

Quantum Dynamics: Models and Mathematics

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Thermodynamic Limit of the Free Energy and Correlation Functions
of Spin Systems*

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Abstract: We give simple proofs of the existence of the thermodynamic limit of the free energy and of equilibrium states for continuous spin systems with "bounded" boundary conditions. For spin- $1/2$ Ising systems we show that the infinite volume limit of a state in which there is a field $h_b > 0$ on the boundaries is the same as that obtained from + boundary conditions (independent of the magnitude of h_b). In an appendix with E. Presutti we present stronger results about the existence and uniqueness of equilibrium states for continuous spin systems.

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I. Introduction

In this lecture I "continue" my review of the thermodynamic properties and equilibrium states of spin lattice systems. I shall be concerned both with continuous spin variables and with the Ising spin- $1/2$ case which I reviewed earlier [1].

The setting will always be the lattice Z^v at each site of which there is a vector 'spin' variable S_i , $i \in Z^v$, $S_i \in \mathbb{R}^n$. Each S_i has associated with it an intrinsic, or free, probability distribution (the same for all S_i) $\rho(S)dS$: $\rho(S)$ depends only on the magnitude of S , $|S|$, and satisfies the bound

$$\int e^{bS^2} \rho(S)dS < \infty, \quad \text{all } b < b_0. \quad (1.1)$$

where, unless otherwise specified, $b_0 = \infty$. I shall designate the class of measures satisfying (1.1) by $\mathcal{S}_1$. Typical examples of such distributions are

$\rho(S) = \exp [-V_j(S)]$ where V_j is an even polynomial in $|S|$, $j=2n$, $V_j(S) = a_0 S^j + a_2 S^{j-2} + \dots + a_{j-2} S^2 + a_j$, $a_0 > 0$. For $j \geq 4$, (1.1) is satisfied with $b_0 = \infty$, while for $j = 2$, the Gaussian case, $b_0 = a_0$.

A more restricted class of distributions, $\mathcal{S}_2 \subset \mathcal{S}_1$, is obtained if $\rho(S)$ is required to have compact support. This may be taken, without loss of generality, to be the unit ball,

$$\rho(S) = 0 \quad \text{if } |S| > 1 \quad (1.2)$$

The simplest case of such a measure is $\rho(S) = K$ (constant) inside

the unit ball. A still more restrictive class of measures,
 $\mathcal{S}_3 \subset \mathcal{S}_2$ is

$$\rho(S) = K \delta(|S| - 1) \quad (1.3)$$

For $n = 1, 2, 3$ these are respectively the Ising spin- $1/2$, the rigid rotator and classical Heisenberg models.

The Hamiltonian of the system in a finite domain $\Lambda \subset \mathbb{Z}^V$ with 'boundary conditions' (b.c.) b_Λ corresponding to specifying the values of the spin variables outside Λ has the form

$$H(S_\Lambda; b_\Lambda) = -\frac{1}{2} \sum_{i \neq j \in \Lambda} J(i-j) S_i S_j - \sum_{i \in \Lambda} h_i S_i, \quad (1.4)$$

$$S_\Lambda = \{S_i; i \in \Lambda\}$$

Here $J(i-j)$ is a symmetric $n \times n$ matrix, $J(i-j) S_i S_j =$

$\sum_{\gamma, \delta} J_{\gamma\delta}(i-j) S_i^\gamma S_j^\delta$, and we assume that the interaction has a

finite range, $J(\underline{x}) = 0$ for $|\underline{x}| > R$: h_i is the "magnetic field"

at site i , $h_i = h + \sum_{j \notin \Lambda} J(i-j) \bar{S}_j$, $h_i S_i = \sum_{\gamma=1}^n h_i^\gamma S_i^\gamma$, and $\{\bar{S}_j\}$

is a specified set of values of S_i for $i \notin \Lambda$. These constitute the set of boundary conditions $\{b_\Lambda\} = b$, i.e. they specify the fields h_i acting on spins near the boundary of Λ . (More generally only the probability distribution of the spins outside Λ need be specified.) The most commonly used b.c. in statistical mechanics are the 'zero' b.c., corresponding to setting $\bar{S}_i = 0$ (this is essentially the Dirichlet b.c. in field theory [2]) and the periodic b.c. We can take the latter into account for Λ a "rectangle", by modifying the definition of $J(\underline{x})$.

