

NUMERICAL EXPERIMENTS IN STOCHASTICITY AND HOMOCLINIC OSCILLATION*

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INTRODUCTION

We report a series of numerical experiments examining the phenomenon often referred to as a "stochastic band," as seen in an island chain in a measure-preserving diffeomorphism on R^2 . Specifically, we employed Henon's area-preserving map, T_0 .[†]

$$x_{n+1} = x_n \cos \alpha - (y_n - x_n^2) \sin \alpha, \quad (1)$$

$$y_{n+1} = x_n \sin \alpha + (y_n - x_n^2) \cos \alpha, \quad (2)$$

with $\cos \alpha = 0.24$ ($\alpha \approx 76.1^\circ$).

This value of α was found by Henon to give a stochastic band of sufficient width to allow investigation, one apparently bounded away from the origin and infinity by closed invariant curves. The origin $(x, y = 0, 0)$ is surrounded by a chain of five hyperbolic points, fixed under $T = T_0^5$, which alternate with five elliptic points to form an island chain with which the band is associated.

FIGURE 1a was obtained by tracing from a scatter plot of a large number of iterates of a single point, and FIGURE 1b is a 20:1 blowup of the vicinity of the hyperbolic point indicated by an arrow in FIGURE 1a. The scatter visible in FIGURE 1b is genuine, not the result of numerical error. And, insofar as it is due to the phenomenon of homoclinic oscillation, it is generic.[‡]

The scatter of iterates is the characteristic feature of a stochastic band. Point sets of zero measure can be defined as a consequence of the presence of homoclinic points with strong ergodic properties.^{2,3} But, to our knowledge, no one has adduced a positive

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†An alternative form that may be more familiar to some is $p_{n+1} = q_n - p_n^2$, $q_n = b - p_n$. This is obtained with $b = \cos^2 \alpha - 2 \cos \alpha$, $x = -(p + \cos \alpha)/\sin \alpha$, and $y = (q + (p + 1) \cos \alpha)/\sin^2 \alpha$.

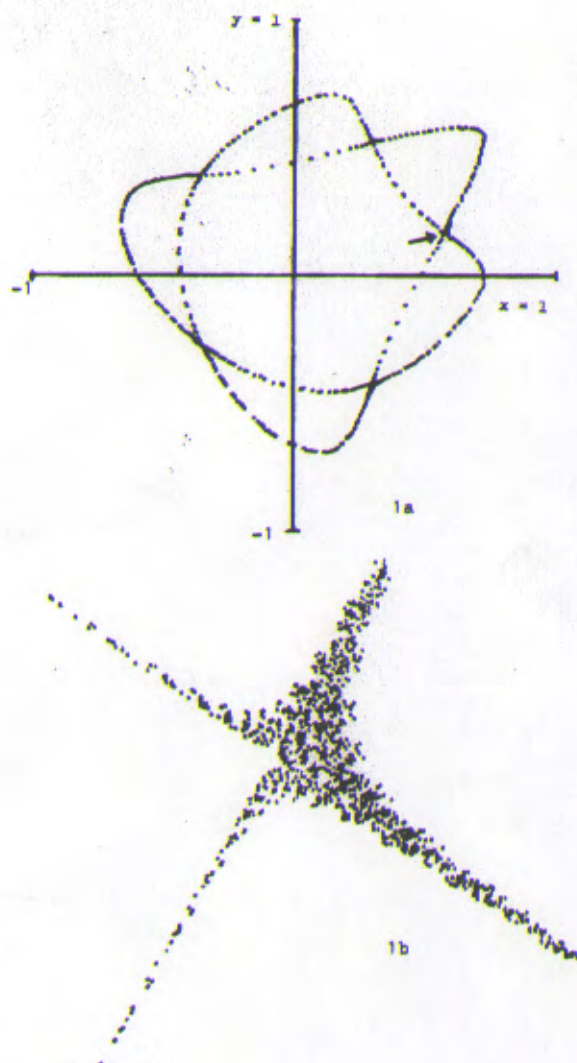


FIGURE 1. (a) A scatter plot of several thousand iterates of $(x, y) = (0.718, 0)$ under Henon's area-preserving quadratic map, T_α , with $\cos \alpha = 0.24$. (b) A 20:1 blowup of (a) in the vicinity of the hyperbolic point indicated by the arrow.

measure set with such properties.²⁻⁴ Hence, it is not at all clear in what sense, if any, these bands deserve the adjective "stochastic." Our aim is to clarify this matter.

