1(R+w_S(R)),..., \omega_k(R+w_S(R))),\\ &\qquad A_\varepsilon(\omega_1R,..., \omega_kR))\,dR\end{aligned}$$

We will use the following general formula: If $f(R)$ is a bounded continuous function on $[0, \infty)$, then for $U, T \to \infty$ with $U/T \to 0$,

$$\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T} f(R)\,dR = \frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T}\frac{1}{U}\int_R^{R+U} f(Q)\,dQ\,dR + O(U/T) \tag{2.19}$$

Indeed, the integral on the RHS is equal to

$$\frac{1}{(c_2-c_1)\,TU}\iint_{\{c_1T\leqslant R\leqslant c_2T,\,0\leqslant Q-R\leqslant U\}} f(Q)\,dQ\,dR$$
$$=\frac{1}{(c_2-c_1)\,TU}\iint_{\{c_1T\leqslant Q\leqslant c_2T,\,0\leqslant Q-R\leqslant U\}} f(Q)\,dQ\,dR + O(U/T)$$
$$=\frac{1}{(c_2-c_1)T}\int_{\{c_1T\leqslant Q\leqslant c_2T\}} f(Q)\,dQ + O(U/T)$$

which coincides with the LHS up to $O(U/T)$. Equation (2.19) implies that

$$\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T} g(P_\varepsilon(R+w_S(R)), P_\varepsilon(R))\,dR = \frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T}\frac{1}{U}\int_R^{R+U} g(P_\varepsilon(Q+w_S(Q)), P_\varepsilon(Q))\,dQ\,dR + O(U/T) \tag{2.20}$$

From (1.18),

$$|w_S(R)-w_S(Q)|\leqslant C\,|R-Q|\,ST^{-2}\leqslant CUST^{-2} \tag{2.21}$$

when $ST^{-2}\ll 1$, $R\leqslant Q\leqslant R+U$, $0\leqslant U\leqslant T$, and $1\ll c_1T\leqslant R\leqslant c_2T$, hence

$$\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T}\frac{1}{U}\int_R^{R+U} g(P_\varepsilon(Q+w_S(Q)), P_\varepsilon(Q))\,dQ\,dR = \frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T}\frac{1}{U}\int_R^{R+U} g(P_\varepsilon(Q+w_S(R)), P_\varepsilon(Q))\,dQ\,dR + O(ST^{-2}) \tag{2.22}$$

Using the ergodic theorem (see, e.g., ref. 16), we have that for every $\delta>0$, there is an $U_0(\delta)$ such that

$$\left|\frac{1}{U}\int_R^{R+U} g(A_\varepsilon(\omega_1(Q+w_S(R)),\ldots,\omega_k(Q+w_S(R))), A_\varepsilon(\omega_1Q,\ldots,\omega_kQ))\,dQ - \int_{\mathbf{T}^k} g(A_\varepsilon(t_1+\omega_1w_S(R),\ldots,t_k+\omega_kw_S(R)), A_\varepsilon(t_1,\ldots,t_k))\,dt\right| < \delta/2 \tag{2.23}$$

when $U > U_0(\delta)$. When this is combined with (2.18)–(2.22), it yields

$$\left| \frac{1}{(c_2 - c_1) T} \int_{c_1 T}^{c_2 T} g(P_\varepsilon(R + w_S(R)), P_\varepsilon(R))\, dR - \int_{\mathbf{T}^k} dt \frac{1}{(c_2 - c_1) T} \int_{c_1 T}^{c_2 T} dR \right.$$
$$\left. \times g(A_\varepsilon(t_1 + \omega_1 w_S(R), \ldots, t_k + \omega_k w_S(R)), A_\varepsilon(t_1, \ldots, t_k)) \right| < \delta \tag{2.24}$$

whenever $T$ and $S^{-1/2} T$ are sufficiently large.

We will prove now that

$$\lim_{S/T \to \infty, S/T^2 \to 0} \frac{1}{(c_2 - c_1) T} \int_{c_1 T}^{c_2 T} g(A_\varepsilon(t + w_S(R)\omega), A_\varepsilon(t))\, dR$$
$$= \int_{\mathbf{T}^k} g(A_\varepsilon(s), A_\varepsilon(t))\, ds \tag{2.25}$$

where $t = (t_1, \ldots, t_k)$, $s = (s_1, \ldots, s_k)$, and $\omega = (\omega_1, \ldots, \omega_k)$. Moreover, the convergence in (2.25) is uniform in $t \in \mathbf{T}^k$.

Define $y = w_S(R)$. From (1.18),

$$y = \frac{S}{2R} (1 + O(S/T^2))$$

$$\frac{dR}{dy} = -\frac{S}{2y^2} (1 + O(S/T^2))$$

when $R \geqslant c_1 T \gg S^{1/2}$, so

$$\frac{1}{(c_2 - c_1) T} \int_{c_1 T}^{c_2 T} g(A_\varepsilon(t + w_S(R)\omega), A_\varepsilon(t))\, dR$$
$$= \frac{1}{(c_2 - c_1) T} \int_{w_L(c_2 T)}^{w_L(c_1 T)} g(A_\varepsilon(t + y\omega), A_\varepsilon(t)) \frac{dR}{dy}\, dy$$
$$= \frac{1}{(c_2 - c_1) T} \int_{(2c_2 T)^{-1} S}^{(2c_1 T)^{-1} S} g(A_\varepsilon(t + y\omega), A_\varepsilon(t)) \frac{S}{2y^2}\, dy + O(S/T^2) \tag{2.26}$$

Define $\tau = S/T$ and

$$\varphi_0(x) = \frac{x_1 x_2}{x^2 (x_2 - x_1)}$$

with

$$x_1 = (2c_2)^{-1}, \qquad x_2 = (2c_1)^{-1}$$

Then $\int_{x_1}^{x_2} \varphi_0(x)\,dx = 1$ and

$$\frac{1}{(c_2-c_1)T}\int_{(2c_2T)^{-1}S}^{(2c_1T)^{-1}S} g(A_\varepsilon(t+y\omega), A_\varepsilon(t))\frac{S}{2y^2}\,dy$$
$$=\frac{1}{\tau}\int_{x_1\tau}^{x_2\tau} g(A_\varepsilon(t+y\omega), A_\varepsilon(t))\,\varphi_0(y/\tau)\,dy$$

By the ergodic theorem the last integral converges uniformly in $t \in \mathbf{T}^k$, when $\tau \to \infty$, to

$$\int_{\mathbf{T}^k} g(A_\varepsilon(s), A_\varepsilon(t))\,ds$$

which implies (2.25).

It now follows from (2.24), (2.25) that

$$\lim_{S/T\to\infty,\, S/T^2\to 0} \frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T} g(P_\varepsilon(R+w_S(R)), P_\varepsilon(R))\,dR$$
$$=\int_{\mathbf{T}^k}\int_{\mathbf{T}^k} g(A_\varepsilon(s), A_\varepsilon(t))\,ds\,dt \tag{2.27}$$

Let

$$v_\varepsilon(B) = \int_{\{t:\, A_\varepsilon(t)\in B\}} dt$$

be the distribution of $A_\varepsilon(t)$. Then we can rewrite (2.27) as

$$\lim_{S/T\to\infty,\, S/T^2\to 0} \frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T} g(P_\varepsilon(R+w_S(R)), P_\varepsilon(R))\,dR$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1, x_2)\,v_\varepsilon(dx_1)\,v_\varepsilon(dx_2) \tag{2.28}$$

By the ergodic theorem,

$$\lim_{T\to\infty} \frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T} g(P_\varepsilon(R))\,dR = \int_{\mathbf{T}^k} g(A_\varepsilon(t))\,dt = \int_{-\infty}^{\infty} g(x)\,v_\varepsilon(dx) \tag{2.29}$$

for every $g(x) \in C_0^\infty(\mathbf{R}^1)$. By Theorem C,

$$\lim_{T\to\infty} \frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T} g(F(R))\,dR = \int_{-\infty}^{\infty} g(x)\,v(dx)$$

and by (2.11),

$$\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T} |g(F(R))\,dR - g(P_\varepsilon(R))\,dR| \leqslant C_0\varepsilon$$

so

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} g(x)\,\nu_\varepsilon(dx) = \int_{-\infty}^{\infty} g(x)\,\nu(dx) \tag{2.30}$$

Now (2.28) together with (2.15), (2.30) implies

$$\lim_{S/T\to\infty,\,S/T^2\to 0}\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T} g(F(R+w_S(R)), F(R))\,dR$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1, x_2)\,\nu(dx_1)\,\nu(dx_2)$$

Hence (2.13) is proved in the particular case when $g(x_1, x_2)\in C_0^\infty(\mathbf{R}^2)$ and $\varphi(c)\equiv 1$.

By implication (2.13) holds in the case when $g(x_1, x_2)\in C_0^\infty$ and $\varphi(c)$ is a stepwise function with a finite number of steps. Now every $\varphi(c)\in L^\infty([c_1, c_2])$ is a limit in $L^1$-norm of stepwise functions; hence, again by implication, (2.13) holds in the case when $g(x_1, x_2)\in C_0^\infty(\mathbf{R}^2)$ and $\varphi(c)\in L^\infty([c_1, c_2])$.