The Gibbs probability distribution of spins in Λ at a temperature $T = \beta^{-1}$ is, for a given external field h and b.c. b_Λ ,

$$\nu(dS_\Lambda, \beta, h; \Lambda, b_\Lambda) = Z^{-1} \exp [-\beta H(S_\Lambda; b_\Lambda)] \prod_{i \in \Lambda} [\rho(S_i) dS_i], \quad (1.5)$$

where

$$Z(\beta, h; \Lambda, b_\Lambda) \equiv \exp[|\Lambda| \Psi(\beta, h; \Lambda, b_\Lambda)] = \int \exp[-\beta H] \prod_{i \in \Lambda} [(S_i) dS_i] \quad (1.6)$$

Here $|\Lambda|$ equals the number of sites in Λ , and Ψ is (except for a factor $-\beta$) the free energy per site.

Let $\nu(dS_\Lambda, \beta, h; \Lambda, b_\Lambda)$ be the projection of the measure (1.5) onto $\Lambda \subset \mathbb{Z}^v$, $S_\Lambda = \{S_i; i \in \Lambda\}$. We are interested in the behaviour of the

$\nu(dS_\Lambda, \beta, h; \Lambda, b_\Lambda)$, or (essentially) equivalently the correlation functions, as $\Lambda \rightarrow \mathbb{Z}^v$ through some sequence of domains Λ_j . We are also interested in the properties of the free energy per site, $\Psi(\beta, h; \Lambda, b_\Lambda)$ in this limit: here we want to require that the Λ_j be reasonable, e.g. the fraction of sites within a distance R of $\partial \Lambda_j$, $|\partial_R \Lambda_j|/|\Lambda_j|$ go to zero as $j \rightarrow \infty$ [3]. We shall denote these limits by $\nu(dS_\Lambda, \beta, h; b)$ and $\Psi(\beta, h; b)$. Some of the questions which naturally arise are the existence, uniqueness, analyticity (in β and h), and cluster properties of these limits. Uniqueness refers both to different ways of letting $\Lambda \rightarrow \mathbb{Z}^v$ for given b.c. and to the dependence of the limit on b.c.

II. Thermodynamic Limit of the Free Energy

Among the above questions the easiest one to tackle is the existence and uniqueness of the limit of $\psi(\beta, h; \Lambda, b_\Lambda)$. It is easy to show, using standard methods, [3] that for $\rho(S) \in \mathcal{S}_2$ (and any reasonable sequence of domain shapes Λ_j), $\psi(\beta, h; \Lambda_j, b_{\Lambda_j}) \rightarrow \psi(\beta, h)$ independent of b.c. The problem is however more difficult in the case where the spins are not bounded. This problem has been treated in some detail by Guerra, Rosen and Simon (GRS) [4] for cases of interest from the point of view of constructive quantum field theory. They were able to prove that various b.c. lead to the same limit for $P(\emptyset)_2$. I shall describe here briefly how to prove such results for the spin lattice systems we are considering here. (I will however not try for the 'best' results which involve considerably more complicated methods: see Appendix for statement of "current" best results). I shall consider first the case of zero b.c., then the case of periodic b.c. and finally the "general" case, which however, as was pointed out by GRS cannot be too general. In all cases the thermodynamic limit will be approached through a sequence of cubes Γ_n , whose sides have lengths $2^n L_0$.

Zero b.c.

Let $Z_n = \exp[|\Gamma_n| \psi_n]$ be the partition function for a cube Γ_n , $|\Gamma_n| = 2^{nv} L_0^v$, with zero b.c. Clearly

$$\psi_n \leq \log \left[\int \exp[\alpha S^2 + hS] \rho(S) dS \right] \quad (2.1)$$

where $\alpha = \sum_{|x| \leq R} \alpha(x)$, $\alpha(x) = \max_{\delta} \sum_{\gamma=1}^n \beta |J_{\gamma\delta}(x)|$. The right side

of (2.1) is a uniform (in n) bound on ψ_n whenever $\alpha < b_0$ in (1.1) which we shall assume to be the case. We next wish to show that

$\psi_{n+1} \geq \psi_n - \epsilon_n$ with $\sum \epsilon_n < \infty$. This will imply the convergence of ψ_n to a limit $\psi(\beta, h; b_0)$ [3].