AUTOCORRELATIONS

We first sought evidence of strongly ergodic behavior in the form of decay of autocorrelation functions computed for individual orbits. (Partly, this was because we could not come up with an implementable definition of an interesting ensemble; the presence of elliptic fixed points of high order in the region in question seemed to rule out any obvious approach.) For f_i , a function evaluated at the i th iterate of some point chosen to give an orbit exhibiting scatter, we evaluated

$$C_M(\tau) = \frac{1}{M} \sum_{i=0}^{M-1} (f_i - \bar{f}) f_{i+\tau}, \quad (3)$$

where

$$\bar{f} = \frac{1}{M} \sum_{i=0}^{M-1} f_i, \quad (4)$$

and looked for decay with increasing τ . Generally, we used $f_i = f(x_i, y_i) = x_i$, and, to eliminate large scale oscillation in C_M due to the more or less steady counterclockwise rotation by 72° produced by T_0 , we looked at iterates of $T = T_0^5$. The result appeared encouraging, at first, in that small values were obtained for local maxima of C_M . E.g., for $(x_0, y_0) = (0.718, 0)$, $M = 6500$, and $250 < \tau < 500$, we found $|C_M| < 0.2$. Unfortunately, increasing M slightly, to $M = 7500$, produced large fractional changes, with some of the local maxima varying by a factor of two. In short, even though we were employing every fifth point out of 4×10^4 iterates under T_0 , computed, as will be discussed, with very high precision, there was no evidence that we were even close to convergence. For this and other reasons, we concluded that autocorrelations may be of doubtful efficacy in the numerical investigation of stochasticity in measure-preserving mappings not dominated by chaotic behavior.

ORBIT PHENOMENOLOGY

In order to understand this null result, we examined the evolution of individual orbits directly, using scatter plots of successive sets of iterates under $T = T_0^5$. FIGURES 2a-f were obtained by plotting $T^n(x, y)$, $n = 1, \dots, 8 \times 10^3$, where $(x, y) = T_0(0.718, 0)$. The figures were traced by hand from computer output, and only their qualitative aspects should be relied upon.

A special effort to overcome the exponential accumulation of numerical error is required when one examines long term properties of individual orbits in a stochastic band. Fortunately, it is possible to deal with this problem for polynomial mappings without having to account for truncation error; only roundoff error need be considered. We have implemented very high precision arithmetic in FORTRAN on a DEC PDP-10 computer, employing a number system with a large base and providing for nominally unbiased handling of residual roundoff.