Assume now that $g(x_1, x_2)$ is a continuous function with compact support. For every $\varepsilon>0$ there exists $g_\varepsilon(x_1, x_2)\in C_0^\infty(\mathbf{R}^2)$ such that

$$\sup_{x_1, x_2} |g_\varepsilon(x_1, x_2) - g(x_1, x_2)| < \varepsilon$$

Hence

$$\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T} |g_\varepsilon(F(R+w_S(R)), F(R)) - g(F(R+w_S(R)), F(R))|\,dR < \varepsilon$$

Thus it follows that (2.8) holds for every continuous function $g(x_1, x_2)$ with compact support.

The condition $F(R)\in B^2$ implies that

$$\lim_{A\to\infty}\limsup_{T\to\infty}\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T} |F(R)|^2\,\chi_{\{|F(R)|\geqslant A\}}(R)\,dR = 0$$

Let $\psi(y)\in C^\infty(\mathbf{R}^1)$ and

$$\psi(y)\begin{cases} =0 & \text{if } |y|>2A \\ =1 & \text{if } |y|<A \\ \in[0, 1] & \text{otherwise}\end{cases}$$

Assume that $g(x)$, $x=(x_1, x_2)$, is a continuous function such that $g(x)=O(|x|^2)$ as $|x| \to \infty$. Then

$$\lim_{A \to \infty} \limsup_{T \to \infty} \left| \frac{1}{(c_2-c_1)T} \int_{c_1 T}^{c_2 T} g(F(R+w_S(R)), F(R)) \right.$$

$$\left. \times [1-\psi(|F(R)|+|F(R)+w_S(R))|)] \, dR \right|$$

$$\leqslant C_0 \lim_{A \to \infty} \frac{1}{(c_2-c_1)T} \int_{c_1 T}^{c_2 T} [|F(R)|^2 \chi_{\{|F(R)| \geqslant A\}}(R)$$

$$+ |F(R+w_S(R))|^2 \chi_{\{|F(R+w_S(R))| \geqslant A\}}(R)] \, dR = 0$$

Hence, by implication, (2.13) holds for every continuous function $g(x_1, x_2)$ with $g(x)=O(|x|^2)$ as $|x| \to \infty$. Theorem 2.1 is proved.

Assume $F(R) \in B^2$. Denote by $v(dx_1, dx_2; z)$ a distribution of the pair $(F(R+z), F(R))$, i.e., for every continuous function $g(x_1, x_2)$ on $\mathbf{R}^2$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T g(F(R+z), F(R)) \, dR = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x_1, x_2) \, v(dx_1, dx_2; z)$$

**Theorem 2.2.** Under the same assumptions as in Theorem 2.1,

$$\lim_{T \to \infty, S/T \to z} \frac{1}{(c_2-c_1)T} \int_{c_1 T}^{c_2 T} g(F(R+w_S(R)), F(R)) \, \varphi(R/T) \, dR$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x_1, x_2) \frac{1}{c_2-c_1} \int_{c_1}^{c_2} dc \, \varphi(c) \, v(dx_1, dx_2; z/(2c)) \tag{2.31}$$

and

$$\lim_{\Delta c \to 0} \lim_{T \to \infty, S/T \to z} \frac{1}{T \Delta c} \int_T^{T(1+\Delta c)} g(F(R+w_S(R)), F(R)) \, dR$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x_1, x_2) \, v(dx_1, dx_2; z/2) \tag{2.32}$$

*Proof.* The proof goes along the same lines as the proof of Theorem 2.1. Moreover, in the derivation of the estimate (2.24) we used only that $T \to \infty$ and $S/T^2 \to 0$ (and not that $S/T \to \infty$), so we can apply (2.24) also in the present proof. Assume that $g(x_1, x_2) \in C_0^\infty(\mathbf{R}^2)$. Since

$$w_S(R) \sim S/(2R) = z/(2c)$$

with $z=S/T$, $c=R/T$, (2.24) implies

$$\lim_{T\to\infty}\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T} g(P_\varepsilon(R+w_S(R)), P_\varepsilon(R))\,dR$$
$$=\int_{\mathbf{T}^k} dt\,\frac{1}{c_2-c_1}\int_{c_1}^{c_2} dc\, g(A_\varepsilon(t+\omega z/(2c)), A_\varepsilon(t))$$
$$=\frac{1}{c_2-c_1}\int_{c_1}^{c_2} dc\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1,x_2)\,\nu_\varepsilon(dx_1,dx_2;z/(2c))$$

where $\nu_\varepsilon(dx_1\,dx_2;z)$ is a joint distribution of $A_\varepsilon(t+\omega z)$ and $A_\varepsilon(t)$. Letting $\varepsilon\to 0$, we come to (2.31).

Since $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1,x_2)\,\nu(dx_1,dx_2;z)$ depends continuously on $z$, (2.32) is a consequence of (2.31). Theorem 2.2 is proved.

We shall also use the following general result:

**Theorem 2.3.** Assume $F(R)\in B^2$. Let $\nu(dx_1\,dx_2;z)$ be a probability distribution of the pair $(F(R), F(R+z))$, and $\nu(dx)$ be a probability distribution of $F(R)$. Then for every continuous function $g(x_1,x_2)$ which is $O(|x_1|^2+|x_2|^2)$ at infinity,

$$I_g(z;F)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1,x_2)\,\nu(dx_1\,dx_2;z) \tag{2.33}$$

is a continuous almost periodic function in $z$, and

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T I_g(z;F)\,dz=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1,x_2)\,\nu(dx_1)\,\nu(dx_2) \tag{2.34}$$

*Proof.* We have

$$I_g(z;F)=\lim_{T\to\infty}\frac{1}{T}\int_0^T g(F(R),F(R+z))\,dR$$

Assume $g(x)\in C_0^\infty(\mathbf{R}^2)$ and $P_\varepsilon(x)$ is a trigonometric polynomial satisfying

$$\|P_\varepsilon(R)-F(R)\|_{B^2}\leqslant\varepsilon$$

Then

$$|I_g(z;F)-I_g(z;P_\varepsilon)|\leqslant C(g)\varepsilon \tag{2.35}$$

Now,

$$P_\varepsilon(R)=A_\varepsilon(\omega_1 R,\ldots,\omega_k R)$$

where $A_\varepsilon$ is a function on $\mathbf{T}^k$ and $\omega_1,\dots,\omega_k$ are incommensurate. It implies that

$$I_g(z; P_\varepsilon) = \int_{\mathbf{T}^k} g(A_\varepsilon(t), A_\varepsilon(t+z\omega))\, dt \tag{2.36}$$

Let $M = \sup_{t\in\mathbf{T}^k} |A_\varepsilon(t)|$. Consider a polynomial $p_\varepsilon(x_1, x_2)$ such that

$$\sup_{|x_1|, |x_2| \leqslant M} |p_\varepsilon(x_1, x_2) - g(x_1, x_2)| \leqslant \varepsilon$$

Then

$$|I_{p_\varepsilon}(z; P_\varepsilon) - I_g(z; P_\varepsilon)| \leqslant \varepsilon \tag{2.37}$$

Since $I_{p_\varepsilon}(z; P_\varepsilon)$ is a trigonometric polynomial in $z$, estimates (2.35), (2.37) prove that if $g(x) \in C_0^\infty(\mathbf{R}^2)$, then $I_g(z; F)$ is an almost periodic function in $z$.

By implication it holds also for every continuous $g(x)$ which grows at infinity as $O(|x|)^2$.

From (2.36) and the ergodic theorem,

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T}\int_0^T I_g(z; P_\varepsilon)\, dz &= \int_{\mathbf{T}^k}\int_{\mathbf{T}^k} g(A_\varepsilon(t), A_\varepsilon(s))\, dt\, ds \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1, x_2)\, \nu_\varepsilon(dx_1)\, \nu_\varepsilon(dx_2)\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain, with the help of (2.30), the formula (2.34). Theorem 2.3 is proved.

## 3. PROOF OF THEOREMS 1.1 AND 1.2

*Proof of Theorem 1.1.* We have

$$\begin{aligned}\Delta N(R, S; \alpha) &= \Delta N(R + w_S(R); \alpha) - \Delta N(R; \alpha) \\ &= [R + w_S(R)]^{1/2} F(R + w_S(R); \alpha) - R^{1/2} F(R; \alpha) \\ &= R^{1/2}[F(R + w_S(R); \alpha) - F(R; \alpha) \\ &\quad + O(SR^{-3/2}|F(R + w_S(R); \alpha)|)\end{aligned} \tag{3.1}$$

since by (1.18),

$$w = w_S(R) \sim S/(2R), \qquad [R + w_S(R)]^{1/2} - R^{1/2} \sim S/(4R^{3/2})$$