Remark: We note that by Jensen's inequality

$$Z_n = \langle \exp [-\beta H_n] \rangle_I \geq \exp [-\beta \langle H_n \rangle_I] = 1 \quad (2.2)$$

so that $\psi_n \geq 0$. Here the subscript I indicates that the expectations are with respect to the intrinsic measure $\prod [\rho(S_i) dS_i]$. Combining (2.1) and (2.2) shows that $|\psi_n|$ is bounded and thus we can always get convergence on subsequences. We are after something stronger however.

To obtain the bound $\psi_{n+1} \geq \psi_n - \epsilon_n$ we note that Γ_{n+1} can be thought of as the union of 2^v cubes $\Gamma_n^{(k)}$, $k = 1, \dots, 2^v$. The Hamiltonian of spins in Γ_{n+1} can therefore be divided into a part which contains only interactions between spins in the same cube $\Gamma_n^{(k)}$ and the interaction part between spins in different cubes $\Gamma_n^{(k)}$,

$$H(S_{\Gamma_{n+1}}; b_0) = \sum_{k=1}^{2^v} H(S_{\Gamma_n^{(k)}}; b_0) + \sum_{k \neq k'=1}^{2^v} U_{k,k'}. \quad (2.2')$$

where

$$U_{k,k'} = - \sum J(i-j) S_i S_j, \quad i \in \Gamma_n^{(k)}, \quad j \in \Gamma_n^{(k')} \quad (2.3)$$

Using Jensen's inequality then yields

$$Z_{n+1} \geq (Z_n)^{2^v} \exp[-\beta \sum \langle U_{k,k'} \rangle_n] \quad (2.4)$$

where $\langle \rangle_n$ indicates expectations with respect to the product of the Gibbs measures in each $\Gamma_n^{(k)}$. Thus,

$$- \langle U_{k,k'} \rangle_n = \sum J(i-j) \langle S_i \rangle_{(\beta, h; \Gamma_n^{(k)}, b_0)} \langle S_j \rangle_{(\beta, h; \Gamma_n^{(k')}, b_0)}, \\ i \in \Gamma_n^{(k)}, \quad j \in \Gamma_n^{(k')} \quad (2.5)$$

Clearly if $\langle u_{k,k'} \rangle_n \leq 0$ then $\Psi_{n+1} \geq \Psi_n$ establishing the desired result. This will be the case if $h = 0$ or if the system is 'properly' ferromagnetic [2]. To deal with the general case it is sufficient to obtain a bound of the form

$$\sum_{i \in \partial R \Gamma_n} \langle S_i^2 \rangle (\beta, h; r_n, b_0) \leq K_n |\partial R \Gamma_n|$$

such that $\sum K_n 2^{-n} < \infty$, since due to the finite range of the interaction we would have

$$\beta |u_{k,k'}| \leq \alpha \left(\sum_{i \in \partial R \Gamma_n(k)} S_i^2 + \sum_{j \in \partial R \Gamma_n(k')} S_j^2 \right) \quad (2.6)$$

giving $\Psi_{n+1} \geq \Psi_n - \epsilon_n, \epsilon_n = \text{Const.} (K_n 2^{-n})$.

We shall now obtain such a bound (using an idea due to E. Lieb: private communication) for the case where $\rho(S) = \exp[-V_\ell(S)], \ell \geq 4$. (The Gaussian case can of course be treated explicitly). We write for a general domain Λ with b.c. b_Λ ,

$$\begin{aligned} \langle [V'_\ell(S_i)]^2 \rangle &= -Z^{-1} \int dS_i V'_\ell(S_i) \frac{d}{dS_i} (e^{-V_\ell(S_i)}) \\ &\quad \exp[-H(S_\Lambda, b_\Lambda)] \prod_{j \neq i} [\rho(S_j) dS_j] \\ &= \langle V'_\ell(S_i) \left[\sum_j J(i-j) S_j + h_i \right] \rangle + \langle V''_\ell(S_i) \rangle \quad (2.7) \\ &\leq \langle V''_\ell(S_i) \rangle + \langle V'_\ell(S_i) h_i \rangle + \alpha \langle [V'_\ell(S_i)]^2 \rangle^{1/2} \langle S_m^2 \rangle^{1/2} \end{aligned}$$

where the prime stands for differentiation, $\langle S_m^2 \rangle = \max_{j \in \Lambda} \langle S_j^2 \rangle$

and we have integrated by parts, used Schwartz's inequality and dropped irrelevant subscripts, etc. Eq. (2.7) remains valid for $i = m$. Using then the fact that for any $k \geq 1$ and $\delta > 0$,