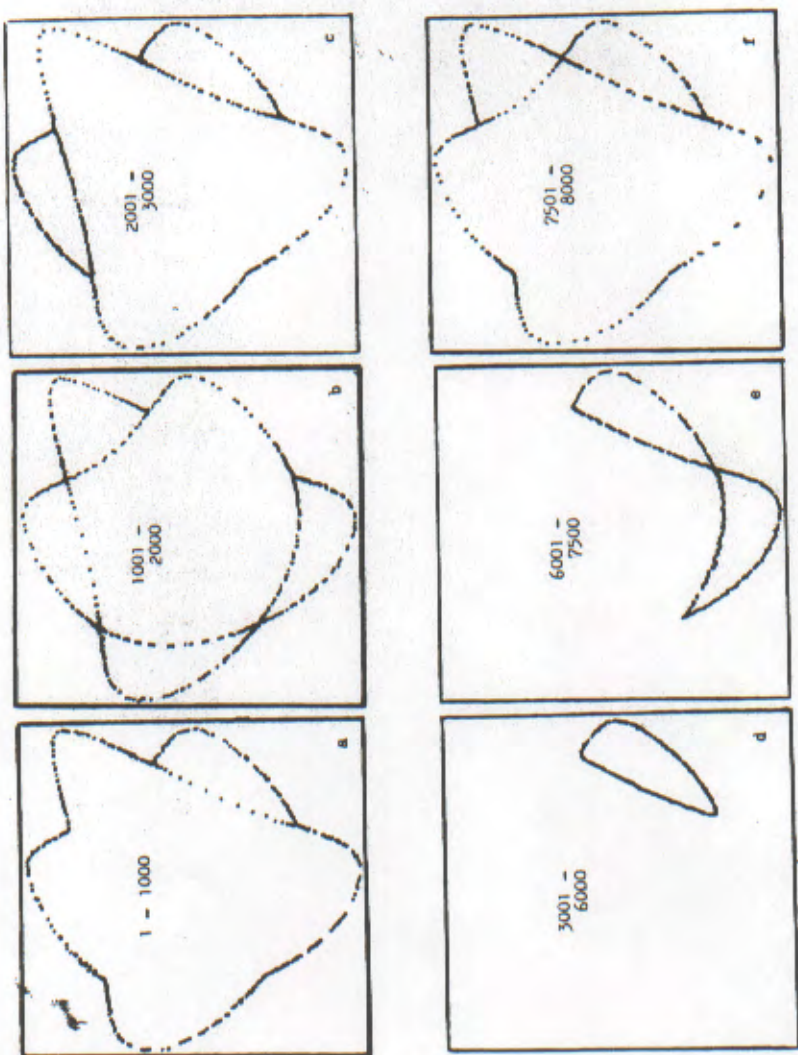


FIGURE 2. 8000 iterates under $T - T_0^2$ of $(x, y) - T_0(0.718, 0)$, illustrating irregular alternation between "free" and "trapped" states. Computed with very high accuracy.

For the particular orbit we examined, 44 digits base 2^{27} were employed, or approximately 358 decimal digits. Experience with orbits shorter by factors of two and four indicated that accuracy to within 10^{-6} may be expected if about 850 iterations under T_0 are allowed per digit. An assessment of accuracy was made for the shorter orbits by recomputing them with much higher precision. It may be worth noting that we examined behavior under the inverse map for several orbits where, by the recomputation test, it was known that a final computed point was in error by an amount $\sim 10^{-6}$ (immensely larger than the roundoff error committed on each step). We found, for the cases examined, that initial conditions were recovered with an error $\leq 10^{-6}$ under orbit reversal.

Figure 2a shows iterates 1-1000, under T . When we examined them in groups of 50, we found that the first 400 (approximately) moved more or less steadily about the outer perimeter of the island chain in the clockwise direction, requiring about 100 steps per revolution; the next 50 moved clockwise about the centroid of the island, where we see iterates on the inner perimeter, completing one revolution; and the final 550 moved as did the first 400. The distinction between these two modes of behavior will appear throughout the remainder of this paper. We shall refer to orbit segments as "trapped" or "free," depending upon whether their motion consists in revolution about a single island or about the entire chain, respectively. This definition will be sharpened later, when we show that a precise and meaningful—albeit somewhat arbitrary—definition can be given for the boundary of an island. Orbit "segments" will be considered to begin and end at crossings of these boundaries, i.e., at "transitions."

In FIGURE 2b, which includes iterates 1001-2000, we see several trapped segments, some probably involving more than one revolution about the island centroid. Despite the impression given by FIGURE 2a, we observe that the typical interval between trapped segments is not necessarily large, nor is the free state necessarily dominant. FIGURE 2c presents iterates 2001-3000, probably dominated by two long trapped segments. Figure 2d, with iterates 3001-6000, drives home the point that very long trapped segments can and do occur. The observation of such long segments suggests that the fraction of segments surviving in a given state will not be well described by a decaying exponential. FIGURE 2e shows iterates 6001-7500, establishing that the very long trapped segment does not last forever and demonstrating a short free segment. Finally, FIGURE 2f, with iterates 7501-8000, illustrates more or less typical behavior, indicating that no "secular" evolution has taken place due to numerical error.