Due to the triangle inequality, (3.1) implies

$$\left(\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T}|\Delta N(R,S;\alpha)|^2\,\varphi(R/T)\,dR\right)^{1/2}$$
$$=\left(\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T}|F(R+w_S(R);\alpha)-F(R;\alpha)|^2\,R\varphi(R/T)\,dR\right)^{1/2}$$
$$+O\left(\left(\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T}(SR^{-3/2})^2\,|F(R+w_S(R);\alpha)|^2\,dR\right)^{1/2}\right)$$
$$=\left(\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T}|F(R+w_S(R);\alpha)-F(R;\alpha)|^2\,R\varphi(R/T)\,dR\right)^{1/2}$$
$$+O(ST^{-3/2})$$

since

$$\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T}|F(R+w_S(R);\alpha)|^2\,dR$$
$$=\frac{1}{(c_2-c_1)T}\int_{c_1T+w_S(c_1T)}^{c_2T+w_S(c_2T)}|F(R';\alpha)|^2\,\frac{dR}{dR'}\,dR'=O(1)$$

where $R'=R+w_S(R)$. Hence

$$\lim_{T\to\infty,\,S/T^2\to 0}\left(T^{-1}\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T}|\Delta N(R,S;\alpha)|^2\,\varphi(R/T)\,dR\right)^{1/2}$$
$$=\lim_{T\to\infty,\,S/T^2\to 0}\left(T^{-1}\frac{1}{(c_2-c_1)T}\right.$$
$$\left.\times\int_{c_1T}^{c_2T}|F(R+w_S(R);\alpha)-F(R;\alpha)|^2\,R\varphi(R/T)\,dR\right)^{1/2}$$

or equivalently,

$$\lim_{T\to\infty,\,S/T^2\to 0}T^{-1}\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T}|\Delta N(R,S;\alpha)|^2\,\varphi(R/T)\,dR$$
$$=\lim_{T\to\infty,\,S/T^2\to 0}T^{-1}\frac{1}{(c_2-c_1)T}$$
$$\times\int_{c_1T}^{c_2T}|F(R+w_S(R);\alpha)-F(R;\alpha)|^2\,R\varphi(R/T)\,dR \qquad (3.2)$$

By Theorem 2.1,

$$\lim_{S/T\to\infty,\,S/T^2\to 0} T^{-1}\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T} |F(R+w_S(R);\alpha)-F(R;\alpha)|^2\,R\varphi(R/T)\,dR$$

$$=J_\varphi \lim_{S/T\to\infty,\,S/T^2\to 0}\frac{1}{(c_2-c_1)T}$$

$$\times\int_{c_1T}^{c_2T} |F(R+w_S(R);\alpha)-F(R;\alpha)|^2\,\varphi_0(R/T)\,dR$$

$$=J_\varphi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1-x_2)^2\,\nu_\alpha(dx_1)\,\nu_\alpha(dx_2)=J_\varphi W(\alpha)$$

where $\varphi_0=J_\varphi^{-1}c\varphi(c)$,

$$J_\varphi=\frac{1}{(c_2-c_1)}\int_{c_1}^{c_2} c\varphi(c)\,dc$$

and

$$W(\alpha)=2\int_{-\infty}^{\infty} x^2\nu_\alpha(dx)$$

Thus

$$\lim_{S/T\to\infty,\,S/T^2\to 0} T^{-1}\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T}|\Delta N(R,S;\alpha)|^2\,dR=J_\varphi W(\alpha)$$

Now Theorem C, the Fourier expansion (1.42), and the Parseval identity (2.9) lead to

$$\int_{-\infty}^{\infty} x^2\nu_\alpha(dx)=(1/2)\,\pi^{-2}\sum_{k=1}^{\infty}|u_\alpha(k)|^2 \tag{3.3}$$

Hence

$$W(\alpha)=\pi^{-2}\sum_{k=1}^{\infty}|u_\alpha(k)|^2$$

Theorem 1.1 is proved.

*Proof of Theorem 1.2.* From an analog of (3.2) where $(S/T)\to\infty$ instead of $(S/T^2)\to 0$ and Theorem 2.2,

$$\lim_{T\to\infty, S/T\to z} T^{-1}\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T}|\Delta N(R,S;\alpha)|^2\,\varphi(R/T)\,dR$$

$$=\lim_{T\to\infty, S/T\to z} T^{-1}\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T}|F(R+w_S(R);\alpha)-F(R;\alpha)|^2\,R\varphi(R/T)\,dR$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x_1-x_2|^2\frac{1}{(c_2-c_1)}\int_{c_1}^{c_2}dc\,c\varphi(c)\,\nu(dx_1,dx_2;z/(2c))$$

$$=\frac{1}{(c_2-c_1)}\int_{c_1}^{c_2}c\varphi(c)\,W(z/c;\alpha)\,dc$$

with

$$W(z;\alpha)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x_1-x_2|^2\,\nu(dx_1,dx_2;z/2)$$
$$=\lim_{T\to\infty}\frac{1}{T}\int_0^T|F(R+z/2;\alpha)-F(R;\alpha)|^2\,dR$$

Now, from (1.42),

$$\begin{aligned}F(R+z/2;\alpha)-F(R;\alpha)\\
&=\pi^{-1}\sum_{k=1}^{\infty}|u_\alpha(k)|\left\{\cos\left[2\pi\left(R+\frac{z}{2}\right)Y_k+\theta_\alpha(k)-\frac{3\pi}{4}\right]\right.\\
&\quad\left.-\cos\left[2\pi RY_k+\theta_\alpha(k)-\frac{3\pi}{4}\right]\right\}\\
&=-2\pi^{-1}\sum_{k=1}^{\infty}u_\alpha(k)\sin\left(\frac{\pi zY_k}{2}\right)\\
&\quad\times\sin\left[2\pi\left(R+\frac{z}{4}\right)Y_k+\theta_\alpha(k)-\frac{3\pi}{4}\right]\end{aligned}\tag{3.4}$$

so, by the Parseval identity (2.9),

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T|F(R+z/2;\alpha)-F(R;\alpha)|^2\,dR$$
$$=2\pi^{-2}\sum_{k=1}^{\infty}|u_\alpha(k)|^2\sin^2\left(\frac{\pi zY_k}{2}\right)$$
$$=\pi^{-2}\sum_{k=1}^{\infty}|u_\alpha(k)|^2\,[1-\cos(\pi zY_k)]$$

and so

$$W(z;\alpha)=\pi^{-2}\sum_{k=1}^{\infty}|u_\alpha(k)|^2\,[1-\cos(\pi z Y_k)]$$

Theorem 1.2 is proved.

## 4. PROOF OF THEOREMS 1.3, 1.4

*Proof of Theorem 1.3.* Assume first that $\gamma\in\Gamma_0$. Then $Y(m)\neq Y(n)$ for $m\neq n$, hence (1.33) reduces to

$$\begin{aligned}W(z;\alpha)&=\pi^{-2}\sum_{n\in\mathbf{Z}^2\setminus\{0\}}|n|^{-3}\,\rho(n)\{1-\cos[\pi Y(n)z]\}\\&=2\pi^{-2}\sum_{n\in\mathbf{Z}^2\setminus\{0\}}|n|^{-3}\,\rho(n)\sin^2[\pi Y(n)z/2]\end{aligned}\tag{4.1}$$

so that

$$z^{-1}W(z;\alpha)=\pi^{-1}\sum_{n\in\mathbf{Z}^2\setminus\{0\}}|\pi nz/2|^{-3}\,\rho(\pi nz/2)\sin^2[Y(\pi nz/2)](\pi z/2)^2$$

Here we used the fact that $\rho(\lambda\xi)=\rho(\xi)$ and $Y(\lambda\xi)=\lambda Y(\xi)$ for every $\lambda>0$, and we reduced $z^{-1}W(z;\alpha)$ to an approximating sum to the integral

$$\pi^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\xi|^{-3}\,\rho(\xi)\sin^2 Y(\xi)\,d\xi$$

In the Appendix we show that

$$\pi^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\xi|^{-3}\,\rho(\xi)\sin^2 Y(\xi)\,d\xi=1\tag{4.2}$$

so

$$\lim_{z\to 0}z^{-1}W(z;\alpha)=1$$

which was stated.

Assume now that $\gamma\in\Gamma_0(H)$. For the sake of simplicity we will consider $H=\{\mathrm{Id},g_1\}$, where $g_1\colon(x_1,x_2)\to(x_1,-x_2)$. The general case is treated in the same way. In the case under consideration

$$m_\alpha(H)=\begin{cases}2 & \text{if}\quad \alpha_2\text{ is half-integer}\\ 1 & \text{if}\quad \alpha_2\text{ is not half-integer}\end{cases}$$

so we have to show that

$$\lim_{z\to 0} z^{-1}W(z;\alpha)=\begin{cases}2 & \text{if } \alpha_2 \text{ is half-integer}\\ 1 & \text{if } \alpha_2 \text{ is not half-integer}\end{cases}$$

Define

$$P=\{(n_1,n_2)\in \mathbf{Z}^2: n_2=0\}$$

so that $P$ consists of fixed points of $g_1$. From (1.26), if $Y(n)=Y_k$, then

$$|u_\alpha(k)|=\begin{cases}2\,|\cos(n_2\alpha_2)|\cdot|n|^{-3/2}\,[\rho(n)]^{1/2} & \text{if } n\notin P\\ |n|^{-3/2}\,[\rho(n)]^{1/2} & \text{if } n\in P\end{cases}$$

so (1.33) reduces to

$$\begin{aligned}W(z;\alpha)&=4\pi^{-2}\sum_{n\notin P}\cos^2(2\pi n_2\alpha_2)\,|n|^{-3}\,\rho(n)\sin^2[\pi Y(n)z/2]\\&\quad+2\pi^{-2}\sum_{n\in P\setminus\{0\}}|n|^{-3}\,\rho(n)\sin^2[\pi Y(n)z/2]\\&=4\pi^{-2}\sum_{n\in \mathbf{Z}^2\setminus\{0\}}\cos^2(2\pi n_2\alpha_2)\,|n|^{-3}\,\rho(n)\sin^2[\pi Y(n)z]\\&\quad-2\pi^{-2}\sum_{n\in P\setminus\{0\}}|n|^{-3}\,\rho(n)\sin^2[\pi Y(n)z/2]\\&=W_0(z;\alpha)-W_1(z;\alpha)\end{aligned}\tag{4.3}$$

with

$$W_0(z;\alpha)=4\pi^{-2}\sum_{n\in \mathbf{Z}^2\setminus\{0\}}\cos^2(2\pi n_2\alpha_2)\,|n|^{-3}\,\rho(n)\sin^2[\pi Y(n)z/2]\tag{4.4}$$

$$W_1(z;\alpha)=2\pi^{-2}\sum_{n_1\in \mathbf{Z}\setminus\{0\}}|n|^{-3}\,\rho(n)\sin^2[\pi Y(n)z/2],\qquad n=(n_1,0)\tag{4.5}$$