$\delta |S_i|^k \geq |S_i|^{k-1} + M_\delta$ we obtain

$$\langle S_i^2 \rangle_\Lambda \leq K_\Lambda, \quad \forall i \in \Lambda \quad (2.8)$$

The dependence on Λ comes in only through the h_i (or $\bar{S}_j$, $j \notin \Lambda$) so that for free boundary conditions (or when the $\bar{S}_i$ are bounded for all $i \in \mathbb{Z}^v$) we have a uniform bound $K_\Lambda = K$ and the result is established;

$$\Psi(\beta, h; \Gamma_n, b_0) \xrightarrow{n \rightarrow \infty} \Psi(\beta, h; b_0), \quad (2.9)$$

for $\rho(S) = \exp [-V_\Lambda(S)]$. (The method is clearly applicable whenever $\log \rho(S) = \Phi(S) \in C^2$ and $[\Phi'(S)]^2 \rightarrow \infty$ as $S \rightarrow \infty$ 'rapidly enough' compared to $|\Phi''(S)|$.)

Periodic b.c.

Let us consider as before the sequence of cubes Γ_n but now with periodic b.c. The "translational invariance" of $H(S_{\Gamma_n}; b_P)$ makes this case simpler to treat. Writing

$$H(S_{\Gamma_{n+1}}; b_P) = \sum_{k=1}^{2^v} H(S_{\Gamma_n^{(k)}}; b_P) + G \quad (2.10)$$

we have immediately that

$$|G| \leq \text{Const.} \sum' S_i^2 \quad (2.11)$$

where the prime indicates that the sum is restricted to i near the "boundaries" of the $\Gamma_n^{(k)}$. Thus

$$\Psi(\Gamma_{n+1}, b_P) \geq \Psi(\Gamma_n, b_P) - K_1 \langle S_i^2 \rangle (\Gamma_n, b_P) / 2^n. \quad (2.12)$$

where we have used the fact that with periodic b.c. $\langle S_i^2 \rangle =$

$\frac{1}{|\Lambda|} \sum_{j \in \Lambda} \langle S_j^2 \rangle (\Lambda, b_P)$ is the same for all i . But it is quite

easy to obtain a uniform bound on the average $\langle S_i^2 \rangle$, which is independent of the volume

$$\frac{\gamma}{|\Lambda|} \sum_{i \in \Lambda} \langle S_i^2 \rangle (\Lambda, b_P) \leq \frac{\gamma}{|\Lambda|} \log \langle \exp \sum S_i^2 \rangle$$

$$\leq \log \left(\frac{\int \exp [(\alpha+\gamma) S^2 + hS] \rho(S) dS}{\int \exp [-(\alpha+\gamma) S^2 + hS] \rho(S) dS} \right)$$

$$= K_2,$$

where we have used the same estimates as in (2.1). Thus

$$\psi(\beta, h; r_n, b_p) \xrightarrow{n \rightarrow \infty} \psi(\beta, h; b_p).$$

General b.c.

We may write generally

$$H(S_\Lambda; b_\Lambda) = H(S_\Lambda; b_0) + U_b$$

where

$$U_b = \sum_{i \in \Lambda} \sum_{j \notin \Lambda} J(i-j) S_i \bar{S}_j$$

so that

$$|U_b| \leq \alpha \left(\sum_{i \in \partial_R \Lambda} S_i^2 + \sum_{j \in \partial_R^+ \Lambda} \bar{S}_j^2 \right)$$

where $\partial_R^+ \Lambda \subset \Lambda^c$ is the set of sites in Λ^c within distance R of Λ .