One of the outstanding features observed here is that segment lengths, both free and trapped, can be quite large. In fact, numerical results will be presented for segments sampled from an appropriate set, showing that the fraction having length t decays roughly as $t^{-3/2}$. Taking that result to be exact and valid for all t , it would follow that segment lengths are unbounded in the mean. Whether the mean is unbounded or merely very large, however, it seems clear that, to the extent that long segments are typical, it will be difficult to obtain convergence in single orbit autocorrelation computations.

CONSEQUENCES OF HOMOCLINIC OSCILLATION

We now consider some consequences of homoclinic oscillation in an island chain in order to refine some of the concepts just employed and to provide a context for later

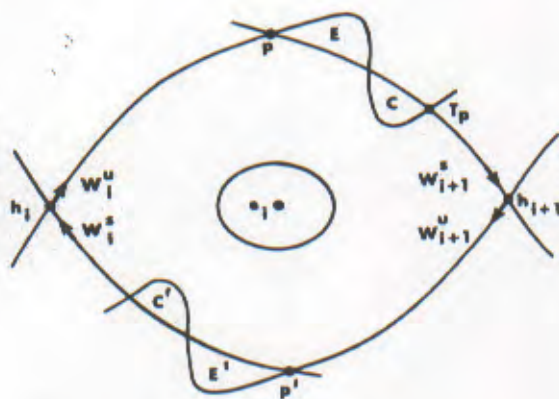
experiments in stochasticity. We assume that we deal with the structure implied by a chain of hyperbolic points, h_i , $i = 1, \dots, n$, fixed under $T = T_0^n$, an area-preserving diffeomorphism ($n = 5$ for our calculations), and that orbits are bounded away from infinity.

The interesting structure arises from the existence, invariance,³⁻⁷ and generically transversal intersection² of the stable and unstable manifolds W_i^s and W_i^u ,

$$W_i^{s,u} = \{x \in R^2; \lim_{m \rightarrow \infty} T^{m,-m}x = h_i\}. \quad (5)$$

FIGURE 3 is a sketch showing two hyperbolic points, h_i and h_{i+1} , portions of the four manifolds, and one of the elliptic fixed points that alternate with the h_i , e_i . Please note that liberties have been taken with the scale, as oscillations are often confined to a very thin band—the situation in the case we examined. While we do not show it, a very complex figure quickly results when the manifolds are extended. One should keep in mind that the image of a point of intersection is also an intersection, that areas are preserved, that this is only one island of a chain, and that, if two manifolds intersect, one must be stable and the other unstable.

FIGURE 3. Portions of the stable and unstable manifolds associated with and used to define one "island" in a chain. An illustration of the early development of heteroclinic oscillation.



Let p be a homoclinic point in $W_i^u \cap W_{i+1}^s$, with the additional property that arcs $h_i p \subset W_i^u$ and $p h_{i+1} \subset W_{i+1}^s$ intersect only at p . Let p' be a similar point in $W_i^s \cap W_{i+1}^u$. For the i th island, A_i , we define the boundary as

$$\sigma_i = h_i p \cup p h_{i+1} \cup h_{i+1} p' \cup p' h_i, \quad (6)$$

and observe that $TA_i = \{x; T^{-1}x \in A_i\}$ is bounded by

$$T\sigma_i = h_i T p \cup (T p) h_{i+1} \cup h_{i+1} T p' \cup (T p') h_i. \quad (7)$$