Assume first that $\alpha_2=0$ or $1/2$. Then

$$W_0(z;\alpha)=4\pi^{-2}\sum_{n\in \mathbf{Z}^2\setminus\{0\}}|n|^{-3}\,\rho(n)\sin^2[\pi Y(n)z/2]$$

which is twice the sum in (4.1), so the same computation as before leads to

$$\lim_{z\to 0} z^{-1}W_0(z;\alpha)=2\tag{4.6}$$

Now,

$$\sum_{n_1=1}^{z^{-1}} n_1^{-3} \sin^2[\pi Y(n_1, 0)z] \leqslant Cz^2 \sum_{n_1=1}^{z^{-1}} n_1^{-1} \leqslant C_0 z^2 |\log z|$$

and

$$\sum_{n_1=z^{-1}}^{\infty} n_1^{-3} \sin^2[\pi Y(n_1, 0)z] \leqslant \sum_{z^{-1}}^{\infty} n_1^{-3} \leqslant C_1 z^2$$

hence

$$0 \leqslant W_1(z;\alpha) \leqslant Cz^2 |\log z| \tag{4.7}$$

From (4.3)–(4.7),

$$\lim_{z\to 0} z^{-1} W(z;\alpha) = 2$$

which was stated.

Consider now the case when $\alpha_2$ is not half-integer. Substituting

$$\cos^2(2\pi n_2 \alpha_2) = [1 + \cos(4\pi n_2 \alpha_2)]/2$$

into (4.4), we obtain that

$$W_0(z;\alpha) = W_2(z;\alpha) + W_3(z;\alpha)$$

with

$$W_2(z;\alpha) = 2\pi^{-2} \sum_{n\in \mathbf{Z}^2\setminus\{0\}} |n|^{-3} \rho(n) \sin^2[\pi Y(n)t/2]$$

and

$$W_3(z;\alpha) = 2\pi^{-2} \sum_{n\in \mathbf{Z}^2\setminus\{0\}} \cos(4\pi n_2\alpha_2)\, |n|^{-3} \rho(n) \sin^2[\pi Y(n)z/2] \tag{4.8}$$

Since $W_2(z;\alpha)$ coincides with $W(z;\alpha)$ in (4.1), we obtain that

$$\lim_{z\to 0} z^{-1} W_2(z;\alpha) = 1$$

Let us prove that

$$\lim_{z\to 0} z^{-1} W_3(z;\alpha) = 0 \tag{4.9}$$

The idea is to apply the Abel summation formula to (4.9). Let us fix $n_1 = j$ and denote

$$a(k) = \cos(4\pi k \alpha_2)$$

and

$$b_j(k) = |n|^{-3}\, \rho(n) \sin^2[\pi Y(n) z/2], \qquad n = (j, k)$$

We have

$$\sum_{k=-\infty}^{\infty} a(k)\, b_j(k) = \sum_{k=-\infty}^{\infty} A(k)[b_j(k) - b_j(k+1)]$$

with

$$A(k) = \begin{cases} a(1) + \cdots + a(k), & k \geqslant 1 \\ 0, & k = 0 \\ -a(k+1) - \cdots - a(0), & k \leqslant -1 \end{cases}$$

Since $\alpha_2$ is not half-integer,

$$|A(k)| = \left| \sum_{i=i_0(k)}^{i_1(k)} \cos(4\pi i \alpha_2) \right|$$

is uniformly bounded in $k$. In addition,

$$|b_j(k) - b_j(k+1)| \leqslant C\{|n|^{-4} \sin^2[\pi Y(n) z/2] + |n|^{-3} z\, |\sin[\pi Y(n) z/2]|\}$$

so

$$|W_3(z; \alpha)| \leqslant C_0 \sum_{n \in \mathbf{Z}^2 \setminus \{0\}} \{|n|^{-4} \sin^2[\pi Y(n) z/2] + |n|^{-3} z\, |\sin[\pi Y(n) z/2]|\} \tag{4.10}$$

Let us show that

$$\begin{aligned} &\sum_{n \in \mathbf{Z}^2 \setminus \{0\}} |n|^{-4} \sin^2[\pi Y(n) z/2] \leqslant C z^2\, |\log z| \\ &\sum_{n \in \mathbf{Z}^2 \setminus \{0\}} |n|^{-3} z\, |\sin[\pi Y(n) z/2]| \leqslant C z^2\, |\log z| \end{aligned} \tag{4.11}$$

We have

$$\sum_{0 \leqslant |n| \leqslant 1/z} |n|^{-4} \sin^2[\pi Y(n) z/2] \leqslant \sum_{0 \leqslant |n| \leqslant 1/z} C\, |n|^{-2} z^2 \leqslant C_1 z^2\, |\log z|$$

and

$$\sum_{|n| \geq 1/z} |n|^{-4} \sin^2[\pi Y(n) z/2] \leq \sum_{|n| \geq 1/z} |n|^{-4} \leq C_1 z^2$$

which proves the first part of (4.11). The second part is established in the same way.

From (4.10), (4.11),

$$W_3(z; \alpha) = O(z^2 |\log z|), \qquad z \to 0$$

which proves (4.9). From (4.9),

$$\lim_{z \to 0} z^{-1} W(z; \alpha) = 1$$

which was stated. Theorem 1.3 is proved.

*Proof of Theorem 1.4.* When $\gamma$ is the circle $\{|x| = \pi^{-1/2}\}$ of area 1, then $Y(n) = \pi^{-1/2} |n|$, $\rho(n) = \pi^{-1/2}$, and (1.33) reduces to

$$W(z; \alpha) = 2\pi^{-5/2} \sum_{k=1}^{\infty} |r_\alpha(k)|^2 k^{-3/2} \sin^2(\pi^{1/2} k^{1/2} z/2) \tag{4.12}$$

with

$$r_\alpha(k) = \sum_{n \in \mathbf{Z}^2:\, n_1^2 + n_2^2 = k} e(n\alpha)$$

Define

$$\psi(\xi) = \xi^{-3/2} \sin^2(\xi^{1/2}), \qquad \Delta = \pi z^2/4 \tag{4.13}$$

Then (4.12) is equivalent to

$$z^{-1} W(z; \alpha) = \pi^{-2} \sum_{k=1}^{\infty} |r_\alpha(k)|^2 \psi(k\Delta) \Delta \tag{4.14}$$

We shall use the following result from ref. 9.

**Theorem D.** For all rational $\alpha \in \mathbf{Q}^2$,

$$\lim_{N \to \infty} (N \log N)^{-1} \sum_{k=1}^{N} |r_\alpha(k)|^2 = C(\alpha) > 0 \tag{4.15}$$

For all Diophantine $\alpha$,

$$\lim_{N \to \infty} N^{-1} \sum_{k=1}^{N} |r_\alpha(k)|^2 = \pi \tag{4.16}$$

*Remark.* Actually in ref. 9 an explicit formula was derived for $C(\alpha)$. For $\alpha=0$ it gives $C(0)=3$.

Consider

$$I(a, a+\varepsilon; \Delta)=\sum_{k:\, a\leqslant k\Delta<a+\varepsilon} |r_\alpha(k)|^2\, \psi(k\Delta)\Delta, \qquad 0<\Delta\ll\varepsilon\ll a$$

Let us replace $\psi(k\Delta)$ by $\psi(a)$ in the RHS of the last formula and estimate the error coming from this replacement:

$$|I(a, a+\varepsilon; \Delta)-\psi(a)\, S(a, a+\varepsilon; \Delta)| \leqslant C\varepsilon\psi_0(a)\, S(a, a+\varepsilon; \Delta) \quad (4.17)$$

where

$$S(a, a+\varepsilon; \Delta)=\sum_{k:\, a\leqslant k\Delta<a+\varepsilon} |r_\alpha(k)|^2\, \Delta$$

and

$$\psi_0(\xi)=\xi^{-3/2}$$

Assume $\alpha\in\mathbf{Q}^2$. Then from (4.15),

$$\begin{aligned} S(a, a+\varepsilon; \Delta) &= C(\alpha)\{(a+\varepsilon)\log[(a+\varepsilon)/\Delta]-a\log(a/\Delta)\}+o(|\log\Delta|) \\ &= C(\alpha)\varepsilon\,|\log\Delta|+o(|\log\Delta|), \qquad \Delta\to 0 \end{aligned} \quad (4.18)$$

(4.17) and (4.18) imply

$$|I(a, a+\varepsilon; \Delta)-C(\alpha)|\log\Delta\; |\psi(a)\varepsilon| \leqslant C\,|\log\Delta|\, \psi_0(a)\,\varepsilon^2, \quad \Delta\leqslant\Delta_0(\varepsilon) \quad (4.19)$$

Summing up this estimate for $N$ adjacent intervals $[a+j\varepsilon, a+(j+1)\varepsilon]$, $j=0,..., N-1$, we obtain that

$$\begin{aligned} &\left|I(a, b; \Delta)-C(\alpha)\,|\log\Delta|\sum_{j=0}^{N-1}\psi(a+j\varepsilon)\varepsilon\right| \\ &\quad\leqslant C\,|\log\Delta|\sum_{j=0}^{N-1}\psi_0(a+j\varepsilon)\,\varepsilon^2, \qquad b=a+N\varepsilon \end{aligned}$$

which implies

$$\left|I(a, b; \Delta)-C(\alpha)\,|\log\Delta|\int_a^b\psi(t)\,dt\right| \leqslant C_0\,|\log\Delta|\,\varepsilon, \qquad C_0=C_0(a) \quad (4.20)$$

It remains to estimate $I(0, a; \Delta)$ and $I(b, \infty; \Delta)$.