Hence, using the same arguments as before, we obtain that when the $\bar{S}_j$ are bounded then

$$\lim_{n \rightarrow \infty} \psi(\beta, h; r_n, b_{r_n}) = \psi(\beta, h; b_0)$$

The same arguments also show that $\psi(\beta, h; b_p) = \psi(\beta, h; b_0)$

III. Equilibrium States

The infinite volume limit of Gibbs states with boundary conditions b will satisfy the Dobrushin, Lanford and Ruelle (DLR) equations,

$$\nu(dS_\Lambda : b) = \int \nu(dS_\Lambda | S_{\Lambda^c}) \nu(dS_{\Lambda^c} : b), \quad (3.1)$$

where

$$\nu(dS_\Lambda | S_{\Lambda^c}) = \exp[-\beta H(S_\Lambda | S_{\Lambda^c})] \prod_{i \in \Lambda} \rho(S_i) dS_i / \text{Normalization} \quad (3.2)$$

is the conditional probability of spins S in Λ given $\{S_i; i \in \Lambda^c\}$ and $H(S_\Lambda | S_{\Lambda^c})$ is given by the right side of (1.4) with

$$h_i = h + \sum_{j \in \Lambda^c} J(i-j) S_j.$$

The converse statement; any state satisfying (3.2) may be obtained as the limit of finite volume Gibbs states given by Eq. (1.5) with suitable general boundary conditions (permitting a specified probability distribution for the $S_i, i \in \Lambda^c$), is also true under mild regularity assumption on the solutions of Eq. (3.2) (see Appendix).

The questions regarding the existence and uniqueness of the infinite volume limits of Gibbs states are therefore equivalent to questions regarding the solutions of the DLR equations, e.g. iff (3.2) has a unique (state) solution (for some values of β and h) then $\nu(dS_\Lambda, \beta, h; \Lambda, b_\Lambda)$ will have a limit independent of (reasonable) b.c. Furthermore under the same type of regularity assumptions, the DLR equations are also equivalent to Kirkwood-Salsburg type equations. The latter type equations are particularly useful for proving uniqueness and analyticity of the state at high tempera-

tures and fields [5].

It is clear that for $\rho(S) \in \mathcal{L}_2$ the measures $\nu(dS_A; \Lambda, b_\Lambda)$, having their support in the unit ball of $\mathbb{R}^{n|A|}$, will have subsequences which approach limits as $\Lambda \rightarrow \infty$. Similar results will hold for $\rho(S) \in \mathcal{L}_1$ with periodic or more general b.c. such that $|h_i| \leq M, \forall i$, since we then have by (2.8) that $\langle S_j^2 \rangle_{\Lambda, b_\Lambda} \leq K$ uniformly in Λ , and this implies that for any $\epsilon > 0$ we can find a ball $B_\epsilon \subset \mathbb{R}^{n|A|}$ such that $S_A = (S_i: i \in A) \in B_\epsilon$ with probability $\geq (1-\epsilon)$.

The requirement that h_i , i.e. $S_j, j \notin \Lambda$, be bounded is however too strong a restriction since it will clearly not be satisfied by the "natural" b.c. occurring in (3.1); see Appendix.

Uniqueness-Analyticity-Decay of Correlations

It has been shown by Israel [5] that for $\rho(S) \in \mathcal{L}_2$ there is a unique equilibrium state at sufficiently high temperatures and large magnetic fields. This state, which being unique is of necessity translation invariant, has correlation functions analytic in β and h , which (for finite range potentials) decay exponentially. Similar results were proven a long time ago for Ising spins and for continuum systems [3] where "large magnetic field" corresponds to "small fugacity". We expect similar results to hold also for $\rho(S) \in \mathcal{L}_1$: indeed they have been proven for some cases of interest in field theory [6].

Since these results do not hold at phase transitions we cannot expect to extend them beyond such a limited region in the β - h plane without further restrictions on the interactions $J(\underline{x})$ and the $\rho(S)$. A particularly interesting class of systems, of importance both in statistical mechanics and in field theory, are ferromagnetic spin systems for which such nice results ought to (and in many cases have been shown to) hold for all $h = 0$. This can indeed be shown to be the case; see Appendix.