Note that $T\sigma_i$ is also a simple closed curve. It is easily seen that

$$h_i p \subset h_i T p, \quad p h_{i+1} \supset (T p) h_{i+1}, \quad h_{i+1} p' \subset h_{i+1} T p', \quad p' h_i \supset (T p') h_i, \quad (8)$$

so that $T\sigma_i$ is coincident with σ_i except for arcs $p T p$ and arcs $p' T p'$. In our drawing, the arcs $p T p$ intersect once between p and $T p$, and it can be shown that there must be an odd number of such intervening intersections. We denote the "lobes" above $p T p \subset \sigma_i$

by E , collectively, and those below by C ; similar remarks lead to lobes E' and C' . Notice that $E \cup E' \subset TA_i$ and $(E \cup E') \cap A_i = \emptyset$, while $(C \cup C') \cap TA_i = \emptyset$ and $C \cup C' \subset A_i$, and that no other elements of area can be so described.

The conclusion is that a set of positive measure will "escape" from the island on each iteration to form $E \cup E'$, a set of equal measure will enter as $C \cup C'$, and points can escape or be captured in no other way. We may say, therefore, that all orbits exhibiting the alternating free/trapped behavior discussed in connection with FIGURE 2 have a point in some C , and hence lie entirely within the invariant set

$$S = \bigcup_{m=-\infty}^{\infty} T^m \left\{ \bigcup_{i=1}^n C(i) \cup C'(i) \right\}, \quad (9)$$

where $C(i)$ is a lobe associated with the i th island. In fact, if free and trapped are relative to σ_n , orbits exhibit this behavior iff they lie entirely within S (save for exceptions of measure zero). There is some risk of triviality here if, e.g., $(T^k C) \cap E \neq \emptyset$ for small $|k|$, as the distinction between trapped and free might then be uninteresting. However, numerical experiments with our system, together with various qualitative features of a general nature, make it seem unlikely that such is a general rule.

Having found an explanation for the alternating free/trapped behavior in terms of heteroclinic oscillation and membership in S , it occurred to us that S is a good candidate for a set exhibiting truly stochastic properties such as ergodicity, mixing, and hyperbolicity, if, indeed, any such sets exist. Two supporting observations may be offered. The first is that S and all stochastic sets depend for their existence on transversal intersection, and the second is that the minimal order of periodic orbits in S is likely to be a large multiple of the order of the chain giving rise to S . In some sense, there is a large lower bound on the degree of complexity to be associated with orbits in S .

EXPERIMENTS ON THE POSSIBLY CHAOTIC SET, S

One of the major advantages of S is that it can be generated from a finite number of lobes that, in turn, are susceptible to numerical definition. We used a binary search procedure to locate about 100 points on each of the four manifolds associated with an island; these points were spaced closely enough to give good visual definition of the lobes corresponding to C , E , C' , and E' when plotted under magnification sufficient to reveal the structure. In the process, we verified that FIGURE 3 is qualitatively correct for our case and found an explanation for the puzzling observation that free segments corresponding to clockwise motion along the outer perimeter of the island chain were seen, but free segments corresponding to counterclockwise motion along the inner perimeter were not. If the lobe E is taken to be near the outer perimeter, it turns out that $\mu(E) \approx 5 \times 10^{-6}$, while $\mu(E') \approx 7 \times 10^{-14}$. Thus, about one part in 10^8 of the points escaping from the island on any one iteration might be expected to wind up moving along the inner perimeter.

We approximated the boundary of C by straight line segments connecting the defining points. Initial conditions for some 7750 orbits were selected in Monte Carlo style from the region thus defined with a FORTRAN-supplied random number

generator, so as to be evenly distributed over C . The measure of the region misclassified under the polygonal approximation for C has been examined numerically; it turns out to be on the order of 1% of the measure of C .

The data to be analyzed was obtained from a set of 7750 orbits, iterated in two stages. In the first stage, we followed the orbits in the trapped state for 1000 applications of the fifth power of the quadratic map, or until they were diagnosed as having escaped, keeping track of the number of steps and the number of revolutions about the island centroid. In the second stage, we continued to follow the 6426 orbits that escaped in less than 1000 steps, now in the free state, for another 1000 iterations or until they were diagnosed as having been captured again, keeping track of the number of additional steps and the number of islands passed.