Assume $a < 1$. Then $\psi(\xi)$ is a decreasing function on $(0, a)$; hence

$$I(a/2, a; \Delta) \leqslant \psi(a/2)\, S(a/2, a; \Delta)$$

Now

$$S(a/2, a; \Delta) \leqslant S(0, a; \Delta) = \sum_{k:\, k\Delta < a} |r_\alpha(k)|^2\, \Delta \leqslant Ca \log(a/\Delta) \leqslant Ca\, |\log \Delta|$$

so

$$I(a/2, a; \Delta) \leqslant C_0\, |\log \Delta|\, \psi(a/2)(a/2)$$

Replacing $a$ by $a2^{-j}$, we obtain that

$$I(a2^{-j-1}, a2^{-j}; \Delta) \leqslant C_0\, |\log \Delta|\, \psi(a2^{-j-1})\, a2^{-j-1}, \qquad j = 0, 1, \ldots, J \quad (4.21)$$

where

$$J = \min\{j \colon a2^{-j} \leqslant \Delta\}$$

Summing up (4.21) in $j = 0, 1, \ldots, J$, we obtain that

$$I(0, a; \Delta) \leqslant C_1\, |\log \Delta| \int_0^a \psi(\xi)\, d\xi \qquad (4.22)$$

Assume $b > 1$. Then along the same way we obtain that

$$I(b, 2b; \Delta) \leqslant C_0\, |\log \Delta|\, \psi_1(b) b, \qquad \psi_1(\xi) = \xi^{-1/2}(1 + \xi^{-1})$$

and then that

$$I(b, \infty; \Delta) \leqslant C_1\, |\log \Delta| \int_b^\infty \psi_1(\xi)\, d\xi \qquad (4.23)$$

Choosing $a \ll 1$ and $b \gg 1$, and then $\varepsilon \ll \min\{a, C_0^{-1}(a)\}$, where $C_0(a)$ is the constant in (4.20), we obtain from (4.20), (4.22), and (4.23) that

$$I(0, \infty; \Delta) = C(\alpha)\, |\log \Delta| \int_0^\infty \psi(\xi)\, d\xi + o(|\log \Delta|), \qquad \Delta \to 0 \quad (4.24)$$

By (4.14), $z^{-1} W(z; \alpha) = \pi^{-2} I(0, \infty; \Delta)$. In addition,

$$\int_0^\infty \psi(\xi)\, d\xi = \int_0^\infty \xi^{-3/2} \sin^2 \xi^{1/2}\, d\xi = 2 \int_0^\infty \eta^{-2} \sin^2 \eta\, d\eta = \pi$$

Therefore

$$z^{-1}W(z;\alpha)=C(\alpha)\,\pi^{-1}\,|\log \Delta|+o(|\log \Delta|)$$

or, since $\Delta=\pi z^2/4$,

$$z^{-1}W(z;\alpha)=2C(\alpha)\,\pi^{-1}\,|\log z|+o(|\log z|)$$

Hence

$$\lim_{z\to 0}\,(z\,|\log z|)^{-1}\,W(z;\alpha)=2C(\alpha)\,\pi^{-1}$$

Thus the first part of Theorem 1.4 is proved.

Assume now that $\alpha$ is Diophantine. Then by (4.16),

$$S(a,a+\varepsilon;\Delta)=\pi\varepsilon+o(1),\qquad \Delta\to 0$$

Hence (4.17) implies that

$$|I(a,a+\varepsilon;\Delta)-\pi\varepsilon|\leqslant C\psi_0(a)\,\varepsilon^2,\qquad \Delta\leqslant\Delta_0(\varepsilon)$$

Now, along the same way as we derived (4.24) for rational $\alpha$, we arrive at

$$I(0,\infty;\Delta)=\pi\int_0^\infty \psi(\xi)\,d\xi+o(1)=\pi^2+o(1),\qquad \Delta\to 0$$

This implies

$$z^{-1}W(z;\alpha)=\pi^{-2}I(0,\infty;\Delta)=1+o(1),\qquad z\to 0$$

which proves Theorem 1.4 for Diophantine $\alpha$.

## 5. PROOF OF THEOREMS 1.5–1.7

*Proof of Theorem 1.5.* By (3.1)

$$F(R,S;\alpha)=F(R+w_S(R);\alpha)-F(R;\alpha)+O(SR^{-2}\,|F(R+w_S(R);\alpha)|)$$

Hence assuming $g(x)\in C_0^\infty(\mathbf{R}^1)$, we obtain that

$$\begin{aligned}&\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T} g(F(R,S;\alpha))\,\varphi(R/T)\,dR\\ &\quad=\frac{1}{(c_2-c_1)T}\int_{c_1T}^{c_2T} g(F(R+w_S(R);\alpha)-F(R;\alpha))\,\varphi(R/T)\,dR+O(ST^{-2})\end{aligned}\tag{5.1}$$

By Theorem 2.1,

$$\lim_{S/T\to\infty, S/T^2\to 0} \frac{1}{(c_2-c_1)T}\int_{c_1 T}^{c_2 T} g(F(R+w_S(R);\alpha)-F(R;\alpha))\,\varphi(R/T)\,dR$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1-x_2)\,\nu_\alpha(dx_1)\,\nu_\alpha(dx_2)=\int_{-\infty}^{\infty} g(x)\,\mu_\alpha(dx)$$

Hence

$$\lim_{S/T\to\infty, S/T^2\to 0} \frac{1}{(c_2-c_1)T}\int_{c_1 T}^{c_2 T} g(F(R, S;\alpha))\,\varphi(R/T)\,dR=\int_{-\infty}^{\infty} g(x)\,\mu_\alpha(dx)$$

By implication this equation holds for every continuous bounded function $g(x)$. Theorem 1.5 is proved.

*Proof of Theorem 1.6.* From (5.1) and Theorem 2.2 we obtain that

$$\lim_{T\to\infty, S/T\to z} \frac{1}{(c_2-c_1)T}\int_{c_1 T}^{c_2 T} g(F(R, S;\alpha))\,\varphi(R/T)\,dR$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1-x_2)\frac{1}{(c_2-c_1)}\int_{c_1}^{c_2} dc\,\varphi(c)\,\nu_\alpha(dx_1\,dx_2;z/(2c))$$

where $\nu_\alpha(dx_1, dx_2; z)$ is a joint distribution of $F(R;\alpha)$ and $F(R+z;\alpha)$. Let $\mu_\alpha(dx; z)$ be a probability distribution of a difference $\xi_1-\xi_2$ of two random variables, whose joint distribution coincides with $\nu_\alpha(dx_1\,dx_2; z/2)$. Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1-x_2)\;\nu_\alpha(dx_1\,dx_2;z/2)=\int_{-\infty}^{\infty} g(x)\,\mu_\alpha(dx;z) \tag{5.2}$$

Hence

$$\lim_{T\to\infty, S/T\to z} \frac{1}{(c_2-c_1)T}\int_{c_1 T}^{c_2 T} g(F(R, S;\alpha))\,\varphi(R/T)\,dR$$
$$=\frac{1}{(c_2-c_1)}\int_{c_1}^{c_2} dc\,\varphi(c)\int_{-\infty}^{\infty} g(x)\,\mu_\alpha(dx;z/c)$$

Theorem 1.6 is proved.

*Proof of Theorem 1.7.* Almost periodicity of

$$\int_{-\infty}^{\infty} g(x)\,\mu_\alpha(dx;z)$$

follows from Eq. (5.2) and the first part of Theorem 2.3. The value of

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dz \int_{-\infty}^{\infty} g(x)\, \mu_\alpha(dx; z)$$

follows from (5.2) and the second part of Theorem 2.3. Theorem 1.7 is proved.