IV. Ferromagnetic Ising Spins with Boundary Fields

From the point of view of the DLR equations we need only consider b.c. which are superpositions of 'pure' b.c., b_Λ , corresponding to a specification of S_i , $i \in \Lambda^c$, with $S_i \in \text{Supp. } \rho(S)$. For some purposes however it is useful to consider also 'unrestricted' b.c. where we simply specify the h_i 's, $i \in \partial_R \Lambda$, without regard as to whether they actually come from properly specified spins outside Λ . Thus zero b.c. for $\rho(S) \in \mathcal{A}_2$ are of this type. Periodic b.c. as well as those of interest in field theory may also be put in this category of b.c. In any case however a state obtained with such b.c. should again satisfy the DLR equations.

We shall now show that for a spin- $1/2$ ferromagnetic Ising system $J(x) \geq 0$, the infinite volume equilibrium state obtained by putting on an external field $h_b > 0$ on the boundary spins is identical to v_+ ; the state obtained from + boundary conditions. To emphasize the fact that we are now dealing with the simple Ising system we shall use $\sigma_i = \pm 1$, instead of S_i , for our variables. Let us denote by $\langle \sigma_A \rangle (h_b; \Lambda)$ the expectation value of $\sigma_A = \prod_{i \in A} \sigma_i$ in a region $A \subset \Lambda$ in the presence of a boundary field h_b . As we already know, when there is an external uniform field $h \neq 0$ or when the temperature is above the critical temperature for spontaneous magnetization there is a unique equilibrium state [1] and hence we need only be concerned with the case $h = 0$, and $\langle \sigma_i \rangle_+ = m > 0$, where $\langle \sigma_1 \rangle_+ = \lim_{h \rightarrow 0} \langle \sigma_1 \rangle (h) = \lim_{\Lambda \rightarrow \infty} \langle \sigma_1 \rangle (h_+; \Lambda)$.

We first note that + boundary conditions correspond to a field h_i^+ for $i \in \partial_R \Lambda$, the boundary of Λ with $h_i^+ < \alpha = h^+$. Thus by the GKS inequalities [1,2,3] $\langle \sigma_A \rangle (h_b; \Lambda) \geq \langle \sigma_A \rangle_+(\Lambda)$ if $h_b \geq h^+$. On the other hand letting $h_b \rightarrow \infty$ clearly corresponds to putting + b.c.

on the region $\Lambda^- = \Lambda \setminus \partial_R \Lambda$. We therefore have, again by G.K.S., that for $A \subset \Lambda^-$,

$$\langle \sigma_A \rangle_+ (\Lambda^-) \geq \langle \sigma_A \rangle (h_b; \Lambda) \geq \langle \sigma_A \rangle_+ (\Lambda) \text{ for } h_b > h^+ \quad (4.1)$$

Letting $\Lambda \rightarrow Z^v$ then yields [7]

$$\langle \sigma_A \rangle (h_b) \equiv \lim_{\Lambda \rightarrow \infty} \langle \sigma_A \rangle (h_b; \Lambda) = \langle \sigma_A \rangle_+ \text{ for } h_b > h^+ \quad (4.2)$$

Our strategy for proving (4.2) for all $h_b > 0$ will be to show that $\langle \sigma_A \rangle (h_b)$ is analytic in h_b for $\text{Re } h_b > 0$. It will actually be sufficient to prove the result just for $\langle \sigma_i \rangle (h_b)$ since by the FKG inequalities [7],

$$0 \leq \langle \sigma_A \rangle_+ (\Lambda^-) - \langle \sigma_A \rangle (h_b; \Lambda) \leq \sum_{i \in A} [\langle \rho_i \rangle_+ (\Lambda^-) - \langle \rho_i \rangle (h_b; \Lambda)] \quad (4.3)$$

where $\rho_i = \frac{1}{2} (1 + \sigma_i)$, $\rho_A = \prod_{i \in A} \rho_i$.

We remark that (i) by GHS $\langle \sigma_i \rangle (h_b)$ is a concave function of h_b so that by (4.2) $\langle \sigma_i \rangle (h_b) > 0$. (ii) If σ_A is replaced by ρ_A then (4.1) and (4.2) remain valid in the presence of non-uniform fields.