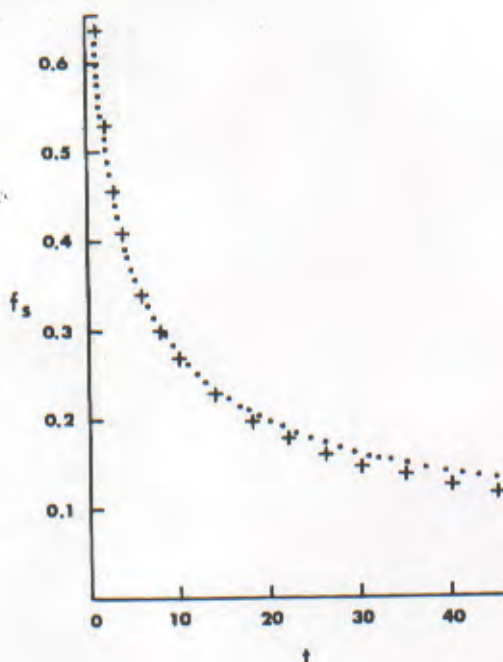


FIGURE 4. The pluses indicate the numerically determined fraction, f_s , of free segments passing t islands or more before being trapped. The dotted line represents a function $\propto 1/(t+1)^{1/2}$ normalized to agree with f_s at $t=1$.

In examining the evolution of individual orbits, it proved helpful to describe them as composed of segments connected by transitions between the "free" and "trapped" states. The segments consist of varying numbers of iterates, but the differences that seem most impressive have to do with how many times a trapped segment revolves about the island centroid or how many islands a free segment passes by in proceeding from one transition to the next. Let time and length refer to the number of islands passed or to the number of revolutions when speaking of segments. We have observed that the distribution of the number of iterates in a segment is rather sharp; e.g., of 16 free segments of length 44, the number of steps ranged between 819 and 882, with nine in the range 841 to 860.

While there probably is not a one-to-one relationship between orbits and sequences of segment lengths, it does not seem that much information is lost by focusing attention on such sequences, and it leads to some tentative—albeit interesting and simply stated—results.

In our discussion of FIGURE 2, we noted that the behavior of the orbit examined suggested that, while short segments were most common, long ones occurred more frequently than an exponential decay model might predict. We looked for and found confirmation on this point. FIGURE 4 shows the fraction, f_t , of the 6426 free segments that survived beyond "time" t , t ranging from 1 to 45. (We have data for segments involving up to 52 island passages, but termination of orbits at the 999th step renders that data incomplete. This effect was taken into account in all the results presented here.) The dotted line represents a function, $P_t \propto (t+1)^{-1/2}$, normalized to agree with the data at $t=1$. We see that agreement is generally good, especially at small t , where the effects of roundoff error should be smallest. We use $t+1$ in this fit instead of t , partly because it yields unambiguously better agreement, and partly because the quantity plotted can be interpreted as the fraction that survives until at least "time" $t+1$. A log-linear plot (not shown) makes it very clear that the data are not consistent with an exponentially decaying survival probability.

Column 3 of the table, headed "all," gives the fraction of the 6426 available free segments that survive beyond the "times" specified in column 1. Forcing agreement between f_t and P_t at $t=1$ gives

$$P_t = \frac{0.9179 \pm 0.0084}{\sqrt{t+1}}, \quad (10)$$

where the uncertainty estimate is derived by computing the rms deviation expected for a binomial distribution characterized by 6426 samples and a success probability of

TABLE I
SURVIVAL PROBABILITY AS A FUNCTION OF "TIME"
(Number of revolutions for trapped segments,
number of islands passed for free segments)