## 6. PROOF OF THEOREMS 1.8, 1.9

*Proof of Theorem 1.8.* For the sake of simplicity we will assume $\gamma \in \Gamma_1$. The case $\gamma \in \Gamma_1(H)$ is treated similarly. By (5.2), $\mu_\alpha(dx; z)$ is a distribution of the almost periodic function $F(R+z/2; \alpha) - F(R; \alpha)$. From (1.41),

$$\begin{aligned} &F(R+z/2;\alpha) - F(R;\alpha) \\ &\quad = -2\pi^{-1} \sum_{n\in \mathbf{Z}^2\setminus\{0\}} e(n\alpha)\, |n|^{-3/2} \,[\rho(n)]^{1/2} \sin[\pi z Y(n)/2] \\ &\qquad \times \sin[2\pi(R+z/4)\, Y(n) - 3\pi/4] \end{aligned} \tag{6.1}$$

Define for every $n \in M$,

$$\begin{aligned} f_n(s; z, \alpha) = -2\pi^{-1} \sum_{k=1}^{\infty} e(kn\alpha)\, |kn|^{-3/2}\, [\rho(n)]^{1/2} \sin[\pi z Y(kn)/2] \\ \times \sin[2\pi k s + \pi z Y(kn)/2 - 3\pi/4] \end{aligned} \tag{6.2}$$

which is a periodic function in $s$ of period 1. Then (6.1) can be rewritten as

$$F(R+z/2;\alpha) - F(R;\alpha) = \sum_{n\in M} f_n(Y(n)R; z, \alpha)$$

By our assumption, the numbers $Y(n)$, $n \in M$, are linearly independent over $\mathbf{Q}$, so by Lemmas 4.4 and 2.5 in ref. 7, $\mu_\alpha(dt; z)$, the distribution of $F(R+z/2;\alpha) - F(R;\alpha)$, coincides with the distribution of the random series

$$\xi_{z\alpha} = \sum_{n\in M} f_n(t_n; z, \alpha) \tag{6.3}$$

where $t_n$ are independent random variables, uniformly distributed on $[0, 1]$. Since $\gamma \in \Gamma_1 \subset \Gamma_0$, all the numbers $\{Y(n), n \in M\}$ are different. Let us order the numbers $\{Y(n), n \in M\}$ in the increasing order, i.e., $Y(n(1)) < Y(n(2)) < \cdots$. Denote

$$a_k(s; z, \alpha) = f_{n(k)}(s; z, \alpha), \qquad a_k(s) = a_k(s; z, \alpha) \tag{6.4}$$

According to Theorem 3.1 in ref. 7, Theorem 1.8 will follow, except the lower bound (1.39), if we prove that

$$\sup_{0\leqslant s\leqslant 1} |a_k(s)| < Jk^{-3/4}, \qquad J>0$$
$$\sum_{j=k}^{\infty} \int_0^1 a_j(s)^2\,ds > J_0 k^{-1/2}, \qquad J_0>0 \tag{6.5}$$

Since $Y(\xi)$ is a positive homogeneous function of second order,

$$C_0 k^{1/2} \leqslant |n(k)| \leqslant C_1 k^{1/2}, \qquad C_0, C_1 > 0 \tag{6.5'}$$

Hence (6.5) is equivalent to the following two estimates:

$$\sup_{0\leqslant s\leqslant 1} |f_n(s; z, \alpha)| < J_1 |n|^{-3/2}, \qquad J_1>0 \tag{6.6}$$

$$\sum_{n\in M:\,|n|>r} \int_0^1 [f_n(s; z, \alpha)]^2\,ds > J_2 r^{-1}, \qquad J_2>0 \tag{6.7}$$

The first estimate follows from (6.2):

$$|f_n(s; z, \alpha)| \leqslant C \sum_{k=1}^{\infty} |kn|^{-3/2} \leqslant C_0 |n|^{-3/2}$$

Let us prove (6.7). Equation (6.2) is a Fourier series in $s$, so

$$\int_0^1 [f_n(s; z, \alpha)]^2\,ds = 2\pi^{-2} \sum_{k=1}^{\infty} |kn|^{-3} \rho(n) \sin^2[\pi z Y(kn)/2]$$
$$\geqslant C|n|^{-3} \sin^2[\pi z Y(n)/2] \tag{6.8}$$

and

$$\sum_{n\in M:\,|n|>r} \int_0^1 [f_n(s; z, \alpha)]^2\,ds \geqslant C \sum_{n\in M:\,|n|>r} |n|^{-3} \sin^2[\pi z Y(n)/2] \tag{6.9}$$

Note that the density of $M$ in the plane is $6/\pi^2$, i.e.,

$$\lim_{r\to\infty} (\pi r^2)^{-1} \sum_{n\in M:\,|n|<r} 1 = 6/\pi^2$$

so (6.7) will follow from (6.9) if we show that $\forall\varepsilon>0$, $\exists\delta>0$ such that the upper density of the set $Q_\delta = \{n\in\mathbf{Z}^2: \sin^2[\pi z Y(n)/2] \leqslant \delta^2\}$ is less than $\varepsilon$. Indeed, then, for large $r$,

$$\sum_{n\in M:\,|n|>r} \int_0^1 [f_n(s; z, \alpha)]^2\,ds \geqslant C \sum_{n\in M\setminus Q_\delta:\,|n|>r} |n|^{-3}\delta^2$$
$$\geqslant C[(6/\pi^2) - 2\varepsilon]\,\delta^2 \sum_{n\in\mathbf{Z}^2:\,|n|>r} |n|^{-3} \geqslant C_0 r^{-1}$$

which gives (6.7). Let $\sin\delta_0 = \delta$. Consider the annuli

$$A_\delta(l) = \{x \in \mathbf{R}^2 \colon |\pi z Y(x)/2 - l| < \delta_0\}$$

and define $Q_\delta(l) = \mathbf{Z}^2 \cap A_\delta(l)$. Then $Q_\delta = \bigcup_{l=1}^{\infty} Q_\delta(l)$. Let $N_\delta(l) = |Q_\delta(l)|$. Due to the 2/3-estimate of Sierpinski (see, e.g., ref. 27 or ref. 15)

$$|N_\delta(l) - \text{Area } A_\delta(l)| \leqslant Cl^{2/3}$$

In addition, Area $A_\delta(l) \leqslant C_0 \delta l$, hence $N_\delta(l) \leqslant C_0 \delta l + Cl^{2/3}$ and

$$\sum_{l=1}^{L} N_\delta(l) \leqslant C_0 \delta L^2 + CL^{5/3}$$

Therefore the upper density $d(Q_\delta)$ of $Q_\delta$ is estimated as

$$d(Q_\delta) \leqslant \limsup_{L \to \infty} C_1 L^{-2} \sum_{l=1}^{L} N_\delta(l) \leqslant C_2 \delta \tag{6.10}$$

This proves the desired estimate of $d(Q_\delta)$, and so Theorem 1.8 is proved, except (1.39). To prove (1.39), we use the following theorem.

**Theorem 6.1**. Assume $a_k(s)$, $k = 1, 2, \ldots$, are continuous functions of period 1, with $\int_0^1 a_k(s)\, ds = 0$ and

$$\sup_k \sup_{0 \leqslant s \leqslant 1} |a_k(s)| < \infty, \qquad \sum_{k=1}^{\infty} \int_0^1 |a_k(s)|^2\, ds < \infty$$

Assume also that $\exists k_0$ and $\delta_0 > 0$ such that $\forall k > k_0$, $\exists G_k \subset \{1, 2, \ldots, k\}$ such that $|G_k| > \delta_0 k$ and $\forall l \in G_k$,

$$C' l^{-\gamma} > \sup_{0 \leqslant s \leqslant 1} |a_l(s)| \geqslant \left( \int_0^1 |a_l(s)|^2\, ds \right)^{1/2} \geqslant C'' l^{-\gamma} \tag{6.11}$$

with some $C'$, $C'' > 0$ and $0 < \gamma < 1$.

Then $\forall \varepsilon > 0$, $\exists C_\varepsilon$, $\lambda_\varepsilon > 0$ such that $\forall x \geqslant 0$,

$$\Pr\left\{ \sum_{k=1}^{\infty} a_k(t_k) > x \right\} > C_\varepsilon \exp(-\lambda_\varepsilon x^{(1+\varepsilon)/(1-\gamma)}) \tag{6.12}$$

$$\Pr\left\{ \sum_{k=1}^{\infty} a_k(t_k) < -x \right\} > C_\varepsilon \exp(-\lambda_\varepsilon x^{(1+\varepsilon)/(1-\lambda)}) \tag{6.13}$$

assuming that $t_k$, $k \geqslant 1$, are independent uniformly distributed random variables on the interval [0, 1].

*Proof.* The proof follows the proof of Theorem 5.1 in ref. 8 and Theorem 3.3 in ref. 7. Define

$$A_l\{s: a_l(s) \geqslant \delta_1 l^{-\gamma}\}, \qquad \delta_1 > 0$$

Then (6.11) and $\int_0^1 a_l(t)\,dt = 0$ imply that $\exists \delta_1, \delta_2 > 0$: $\forall l \in G_k$, mes $A_l > \delta_2$ [otherwise $L^2$-norm of $a_l(t)$ is too small]. For $l \notin G_k$ define

$$A_l = \{t: a_l(t) \geqslant -l^{-5}\}$$

Again mes $A_l > l^{-10}$, $l \geqslant l_0$ [otherwise $\int_0^1 a_l(t)\,dt < 0$]. Define

$$D_k = \left\{(t_1, t_2,...): t_l \in A_l, l = k_0,..., k; \left| \sum_{l=k+1}^{\infty} a_l(t_l) \right| < 1 \right\}$$

For large $k$,

$$\mathrm{Var} \sum_{l=k+1}^{\infty} a_l(t_l) = \sum_{l=k+1}^{\infty} \int_0^1 |a_l(s)|^2\,ds < 1/2$$

so by Chebyshev's inequality

$$\Pr\left\{ \left| \sum_{l=k+1}^{\infty} a_l(t_l) \right| > 1 \right\} \leqslant 1/2$$

and

$$\Pr D_k \geqslant (1/2) C \prod_{l=1}^{k} l^{-10} \geqslant \exp(-\gamma_\varepsilon k^{1+\varepsilon}), \qquad \gamma_\varepsilon > 0, \quad k \geqslant k_1(\varepsilon)$$

For $(t_1, t_2,...) \in D_k$,

$$\sum_{l=1}^{\infty} a_l(t_l) \geqslant \left(\delta_1 \sum_{l \in G_k} l^{-\gamma}\right) - C \geqslant \delta_0 \delta_1 k^{1-\gamma} - C \geqslant \delta_3 k^{1-\gamma}, \qquad k \geqslant k_2$$

Thus

$$\Pr\left\{ \sum_{l=1}^{\infty} a_l(t_l) \geqslant x = \delta_3 k^{1-\gamma} \right\} \geqslant \Pr D_k \geqslant \exp(-\gamma_\varepsilon k^{1+\varepsilon})$$

$$= \exp(-\lambda_\varepsilon x^{(1+\varepsilon)/(1-\gamma)}), \qquad x \geqslant x_0$$

which proves (6.12). Condition (6.13) is established along the same way. Theorem 6.1 is proved.