The argument so far is valid for all measures in $\mathcal{L}_2$. To obtain analyticity in h_b we have to restrict ourselves to the Ising spin $1/2$ case. We use the Lee-Yang theorem in a manner similar to its use by Lebowitz and Penrose [8] for proving analyticity in a uniform external field. Letting $\eta = \exp [2\beta h_b]$, $\xi_i = \exp [2\beta h_i]$ (assuming for the moment that there is an external field present at site i) the grand partition function in Λ can be written (up to factors which are irrelevant) in the form

$$E(\eta, \xi_i; \Lambda) = P(\eta, \Lambda) + \xi_i Q(\eta; \Lambda) \quad (4.4)$$

where P and Q are polynomials in η .

We then obtain [8], when $\xi_1 = 1$,

$$\langle \sigma_i \rangle (h_b; \Lambda) = [1 - \phi(\eta; \Lambda)] / [1 + \phi(\eta; \Lambda)]$$

where $\phi(\eta; \Lambda) = P(\eta, \Lambda) / Q(\eta; \Lambda)$. By the Lee-Yang theorem $\Xi(\eta, \xi_1; \Lambda) \neq 0$ if $|\xi_1| > 1$ and $|\eta| > 1$. Hence $|\phi(\eta; \Lambda)| \leq 1$ when $|\eta| > 1$: $\phi(\eta; \Lambda)$ is thus a bounded analytic function outside the unit disk and

$$\phi(\eta; \Lambda) = [1 - \langle \sigma_i \rangle (h_b; \Lambda)] / [1 + \langle \sigma_i \rangle (h_b; \Lambda)] \xrightarrow{\Lambda \rightarrow \infty} \frac{1-m}{1+m}, \quad h_b \geq h^+$$

Therefore by Vitali's theorem $\lim_{\Lambda \rightarrow \infty} \phi(\eta; \Lambda)$ is constant for $|\eta| > 1$ which proves the result.

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Appendix:

STATISTICAL MECHANICS OF CONTINUOUS SPIN SYSTEMS

by

J.L. LEBOWITZ AND E. PRESUTTI*

Abstract:

We present results relating to the existence and uniqueness of the free energy equilibrium states for classical continuous spin systems with superstable interactions.

We consider the lattice Z^v at each site of which there is a vector spin variable S_x , $x \in Z^v$, $S_x \in R^d$. We denote by $\underline{S} \in (S)$ a spin configuration on Z^v . Each S_x has associated with it an intrinsic positive measure $\mu(dS)$, the same for all sites, such that $\int \mu(dS) e^{-\alpha S^2} < \infty$ for $\alpha > 0$. The energy of a given spin configuration S_Λ in $\Lambda \subset Z^v$ consists of both pair and self interactions and satisfies the following conditions:

a) Superstability There exists $A > 0$, $C \in R$ such that

$$U(S_\Lambda) \geq \sum_{x \in \Lambda} [A S_x^2 - C] \quad (1)$$

where S_Λ is a configuration in Λ .

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b) Regularity If Λ_1, Λ_2 are disjoint then their interaction energy $W(S_{\Lambda_1} | S_{\Lambda_2}) = U(S_{\Lambda_1 \cup \Lambda_2}) - U(S_{\Lambda_1}) - U(S_{\Lambda_2})$ has the bound

$$|W(S_{\Lambda_1} | S_{\Lambda_2})| \leq \frac{1}{2} K \sum_{x \in \Lambda_1} \sum_{y \in \Lambda_2} |S_x| |S_y| |x-y|^{-v-\epsilon} \quad (2)$$

where $|x| = \max_{1 \leq i \leq v} |x^i|$, $|S| = [\sum_{i=1}^d (S^i)^2]^{1/2}$

For Λ bounded in $\mathbb{Z}^v$ we consider the restriction, S_{Λ^c} of $\underline{S}$ to Λ^c and define the partition function $Z(\Lambda | S_{\Lambda^c})$ and free energy per site $F(\Lambda | S_{\Lambda^c})$ with 'boundary conditions' determined by $\underline{S}$ as

$$Z(\Lambda | S_{\Lambda^c}) = \int \mu_{\Lambda}(dS_{\Lambda}) \exp [-U(S_{\Lambda}) - W(S_{\Lambda} | S_{\Lambda^c})]$$

$$F(\Lambda | S_{\Lambda^c}) = |\Lambda|^{-1} \ln Z(\Lambda | S_{\Lambda^c})$$

where $\mu_{\Lambda}(dS_{\Lambda}) = \prod_{x \in \Lambda} \mu(dS_x)$, $|\Lambda| = \#$ of sites in Λ . (The dependence on temperature and magnetic field is included in U and W ; it will be made explicit when necessary.)