"Time"	Trapped	Free					
		all	$t_e = 1$	$t_e > 1$	$t_e > 2$	$t_e > 3$	$t_e > 4$
1	.6403	.6491	.6786	.6264	.6187	.6183	.6173
2	.5289	.5430	.5667	.5247	.5186	.5149	.5150
3	.4517	.4743	.5018	.4533	.4461	.4378	.4398
4	.3991	.4177	.4455	.3964	.3924	.3822	.3827
5	.3618	.3816	.4118	.3584	.3553	.3441	.3437
10	.2628	.2772	.3045	.2561	.2548	.2462	.2493
15	.2166	.2280	.2504	.2108	.2076	.1971	.2001
20	.1874	.1950	.2159	.1789	.1773	.1727	.1752
25	—	.1757	.1955	.1605	.1582	.1534	.1560
30	—	.1584	.1765	.1446	.1413	.1373	.1396
35	—	.1439	.1614	.1306	.1276	.1254	.1278
40	—	.1351	.1521	.1220	.1200	.1176	.1204
45	—	.1262	.1402	.1154	.1135	.1102	.1131
No. of orbits:	7750	6429	2788	3638	2775	2177	1769

0.6491. Despite the fact that this fit does very well out to $t = 10$, it clearly does not extrapolate back to $t = 0$ (by definition, $f_t(0) = 1$). It would be interesting to know how much normalization depends on parameters of the chain.

The deviations seen at the larger values of t may be significant at the 2-sigma level, but the point is moot, if only because the roundoff error is likely to be a significant factor. These computations were done with FORTRAN-supplied double precision arithmetic implemented on an LSI 11/3 minicomputer (giving a 56 bit mantissa); it was not considered feasible to employ very high precision arithmetic on that machine. Experience with higher precision calculations suggests that we can iterate the quadratic map about 1700 times and look for accuracy to within 10^{-6} . This takes us out to about $t = 15$ in FIGURE 4.

For larger values of t , we may hope for some help from the shadow orbit theorem,⁸⁻⁹ which was discussed in a numerical context by Benettin et al.¹⁰ That is, while computation may produce iterates that differ significantly from the actual orbit corresponding to the given initial conditions, it would not be surprising to find nearby initial conditions corresponding to an actual orbit having behavior similar to that computed. This argument persuades us not to suppress results for the longer segments. Nonetheless, it is entirely possible that roundoff error results in a biased set of shadow orbits, perhaps enhancing the relative probability of an early transition.

We also looked at the distribution of segment lengths (numbers of revolutions) for the 7750 trapped segments. Recall that all of the orbits began in the trapped state. Rather unexpectedly, we found that the fraction of trapped segments surviving after t revolutions is difficult to distinguish from the fraction of free segments surviving after passing t islands. Column 2 of the table, headed "trapped," gives the fraction surviving beyond selected values of t , up to $t = 20$.

Finally, we looked for dependence of segment length distributions on the length of the preceding segment. Columns 4-8 show the fractions surviving in the free state based on various subsets of the available 6426. We considered the subset consisting of those which had escaped on the first revolution ($t_s = 1$), and the subsets corresponding to escape after $t_s = 1, 2, 3$, and 4. The data for each column are rather well approximated by $P_t = P_s(0)/(1 + t)^{1/2}$. (It should be pointed out that such behavior cannot continue indefinitely if motion is, in fact, bounded.) However, the normalizing $P_s(0)$ varies from 0.873 ± 0.012 for $t_s > 4$ to 0.960 ± 0.009 for $t_s = 1$. Hence, it would appear that the distribution of sojourn "time" in the free state has some small dependence on "time" spent in the preceding trapped state.

We conclude that the data suggests (as approximations) two hypotheses concerning statistics governing sequences of segment lengths for orbits in S :

1. There is no distinction to be made between free and trapped.
2. Successive segment lengths are statistically independent.

We will continue our investigation of these phenomena and their relationship to the various concepts ordinarily invoked for the analysis of stochastic behavior in deterministic dynamical systems.^{3,4,11,12}

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