**Lemma 6.2.** $\exists \delta > 0$ such that $a_k(s) = f_{n(k)}(s; z, \alpha)$ satisfies the condition of Theorem 6.1 with $G_k = \{1 \leqslant l \leqslant k: n(l) \in M \backslash Q_\delta\}$ and $\gamma = 3/4$.

*Proof.* The first estimate in (6.11) follows from (6.5). To prove the second estimate, remark that by (6.8) and (6.5′),

$$\int_0^1 (f_{n(l)}(s; z, \alpha))^2 \, ds \geq Cl^{-3/2} \sin^2[\pi z Y(n(l))/2]$$

If $n(l) \in M \backslash Q_\delta$, then $\sin^2[\pi z Y(n(k))/2] \geq \delta^2$, so

$$\int_0^1 (f_{n(l)}(s; z, \alpha))^2 \, ds \geq Cl^{-3/2}\delta^2$$

which proves the second estimate in (6.11). From (6.10) we obtain that

$$|G_k| \geq \delta_0 k, \qquad \delta_0 > 0$$

Lemma 6.2 is proved.

Theorem 6.1 and Lemma 6.2 imply (1.39), so Theorem 1.8 is proved. Proof of Theorem 1.9 goes along the same lines, so we omit it.

## 7. PROOF OF THEOREMS 1.10, 1.11

*Proof of Theorem 1.10.* Assume $\gamma \in \Gamma_1$. As was shown in Section 6, $\mu_\alpha(dx; z)$ is a distribution of the random series (6.3), so to prove Theorem 1.10 we have to prove that the distribution of $\xi_{z\alpha}/(\operatorname{Var} \xi_{z\alpha})^{1/2}$ converges to a standard normal distribution as $z \to 0$. To that end we shall check that the Lindeberg condition holds for the random series (6.3).

By the Parseval formula,

$$\operatorname{Var} \xi_{z\alpha} = \int_{-\infty}^{\infty} x^2 \mu_\alpha(dx; z) = \|F(R + z/2; \alpha) - F(R; \alpha)\|_{B^2}^2$$

$$= 2\pi^{-2} \sum_{n \in \mathbf{Z}^2 \backslash \{0\}} |n|^{-3} \rho(n) \sin^2[\pi z Y(n)/2]$$

By (4.1) this coincides with $W(z; \alpha)$, so by Theorem 1.3,

$$\operatorname{Var} \xi_{z\alpha} = z + o(z), \qquad z \to 0 \tag{7.1}$$

Let

$$\sigma_{z\alpha}(n) = [\operatorname{Var} f_n(t_n; z, \alpha)]^{1/2} = \left(\int_0^1 |f_n(s; z, \alpha)|^2 \, ds\right)^{1/2}$$

Then the Lindeberg condition is that for every $n \in M$,

$$\lim_{z \to 0} \sigma_{z\alpha}^2(n)/\operatorname{Var} \xi_{z\alpha} = 0 \tag{7.2}$$

and for every $\varepsilon > 0$,

$$g_{z\alpha}(\varepsilon) = (\text{Var}\ \xi_{z\alpha})^{-1} \sum_{n \in M} \int_{|s| \geqslant \varepsilon(\text{Var}\ \xi_{z\alpha})^{1/2}} s^2 f_n(s; z, \alpha)\, ds \to 0 \tag{7.3}$$

as $z \to 0$. From (6.2),

$$|f_n(s; z, \alpha)| \leqslant C\, |n|^{-3/2} \sum_{k=1}^{\infty} k^{-3/2}\, |\sin[\pi z Y(kn)/2]|$$

It implies that

$$|f_n(s; z, \alpha)| \leqslant \begin{cases} C_0\, |n|^{-1}\, z^{1/2} & \text{when} \quad |n|\, z < 1 \\ C_0\, |n|^{-3/2} & \text{when} \quad |n|\, z > 1 \end{cases} \tag{7.4}$$

Indeed, if $|n|\, z < 1$, then

$$\sum_{k=1}^{\infty} k^{-3/2}\, |\sin[\pi z Y(kn)/2]|$$
$$= \Delta^{1/2} \sum_{k=1}^{\infty} |k\Delta|^{-3/2} \sin(k\Delta)\Delta \leqslant C_1\, |n|^{1/2}\, z^{1/2}, \qquad \Delta = \pi z Y(n)/2$$

which proves the first part of (7.4). The second part follows from the evident inequality

$$\sum_{k=1}^{\infty} k^{-3/2}\, |\sin[\pi z Y(kn)/2]| \leqslant \sum_{k=1}^{\infty} k^{-3/2}$$

Similarly,

$$\sigma_{z\alpha}^2(n) = \int_0^1 |f_n(s; z, \alpha)|^2\, ds \leqslant C\, |n|^{-3} \sum_{k=1}^{\infty} k^{-3}\, |\sin[\pi z Y(kn)/2]|^2$$

hence

$$\sigma_{z\alpha}^2(n) \leqslant \begin{cases} C_0\, |n|^{-1}\, z^2\, |\log z| & \text{when} \quad |n|\, z < 1 \\ C_0\, |n|^{-3} & \text{when} \quad |n|\, z > 1 \end{cases} \tag{7.5}$$

Comparing (7.1) with (7.5), we obtain (7.2).

To prove (7.3), remark that (7.4) implies that for every $\varepsilon > 0$ the inequality

$$\sup_{0 \leqslant s \leqslant 1} |f_n(s; z, \alpha)| \geqslant \varepsilon(\text{Var}\ \xi_{z\alpha})^{1/2} = \varepsilon z^{1/2}(1 + o(1))$$

holds only for a finite set $n \in M_\varepsilon \subset M$, which does not depend on $z$. By (7.3),

$$g_{z\alpha}(\varepsilon) \leqslant (\operatorname{Var} \xi_{z\alpha})^{-1} \sum_{n \in M_\varepsilon} \sigma_{z\alpha}^2(n)$$

and hence the condition $\lim_{z \to 0} g_{z\alpha}(\varepsilon) = 0$ follows from (7.2).

Thus, the random series (6.3) satisfies the Lindeberg condition, which implies that the distribution of $\xi_{z\alpha}/(\operatorname{Var} \xi_{z\alpha})^{1/2}$ converges to a standard normal distribution as $z \to 0$ (see, e.g., ref. 24). This proves Theorem 1.10 in the case when $\gamma \in \Gamma_1$. The same arguments work in the case when $\gamma \in \Gamma_1(H)$, so Theorem 1.10 is proved.

*Proof of Theorem 1.11.* For $\gamma = \{|z| = \pi^{-1/2}\}$, (1.41) reduces to

$$\begin{aligned} F(R; \alpha) &= \pi^{-1} \sum_{n \in \mathbf{Z}^2 \setminus \{0\}} |n|^{-3/2} \pi^{-1/4} \cos[2\pi^{1/2} R\, |n| + \phi(n; \alpha)] \\ &= \Re \left\{ \pi^{-5/4} \sum_{n \in \mathbf{Z}^2 \setminus \{0\}} |n|^{-3/2} \exp[2\pi^{1/2} R\, |n|\, i + \phi(n; \alpha) i] \right\} \\ &= \pi^{-5/4} \sum_{k=1}^{\infty} r_\alpha(k)\, k^{-3/4} \cos(2\pi^{1/2} R k^{1/2} - 3\pi/4) \end{aligned} \tag{7.6}$$

with

$$r_\alpha(k) = \sum_{n \in \mathbf{Z}^2 :\, n_1^2 + n_2^2 = k} e(n\alpha) \tag{7.7}$$

Hence

$$\begin{aligned} &F(R + z/2; \alpha) - F(R; \alpha) \\ &\quad = -2\pi^{-5/4} \sum_{k=1}^{\infty} r_\alpha(k)\, k^{-3/4} \sin(\pi^{1/2} z k^{1/2}/2) \sin[2\pi^{1/2}(R + z/4)\, k^{1/2} - 3\pi/4] \\ &\quad = \sum_{\text{square free } k} f_k(k^{1/2} R; z, \alpha) \end{aligned} \tag{7.8}$$

with

$$\begin{aligned} f_k(s; z, \alpha) = -2\pi^{-5/4} \sum_{l=1}^{\infty} r_\alpha(l^2 k)(l^2 k)^{-3/4} \sin[\pi^{1/2} z (l^2 k)^{1/2}/2] \\ \times \sin[2\pi^{1/2}(ls + zlk^{1/2})/4 - 3\pi/4] \end{aligned} \tag{7.9}$$

$k \in \mathbf{N}$ is square free if $k = k'l^2$ implies $l = 1$. The numbers $\{k^{1/2}, k \text{ is square free}\}$ are linearly independent over $\mathbf{Q}$, so $\mu_\alpha(dx; z)$, the distribution of $F(R + z/2; \alpha) - F(R; \alpha)$, coincides with the distribution of the random series

$$\xi_{z\alpha} = \sum_{\text{square free } k} f_k(k^{1/2} t_k; z, \alpha) \tag{7.10}$$

where $t_k$ are independent random variables, uniformly distributed on $[0, 1]$.[8,7] Let us check the Lindeberg condition for the random series (7.10) as $z \to 0$.