Theorem 1. Let (1) and (2) hold and let $\underline{S} \in \mathcal{H}_a$:

$\mathcal{H}_a = \{\underline{S} | S_y^2 \leq a \text{ and } |y| \text{ for } |y| > 1\}$. Let $\{\Lambda\}$ be a sequence of increasing domains tending to $\mathbb{Z}^v$ in the sense of Van Hove [1] then $\lim_{n \rightarrow \infty} F(\Lambda_n | S_{\Lambda_n^c}) = F$ exists and is independent of the sequence $\{\Lambda_n\}$ and of the b.c. $S_{\Lambda_n^c}$.

Remark: "Zero" b.c. correspond to, $S_x = 0$ for $x \in \Lambda^c$. The thermodynamic limit of the "periodic" b.c. free energy can also be shown to exist and be equal to F .

A probability measure ν on the configuration space (S) is said to be regular if it satisfies the following condition: There

exists $\gamma > 0$, $\delta \geq 0$, such that for every Λ bounded in Z^V and $N^2 > 0$ the following holds:

$$\nu[B(N^2|\Lambda)] \leq \exp[-|\Lambda|(\gamma N^2 - \delta)]$$

where

$$B(N^2|\Lambda) = \{ \underline{S} \mid \sum_{x \in \Lambda} S_x^2 \geq N^2 |\Lambda| \}$$

For Λ bounded in Z^V we denote by $F(\Lambda|\nu)$ the free energy in Λ for boundary conditions specified by the measure ν as

$$F(\Lambda|\nu) = \int \nu(d\underline{S}) F(\Lambda|\underline{S}_{\Lambda^c})$$

Theorem 2.

Let (1) and (2) hold and let Λ_n be as in theorem 1 then for ν regular $\lim_{n \rightarrow \infty} F(\Lambda_n|\nu) = F$.

The finite volume equilibrium measure with boundary conditions $\underline{S}_{\Lambda^c}$, $\nu_{\Lambda}(d\underline{S}_{\Lambda}|\underline{S}_{\Lambda^c})$ is given by

$$\nu_{\Lambda}(d\underline{S}_{\Lambda}|\underline{S}_{\Lambda^c}) = Z^{-1}(\Lambda|\underline{S}_{\Lambda^c}) \mu(d\underline{S}_{\Lambda}) \exp[-U(\underline{S}_{\Lambda}) - W(\underline{S}_{\Lambda}|\underline{S}_{\Lambda^c})] \quad (3)$$

A measure ν on $\{S\}$ is said to be an equilibrium measure (for our system) if its conditional probabilities $\nu(d\underline{S}_{\Lambda}|\underline{S}_{\Lambda^c})$ satisfy the Dobrushin, Lanford and Ruelle (DLR) [2] equations, i.e. eq. (3).

Theorem 3.

Let the conditions of Theorem 1 be satisfied and let $\nu_{\Lambda_n}(d\underline{S}_{\Lambda_n}|\underline{S}_{\Lambda_n^c})$ be finite volume equilibrium states: it is always

possible to choose subsequences n_i (which may depend on the b.c.) such that $\nu_{\Lambda_{n_i}}(d\underline{S}_{\Lambda_{n_i}}|\underline{S}_{\Lambda_{n_i}^c}) \rightarrow \nu$ a regular equilibrium measure on $\{S\}$, [3].

The one component spin system, $S_x \in \mathbb{R}$, will be called ferromagnetic with translation invariant interaction if

$$U(S_\Lambda) = -\frac{1}{2} \sum_{x \neq y \in \Lambda} J(x-y) S_x S_y - \sum_{x \in \Lambda} \phi(S_x) - h \sum_{x \in \Lambda} S_x; J(x) \geq 0,$$

Theorem 4.

Let ν be a regular equilibrium measure of a ferromagnetic system in an external field h whose interactions satisfy (1) and (2) then ν is unique (and hence translation invariant) whenever the infinite volume free energy $F(h)$ is differentiable with respect to h [4].

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