From (7.8),

$$\begin{aligned}
\operatorname{Var} \xi_{z\alpha} &= \int_{-\infty}^{\infty} x^2 \mu_\alpha(dx; z) \\
&= \| F(R + z/2) - F(R; \alpha) \|_{B^2}^2 \\
&= 2\pi^{-5/2} \sum_{k=1}^{\infty} |r_\alpha(k)|^2 k^{-3/2} \sin^2(\pi^{1/2} z k^{1/2}/2) \\
&\geqslant C_0 \sum_{k=1}^{z^{-2}} |r_\alpha(k)|^2 k^{-1/2} z^2 \geqslant C_0 z^3 \sum_{k=1}^{z^{-2}} |r_\alpha(k)|^2
\end{aligned} \tag{7.11}$$

By Theorem D in ref. 8,

$$\sum_{k=1}^{z^{-2}} |r_\alpha(k)|^2 \geqslant z^{-2} |\log z|^{-1}, \qquad z \leqslant z_0(\alpha) \tag{7.12}$$

Hence

$$\operatorname{Var} \xi_{z\alpha} \geqslant Cz |\log z|^{-1}, \qquad z \leqslant z_0(\alpha) \tag{7.13}$$

Let

$$\sigma_{z\alpha}(k) = [\operatorname{Var} f_k(t_k; z, \alpha)]^{1/2} = \left( \int_0^1 |f_k(s; z, \alpha)|^2 \, ds \right)^{1/2}$$

Then the Lindeberg condition is that for every $k \in \mathbf{N}$,

$$\lim_{z \to 0} \sigma_{z\alpha}^2(k) / \operatorname{Var} \xi_{z\alpha} = 0 \tag{7.14}$$

and for every $\varepsilon > 0$,

$$g_{z\alpha}(\varepsilon) = (\operatorname{Var} \xi_{z\alpha})^{-1} \sum_{k=1}^{\infty} \int_{|s| \geqslant \varepsilon (\operatorname{Var} \xi_{z\alpha})^{1/2}} s^2 f_k(s; z, \alpha) \, ds \to 0 \tag{7.15}$$

as $z \to 0$. Since

$$|r_\alpha(k)| \leqslant C_\delta k^\delta, \qquad \forall \delta > 0$$

(see, e.g., ref. 21), we obtain from (7.9) that

$$|f_k(s; z, \alpha)| \leqslant C_\delta \sum_{l=1}^{\infty} (l^2 k)^{-(3/4)+\delta} |\sin(\pi^{1/2} z l k^{1/2}/2)|$$

It implies that

$$|f_k(s; z, \alpha)| \leqslant \begin{cases} C_0(\delta)\, z^{(1/2)-2\delta} k^{-1/2} & \text{when} \quad k^{1/2} z < 1 \\ C_0(\delta)\, k^{-(3/4)+\delta} & \text{when} \quad k^{1/2} z > 1 \end{cases} \tag{7.16}$$

Indeed, if $k^{1/2} z < 1$, then

$$\sum_{l=1}^{\infty} (l^2 k)^{-(3/4)+\delta} |\sin(\pi^{1/2} z l k^{1/2}/2)|$$
$$= \Delta^{(1/2)-2\delta} k^{-(3/4)+\delta} \sum_{l=1}^{\infty} |l\Delta|^{-(3/2)+2\delta} |\sin(l\Delta)|\, \Delta, \qquad \Delta = \pi^{1/2} z k^{1/2}/2$$

which proves the first part of (7.16). The second part follows from the evident inequality

$$\sum_{l=1}^{\infty} (l^2 k)^{-(3/4)+\delta} |\sin(\pi^{1/2} z l k^{1/2}/2)| \leqslant k^{-(3/4)+\delta} \sum_{l=1}^{\infty} l^{-(3/2)+2\delta}$$

Similarly,

$$\sigma_{z\alpha}^2(k) = \int_0^1 |f_k(s; z, \alpha)|^2\, ds \leqslant C_\delta \sum_{l=1}^{\infty} (l^2 k)^{-(3/2)+2\delta} \sin^2(\pi^{1/2} z l k^{1/2}/2)$$

Hence

$$\sigma_{z\alpha}^2(k) \leqslant \begin{cases} C_0(\delta)\, k^{-1/2} z^{2-4\delta} & \text{when} \quad k^{1/2} z < 1 \\ C_0(\delta)\, k^{-(3/2)+2\delta} & \text{when} \quad k^{-1/2} z > 1 \end{cases} \tag{7.17}$$

Comparing (7.13) with (7.17), we obtain (7.14).

Let us prove (7.15). By (7.13) and (7.16), the inequality

$$\sup_{0 \leqslant s \leqslant 1} |f_k(s; z, \alpha)| \geqslant (\operatorname{Var} \xi_{z\alpha})^{1/2}$$

can hold only if

$$C_0(\delta)\, z^{(1/2)-2\delta} k^{-1/2} \geqslant C\varepsilon z^{1/2} |\log z|^{-1/2}$$

which implies

$$k \leqslant z^{-6\delta}, \qquad z \leqslant z_0(\alpha, \delta, \varepsilon) \tag{7.18}$$

From (7.15), (7.13), and (7.18),

$$g_{z\alpha}(\varepsilon) \leqslant z^{-1-\delta} \sum_{k=1}^{z^{-6\delta}} \sigma_{z\alpha}(k), \qquad z \leqslant z_1(\alpha, \delta, \varepsilon)$$

so by (7.17),

$$g_{z\alpha}(\varepsilon) \leqslant z^{-1-\delta} \sum_{k=1}^{z^{-6\delta}} k^{-1/2} z^{2-5\delta} \leqslant z^{1-20\delta}, \qquad z \leqslant z_2(\alpha, \delta, \varepsilon)$$

which proves that $\lim_{z \to 0} g_{z\alpha}(\varepsilon) = 0$.

Thus the Lindeberg condition (7.14), (7.15) holds, and so the distribution of $(\operatorname{Var} \xi_{z\alpha})^{-1/2} \xi_{z\alpha}$ converges to a standard normal distribution as $z \to 0$. Theorem 1.11 is proved.

## APPENDIX. PROOF OF THE FORMULA (4.2)

We prove in this Appendix the formula

$$I = \pi^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\xi|^{-3} \rho(\xi) \sin^2 Y(\xi)\, d\xi = \operatorname{Area}\{\operatorname{Int} \gamma\} \tag{A.1}$$

which reduces to (4.2) when $\operatorname{Area}\{\operatorname{Int} \gamma\} = 1$. To prove (A.1), let us rewrite the expression $I$ in a polar coordinate system. Given an angle $\varphi$, let $\xi(\varphi)$ be the unit vector in $\mathbf{R}^2$ with this angle, and $\rho_0(\varphi) = \rho(\xi(\varphi))$ and $Y_0(\varphi) = Y(\xi(\varphi))$.

Then

$$\begin{aligned} I &= \pi^{-1} \int_0^{\infty} \int_0^{2\pi} r^{-2} \rho_0(\varphi) \sin^2[r Y_0(\varphi)]\, d\varphi\, dr \\ &= \pi^{-1} \int_0^{2\pi} \rho_0(\varphi)\, Y_0(\varphi)\, d\varphi \int_0^{\infty} \frac{\sin^2 u}{u^2}\, du = \frac{1}{2} \int_0^{2\pi} \rho_0(\varphi)\, Y_0(\varphi)\, d\varphi \end{aligned} \tag{A.2}$$

Given a nonzero vector $\xi$ in $\mathbf{R}^2$ with angle $\varphi$, let $\psi(\varphi)$ be the angle of the point $x = x(\varphi) \in \gamma$ for which the outer normal is parallel to $\xi$. Define the sector

$$V(\psi_1, \psi_2) = \{x \in \mathbf{R}^2 \colon \psi_1 < \text{the angle of the vector } x < \psi_2\}$$

Then

$$\rho_0(\varphi_1)\, Y_0(\varphi_1)(\varphi_2 - \varphi_1) = 2 \operatorname{Area}\{(V(\psi(\varphi_1); \psi(\varphi_2)) \cap \operatorname{Int} \gamma)\} + o(|\varphi_1 - \varphi_2|)$$

The last relation holds, since the set $V(\psi(\varphi_1); \psi(\varphi_2)) \cap \operatorname{Int} \gamma$ can be well approximated by a triangle with the baseline $\Delta s = (\varphi_2 - \varphi_1)\, \rho_0(\varphi_1)$ and the height $Y_0(\varphi)$. Hence, we get, by approximating the integral by the usual approximating sum that

$$\int_0^{2\pi} \rho_0(\varphi)\, Y_0(\varphi)\, d\varphi = 2 \text{ Area Int } \gamma$$

The last identity together with (A.2) imply formula (A.1).

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